The wave equation (IBVP)

Conservation of Energy (Dirichlet data)

$$\begin{cases} \ddot{u} - u'' = 0, & \text{in } (0,1) \quad \text{(DE)} \\ u(0,t) = u(1,t) = 0 & \text{(BC)} \\ u(x,0) = u_0(x) & \text{(IC)} \\ \dot{u}(x,0) = v_0(x) & \end{cases}$$

(Same as $u_{tt} - u_{xx} = 0$)

Recall:

$$\|w\|^2 = \|w(\cdot,t)\|^2 = \int_0^1 |w(x,t)|^2 dx$$

Conservation of energy:

Multiply the (DE) with $\dot{u} = \frac{du}{dt}$ and integrate over $I := (0,1) \Rightarrow$

$$\int_0^1 \ddot{u}\dot{u} - \int_0^1 u''\dot{u} = 0 \overset{PI}{\Longrightarrow} \int_0^1 \frac{1}{2}\frac{d}{dt}(\dot{u})^2 dx + \int_0^1 u'\dot{u}' dx - \underbrace{\Big[u'(x,t)\dot{u}(x,t)\Big]_0^1}_{(=0 \text{ using } \frac{d}{dt}(BC))} = 0$$

$$\Leftrightarrow \int_0^1 \frac{1}{2}\frac{d}{dt}(\dot{u}^2)\,dx + \int_0^1 \frac{1}{2}\frac{d}{dt}(u'^2)\,dx = 0$$

$$\Leftrightarrow \frac{1}{2}\frac{d}{dt}\left(\|\dot{u}\|^2 + \|u'\|^2\right) = 0. \quad \text{i.e.}$$

$$\frac{1}{2}\|\dot{u}\|^2 + \frac{1}{2}\|u'\|^2 = \{\text{Constant, independent of } t\} = \frac{1}{2}\|v_0\|^2 + \frac{1}{2}\|u_0'\|^2.$$

Therefore the total energy is conserved

$\frac{1}{2}\|\dot{u}\|^2$, is kinetic energy

$\frac{1}{2}\|u'\|^2$, is potential (elastic) energy.

Exercise 1.

Show that $\|(\dot{u})'\|^2 + \|u''\|^2 = \text{Constant}.$

Hint: Multiply the (DE) by $-(\dot{u})''$ and integrate over I.

Alternative: differentiate w.r.t. x and multiply by $\dot{u}$....

Construction CG(1)–cG(1), FEM for the Wave equation
Mixed boundary Conditions — 2 —

M.A.
Lecture 10/-2-

We seek piecewise linear continuous $\overbrace{\text{polynomial}}^{\text{space-time}}$ approximation for

$$\begin{cases} \ddot{u} - u'' = 0, & 0<x<1, \quad t>0 \\ u(0,t)=0, \quad u'(1,t)=g(t), & t>0 \\ u(x,0)=u_0(x), \quad \dot{u}(x,0)=v_0(x), & 0<x<1 \end{cases}$$

Reformulate the problem as a system of PDEs:

$$(*)\quad \begin{cases} \dot{u} - v = 0 & \text{(Convection)} \\ \dot{v} - u'' = 0 & \text{(Diffusion)} \end{cases}$$

Let $w=\begin{pmatrix} u \\ v \end{pmatrix}$, then $(*)$ can be written as $\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ -\frac{d^2}{dx^2} & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\underline{\dot{w} + Aw = 0} \quad \text{with} \quad A = \begin{pmatrix} 0 & -1 \\ -\frac{d^2}{dx^2} & 0 \end{pmatrix}. \qquad \begin{pmatrix} \text{compare with} \\ \dot{u} + au = 0 \end{pmatrix}$$

Let now for each n, the $\overbrace{\text{piecewise}}^{\text{space-time}}$ linear approximation

$$\begin{cases} U(x,t) = U_{n-1}(x)\Psi_{n-1}(t) + U_n(x)\Psi_n(t), & t\in I_n = (t_{n-1}, t_n) \\ V(x,t) = V_{n-1}(x)\Psi_{n-1}(t) + V_n(x)\Psi_n(t) & \text{—} \;\; " \;\; \text{—} \end{cases}$$

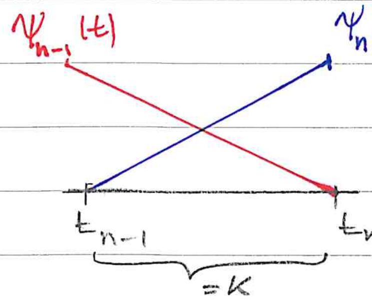
where

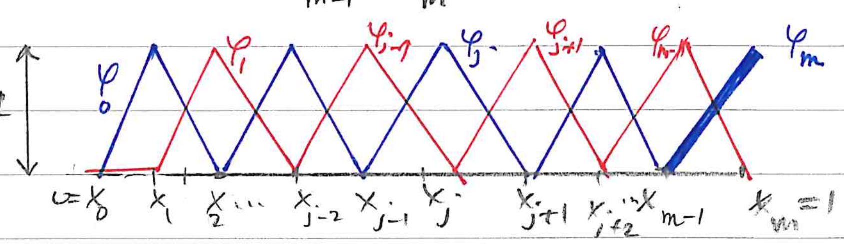
$$U_{\tilde{n}}(x) = U_{\tilde{n},1}\varphi_1(x) + U_{\tilde{n},2}\varphi_2(x) + \dots + U_{\tilde{n},m}\varphi_m(x),$$

likewise $\quad V_{\tilde{n}}(x) = V_{\tilde{n},1}\varphi_1(x) + \dots + V_{\tilde{n},m}\varphi_m(x) \qquad \tilde{n} = n-1, n$

OBS! Just for the "uniformity" we choose $m+1$ partition points in x:

$$0 = x_0 < x_1 < \dots < x_{m-1} < x_m = 1$$





Now for $\dot{u}-v=0$, $t\in I_n=(t_{n-1},t_n]$ we write the general VF:

$$(I): \int_{I_n}\int_0^1 \dot{u}\varphi\,dx\,dt-\int_{I_n}\int_0^1 v\varphi\,dx\,dt=0, \quad \text{for all } \varphi(x,t) \text{ (how!)}$$

Similarly for $\dot{v}-u''=0$ & $u'(1,t)=g(t)$ we have the VF:

$$[PI] \Rightarrow \int_{I_n}\int_0^1 \dot{v}\varphi+\int_{I_n}\int_0^1 u'\varphi'-\int_{I_n} u'\varphi\Big|_{x=0}^{x=1} dt=0 \iff$$

$$(II): \int_{I_n}\int_0^1 \dot{v}\varphi\,dx\,dt+\int_{I_n}\int_0^1 u'\varphi'\,dx\,dt=\int_{I_n} g(t)\,\varphi(1,t)\,dt,$$

for all φ such that $\varphi(0,t)=0$.

We therefore seek $U(x,t)$ and $V(x,t)$ such that

$$(I): \int_{I_n}\int_0^1 \frac{U_n(x)-U_{n-1}(x)}{k}\varphi_j(x)-\int_{I_n}\int_0^1\Big(V_{n-1}(x)\Psi_{n-1}(t)+V_n(x)\Psi_n(t)\Big)\varphi_j(x)=0$$

$j=1,2,\dots,m$

$$(II): \int_{I_n}\int_0^1 \frac{V_n(x)-V_{n-1}(x)}{k}\varphi_j(x)+\int_{I_n}\int_0^1\Big(U'_{n-1}(x)\Psi_{n-1}(t)+U'_n(x)\Psi_n(t)\Big)\varphi'_j(x)=\int_{I_n} g(t)\varphi_j(1)$$

This is reduced to the iterative forms:

$$\underbrace{\int_0^1 U_n(x)\varphi_j(x)}_{MU_n}-\frac{k}{2}\underbrace{\int_0^1 V_n(x)\varphi_j(x)}_{MV_n}=\underbrace{\int_0^1 U_{n-1}(x)\varphi_j(x)}_{MU_{n-1}}+\frac{k}{2}\underbrace{\int_0^1 V_{n-1}(x)\varphi_j(x)}_{MV_{n-1}}, \quad j=1,2,\dots,m.$$

and

$$\underbrace{\int_0^1 V_n(x)\varphi_j(x)}_{MV_n}+\frac{k}{2}\underbrace{\int_0^1 U'_n(x)\varphi'_j(x)}_{SU_n}=\underbrace{\int_0^1 V_{n-1}(x)\varphi_j(x)}_{MV_{n-1}}-\frac{k}{2}\overbrace{\int_0^1 U'_{n-1}(x)\varphi'_j(x)}^{SU_{n-1}}+g_n;$$

$j=1,2,\dots,m,$

respectively, where

$$S = \frac{1}{h}\begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ & & \ddots & \ddots & \ddots & \\ & & & -1 & 2 & -1 \\ & & & 0 & -1 & 1 \end{bmatrix} \quad \& \quad M = \frac{h}{6}\begin{bmatrix} 4 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 4 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 4 & 1 & \cdots & 0 \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 4 & 1 \\ & & & 0 & 1 & 2 \end{bmatrix}$$

(due to half hat function : φ_m)

$$\mathcal{U}_n = \begin{bmatrix} \mathcal{U}_{n,1} \\ \mathcal{U}_{n,2} \\ \vdots \\ \mathcal{U}_{n,m} \end{bmatrix}, \quad V_n = \begin{bmatrix} V_{n,1} \\ V_{n,2} \\ \vdots \\ V_{n,m} \end{bmatrix}, \quad g = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g_{n,m} \end{bmatrix}, \quad g_{n,m} = \int_{I_n} g(t)\,dt.$$

In the compact form the vectors $\mathcal{U}_n$ & V_n are determined through solving the linear system of equations:

$$\begin{cases} M\mathcal{U}_n - \frac{k}{2} M V_n = M\mathcal{U}_{n-1} + \frac{k}{2} M V_{n-1} \\ \frac{k}{2} S\mathcal{U}_n + M V_n = -\frac{k}{2} S\mathcal{U}_{n-1} + M V_{n-1} + g_n \end{cases}$$

This is a system of 2m equations with 2m unknowns, which in matrix form is written as

$$\underbrace{\begin{bmatrix} M & -\frac{k}{2} M \\ \frac{k}{2} S & M \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \mathcal{U}_n \\ V_n \end{bmatrix}}_{W} = \underbrace{\begin{bmatrix} M & \frac{k}{2} M \\ -\frac{k}{2} S & M \end{bmatrix} \begin{bmatrix} \mathcal{U}_{n-1} \\ V_{n-1} \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ g_n \end{bmatrix}}_{b}.$$

With $W = A \backslash b$, $\mathcal{U}_n = W(1:m)$ & $V_n = W(m+1:2m)$.