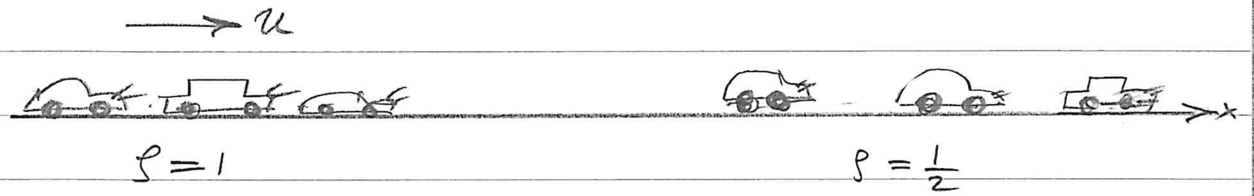
The convection problem:

Example; The traffic flow in a highway:



Let $\rho = \rho(x,t)$ be the density of cars $(0 \le \rho \le 1)$ and $u = u(x,t)$ the velocity (speed vector) of the cars.

For a highway path (a, b) the difference between the traffic inflow $u(a)\rho(a)$ at the point $x = a$ and outflow $u(b)\rho(b)$ at $x = b$ gives the density variation on the interval (a,b):

$$\frac{d}{dt}\left(\int_a^b \rho(x,t)\,dx\right) = \int_a^b \dot{\rho}(x,t)\,dx = \rho(a)u(a) - \rho(b)u(b)$$

$$= -\int_a^b (u\rho)'\,dx.$$

$$\Longleftrightarrow \quad \int_a^b \{\dot{\rho} + (u\rho)'\}\,dx = 0.$$

Since a & b can be choosen arbitrary. Thus

$$(*) \qquad \underline{\dot{\rho} + (u\rho)' = 0.}$$

Let now $\underline{u = 1 - \rho}$ (motivate) then $\dot{\rho} + ((1-\rho)\rho)' = 0$,

i.e.

$$\boxed{\dot{\rho} + (1 - 2\rho)\rho' = 0}$$

A non-linear Convection-equation

As an alternative let now $u = c - \varepsilon\left(\rho'/\rho\right)$ in $(*)$, $c>0$, $\varepsilon>0$ (Motivate). Then we get

$$\dot\rho + \left(\left(c - \varepsilon\frac{\rho'}{\rho}\right)\rho\right)' = 0 \iff \underline{\dot\rho + c\rho' - \varepsilon\rho'' = 0}.$$

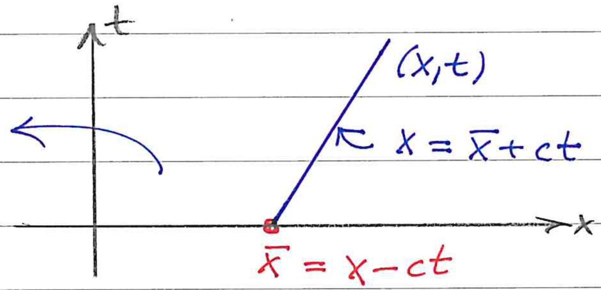
i.e., A time dependent convection – diffusion equation (Convection dominated if $c > \varepsilon$)

For $\varepsilon = 0$ the solution is given by the exact transport:

$\rho(x,t) = \rho_0(x - ct)$, because then $\rho =$ constant on the $(c,1)$ – direction:

$\iff (c,1)\cdot\left(\frac{\partial\rho}{\partial x}, \frac{\partial\rho}{\partial t}\right) = 0$;

$$\left(\begin{array}{l} \rho(x,t) = \rho(\bar x + ct, t) \\ \frac{\partial\rho}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial\rho}{\partial t} = 0 \Rightarrow \\ c\rho' + \dot\rho = 0 \quad \dots \end{array}\right)$$



Change of notation $\rho \rightsquigarrow u \overset{c \rightsquigarrow \beta}{\Longrightarrow} \underline{\dot u + \beta u' - \varepsilon u'' = 0}.$

Compare to the Navier–Stokes equation for incompressible flow:

$$\begin{cases} \dot u + (\beta\cdot\nabla)u - \varepsilon\,\Delta u + \Delta P \equiv 0 \\ \qquad\qquad \operatorname{div} u = 0 \end{cases}$$

where $\beta = u$, $\underline{u} = (\overset{\text{mass}}{u_1}, \underset{\text{momentum}}{u_2}, \underset{\text{energy}}{u_3})$ is the velocity vector,

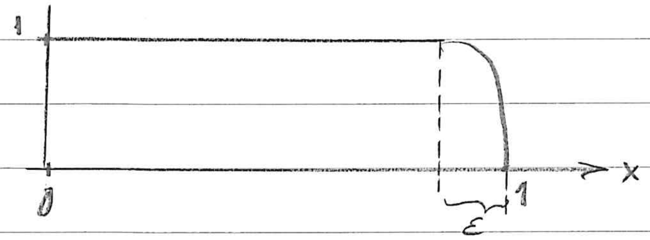
P is the pressure and $\varepsilon = \frac{1}{Re}$, with Re being the Reynolds' number.

These equations are not easily solvable for $\varepsilon > 0$ small, because of boundary layer and turbulence. A typical range of Re is between 10^5 and 10^7.

Example Consider the (BVP):
$$\begin{cases} u' - \varepsilon u'' = 0, & 0<x<1 \\ u(0)=1, \quad u(1)=0. \end{cases}$$

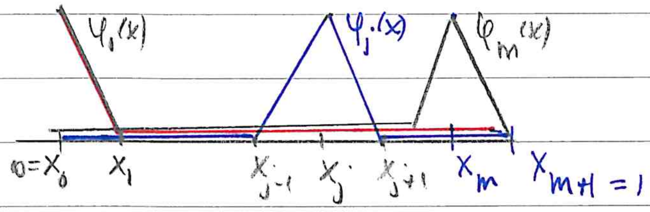
The exact solution is given by

$$u(x) = c\left(e^{\frac{1}{\varepsilon}} - e^{\frac{x}{\varepsilon}}\right), \qquad \text{with } c = \frac{1}{\left(e^{\frac{1}{\varepsilon}}-1\right)}.$$



Note an outflow boundary layer of width $\sim \varepsilon$

FEM. Set $U(x) = \varphi_0(x) + U_1\varphi_1(x) + \dots + U_m\varphi_m(x)$



Inserting in corresponding discrete variational formulation:

(C is convection matrix)
$$\underbrace{\int_0^1 U'\varphi_j}_{CU} + \varepsilon \underbrace{U'\varphi_j'}_{SU} = 0, \quad j=1,\dots,m$$

yields (row j): $\int_0^1 \varphi_j'\varphi_j = 0$, $\int_0^1 \varphi_{j-1}'\varphi_j = -\frac{1}{2}$, $\int_0^1 \varphi_{j+1}'\varphi_j = \frac{1}{2}$

$$(**)\quad \boxed{\frac{1}{2}\left(U_{j+1} - U_{j-1}\right) + \frac{\varepsilon}{h}\left(2U_j - U_{j-1} - U_{j+1}\right) = 0}, \quad j=1,\dots,m$$

, where $U_0 = 1$ & $U_{m+1} = 0$

Notice that we can also use central-differencing and write:

$$\underbrace{\frac{U_{j+1}-U_{j-1}}{2h}}_{\text{Corresp. to } U'(x_j)} - \varepsilon \underbrace{\frac{U_{j+1} - 2U_j + U_{j-1}}{h^2}}_{\text{Corresp. to } U''(x_j)} = 0 \quad \left(\Leftrightarrow \frac{(**)}{h}\right) \text{ OBS!}$$

For ε very small this gives that $U_{j+1} \approx U_{j-1}$, given, for even m values alternating 0 and 1 as the solution values at the nodes:

i.e. Oscillation in U are transported "upstreams" making U a globally "bad" approximation of u.

A better way is to approximate $U'(x_j)$ by an "upwind" derivative:

$$U'(x_j) \approx \frac{U_j - U_{j-1}}{h},$$

which, formally, gives a better stability, however with low accuracy. The example itself demonstrates that a high accuracy without stability is indeed useless.

A more systematic method of making FE-Solution of the fluid problems stable; is through using:

The streamline-diffusion method (SDM):

The idea is to choose the test functions of the form

$$(v + \tfrac{1}{2}\beta h v'), \quad \text{instead of just } v.$$

Then, e.g., for our model problem (example above with source $f(x)$) we obtain ($\beta = 1$) the variational formulation:

$$\text{(VF)} \quad \int_0^1 \left[U'(v+\tfrac{1}{2}hv') - \varepsilon U''(v+\tfrac{1}{2}hv')\right]dx = \int_0^1 f(v+\tfrac{1}{2}hv')\,dx.$$

Here in the discrete version of the variational formulation, One should interpret the term $\int_0^1 U''v'\,dx$ as

$$\sum_j \int_{I_j} U''v' \quad (= 0 \text{ in case of approx. with p.L.})$$

Note that with $v=\varphi_j$ in discrete version of (VF):

$$\int_0^1 U' \tfrac{1}{2} h \varphi_j' = U_j - \tfrac{1}{2} U_{j+1} - \tfrac{1}{2} U_{j-1}$$ combined with (stiffness matrix row j)

$$\int_0^1 U' \varphi_j = \frac{U_{j+1} - U_{j-1}}{2}$$ (conv. matrix row j)

gives $(U_j - U_{j-1})$ approx. the 1st integral in (VF); corresponding to the upwind scheme.

The SDM can also be interpreted as a sort of least-square.

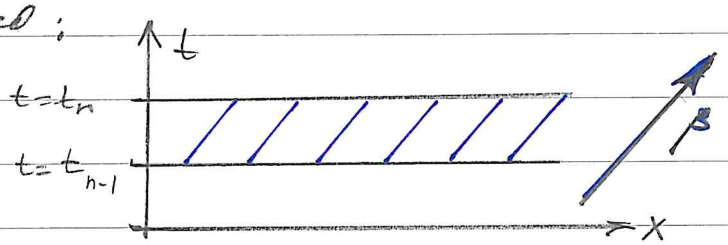
Let $A = \frac{d}{dx}$, $A^t = -\frac{d}{dx}$, then u minimizes

$\|w' - f\|$ if $u' = Au = f \iff A^t A u = A^t f$

$\iff$ {mult. v & $\int$} $\iff$ $-u'' = (-f' = g)$ (continuous)

$\iff$ {$\int$ P.I. in both sides with Dirichlet data} $\iff \int U'v' = \int f v'$ (weak form).

For the time dependent convection problem, the oriented space-time elements are used:



Set $U(x,t)$ such that U is piecewise linear in x and piecewise constant in the $(\beta,1)$-direction. Combine with SDM and add up some artificial viscosity $\hat{\varepsilon}$ depending on the residual term to get for each time interval:

$$\int_{I_n}\int_{\Omega} \{(\dot{U} + \beta U')(v + \tfrac{\beta}{2} h \dot{v}) + \hat{\varepsilon} U' v'\} = \int_{I_n}\int_{\Omega} f\,(v + \tfrac{\beta}{2} h v'). \dots$$