Variational formulation in $\mathbb{R}^2$

All the previous studies (1-dimensional) can be extended to $\mathbb{R}^n$, then the mathematics of computation becomes more cumbersome. On the other hand 2 and 3 dimentional cases are of both physical relevance and practical interest.

A typical problem to study is, e.g.

$$\begin{cases} -\Delta u + \underbrace{c \cdot \nabla u}_{\text{(Convection)}} + \underbrace{au}_{\text{(absorption)}} = S, & x \in \Omega \subset \mathbb{R}^3 \\ u(x) = 0 \quad \text{OR} \quad \mathbf{n} \cdot u(x) = 0, & x \in \partial\Omega \end{cases}$$

OR

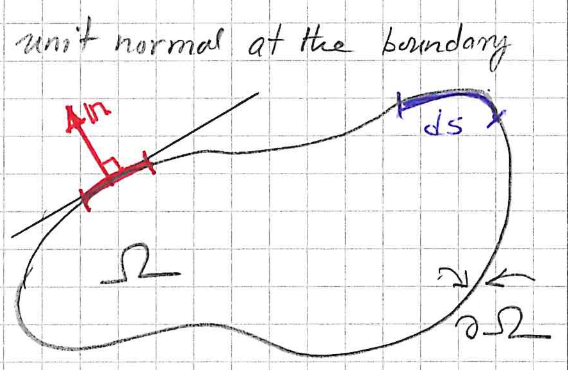
$$\begin{cases} -\Delta u + au = f, & x = (x,y) \in \Omega \subset \mathbb{R}^2 \\ u(x,y) = 0, & \text{for } x = (x,y) \in \Gamma := \partial\Omega \end{cases}$$

The only difference with the 1-dimensional case is the performance of the partial integrations.

Green's formula Let $u \in C^2(\Omega)$ and $v \in C(\Omega)$, then

$$\iint_{\Omega} \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) v \, dx\, dy = \int_{\partial\Omega} \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right) \cdot \mathbf{n}(x,y) \, v(x,y) \, ds$$
$$- \iint_{\Omega} \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right) \cdot \left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \right) dx\, dy,$$

where $\mathbf{n}(x,y)$ is the outward unit normal at the boundary point $x = (x,y) \in \partial\Omega$ and ds is a curved element on the boundary $\partial\Omega$.



Concise form of the Green's formula:

$$(G1) \quad \int_{\Omega} (\Delta u)\, v \, d\mathbf{x} = \int_{\partial\Omega} (\nabla u \circ \mathbf{n})\, v \, ds - \int_{\Omega} \nabla u \circ \nabla v \, d\mathbf{x}.$$

Interchanging u and v (assuming now also $v \in C^2(\Omega)$);

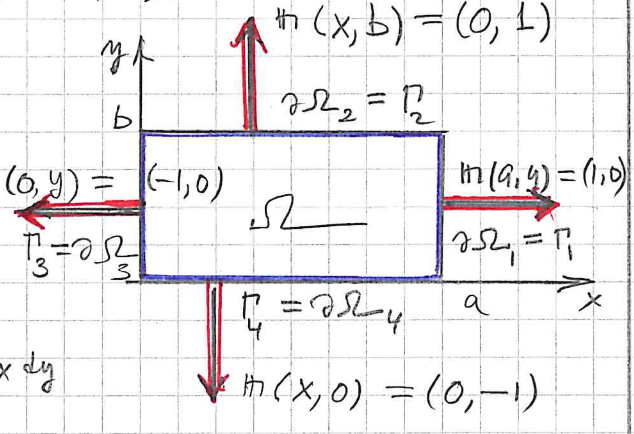
$$(G2) \quad \int_{\Omega} (\Delta v)\, u \, d\mathbf{x} = \int_{\partial\Omega} (\nabla v \circ \mathbf{n})\, u \, ds - \int_{\Omega} \nabla v \circ \nabla u \, d\mathbf{x}.$$

(G1) − (G2) $\Rightarrow$ The Alternative Green's formula:

$$\int_{\Omega} \Big((\Delta u)\, v - (\Delta v)\, u\Big)\, d\mathbf{x} = \int_{\partial\Omega} \Big((\nabla u \circ \mathbf{n})\, v - (\nabla v \circ \mathbf{n})\, u\Big)\, ds.$$

Proof The proof of the Green's formula for a general domain $\Omega \subset \mathbb{R}^2$ can be found in any text-book in Calculus of several variables. Here we give a simple proof for Ω being a rectangle:

$\Omega := (0,a) \times (0,b)$:



Then we have [PI] w.r.t. x:

$$\iint_{\Omega} \frac{\partial^2 u}{\partial x^2}\, v \, dx\, dy =$$

$$= \int_0^b \int_0^a \frac{\partial^2 u}{\partial x^2}(x,y)\, v(x,y)\, dx\, dy$$

$$= \int_0^b \left(\left[\frac{\partial u}{\partial x}(x,y)\, v(x,y) \right]_{x=0}^{x=a} - \int_0^a \frac{\partial u}{\partial x}(x,y)\, \frac{\partial v}{\partial x}(x,y)\, dx \right) dy$$

$$\overset{(*)}{=} \int_0^b \left(\frac{\partial u}{\partial x}(a,y)\, v(a,y) - \frac{\partial u}{\partial x}(0,y) \right) dy - \iint_{\Omega} \frac{\partial u}{\partial x}\, \frac{\partial v}{\partial x}\, dx\, dy.$$

$\rightsquigarrow$ {pf of Green's formula for rectangular domain Ω } $\rightarrow$

Now we consider the boundary terms above:

On Γ_1: $\displaystyle\int_0^b \frac{\partial u}{\partial x}(a,y)\, v(a,y)\, dy = \int_0^b \left(\frac{\partial u}{\partial x}(a,y), \frac{\partial u}{\partial y}(a,y)\right) \underbrace{(1,0)}_{= n(a,y)} v(a,y)\, dy$

$$= \underbrace{\int_{\Gamma_1} \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \circ n(x,y)\, v(x,y)\, ds} \qquad (\Gamma_1)$$

On Γ_3: $\displaystyle\int_0^b -\frac{\partial u}{\partial x}(0,y)\, v(0,y)\, dy = \int_0^b \left(\frac{\partial u}{\partial x}(0,y), \frac{\partial u}{\partial y}(0,y)\right) \underbrace{(-1,0)}_{= n(0,y)} v(0,y)\, dy$

$$= \underbrace{\int_{\Gamma_3} \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \circ n(x,y)\, v(x,y)\, ds} \qquad (\Gamma_3)$$

Inserting the contribution of the boundary integrals (Γ_1) and (Γ_3) in $(*)$ we get

$$\text{(I)} \quad \iint_\Omega \frac{\partial^2 u}{\partial x^2} v\, dx\, dy = \int_{\Gamma_1 \cup \Gamma_3} \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \circ n(x,y)\, v(x,y)\, ds - \iint_\Omega \frac{\partial u}{\partial x} \frac{\partial v}{\partial x}\, dx\, dy.$$

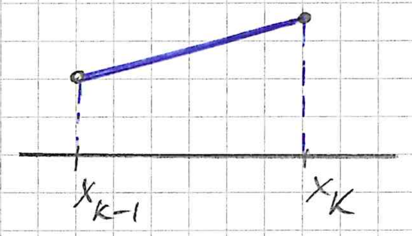
Similarly for the y-direction we have

$$\text{(II)} \quad \iint_\Omega \frac{\partial^2 u}{\partial y^2} v\, dy\, dy = \int_{\Gamma_2 \cup \Gamma_4} \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \cdot n(x,y)\, ds - \iint_\Omega \frac{\partial u}{\partial y} \frac{\partial v}{\partial y}\, dx\, dy.$$

Now (I) + (II) $\Rightarrow$ The desired result. $\boxtimes$

Basis functions for the piecewise linear case in 2 d.

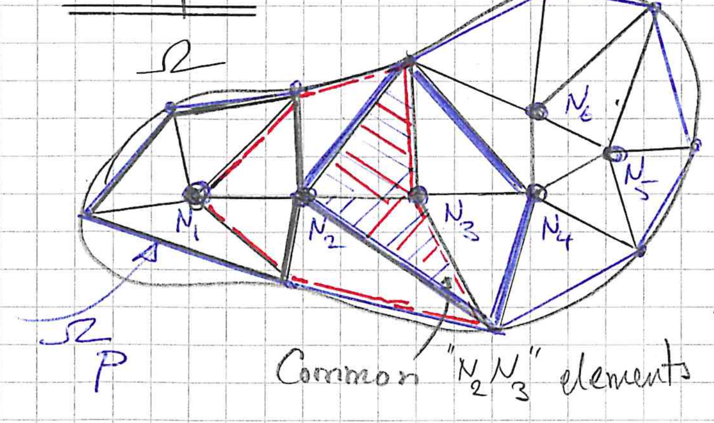
In the 1-dimensional case a function which is linear on an interval is uniquely determined by its values at two points, e.g., the two end points of the interval (because there is only one straight line connecting two points)



Similarly a plane in $\mathbb{R}^3$ is uniquely determined by three points. Therefore it is natural to make partitions of 2-dimensional domains using triangular elements and letting the sides of the triangles to correspond to the end-points of the intervals in the 1-dimensional case.

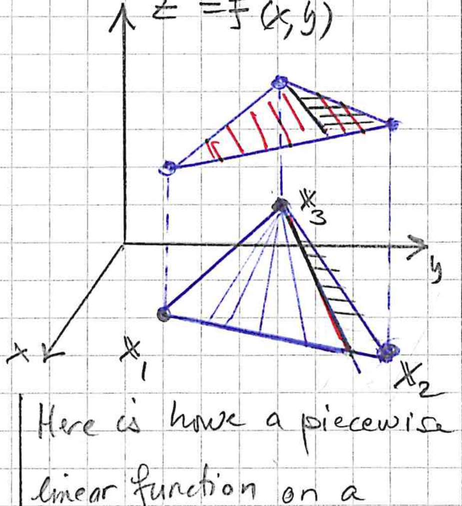
Observe that this "partitioning", called triangulation, works only for the domains with polygonal boundary.

Example



Common "$N_2 N_3$" elements

Here we have 6 internal nodes N_i and $\Omega_P \subset \Omega$ is the triangulated polygonal domain.



Here is how a piecewise linear function on a triangle is determined by its values at the vertices.

⇝ Example (cont) ⇝

Now for every linear function $U : \Omega_p \to \mathbb{R}$ we have

$$U(x) = U_1\varphi_1(x) + U_2\varphi_2(x) + \cdots + U_6\varphi_6(x),$$

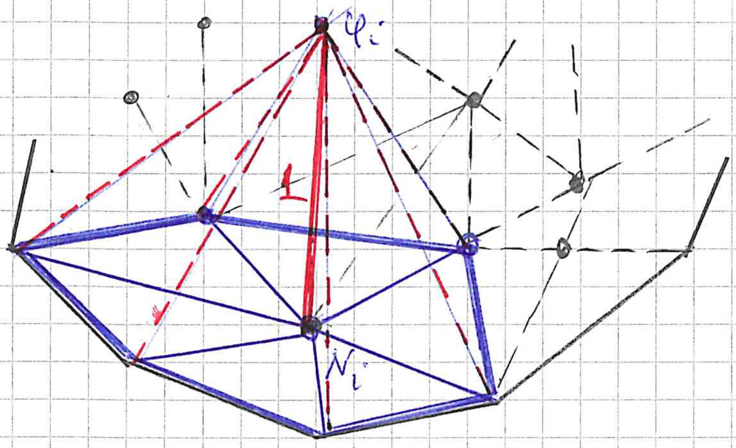
where $U_i = U(N_i)$, $i = 1, 2, \ldots, 6$ are real numbers and

$\varphi_i(N_i) = 1$, $\varphi_i(N_j) = 0$, for $j \neq i$, $\varphi_i(x)$ is linear;

In other words; φ_i is affine and

$$\varphi_i(N_j) = \begin{Bmatrix} 1 & j = i \\ 0 & j \neq i \end{Bmatrix} = \delta_{ij}$$

on every triangle/element and $\varphi_i = 0$ on $\partial\Omega_p$.



Therefore, e.g., given a differential equation to determine the approximate solution U is now reduced to find the coefficients $U_1, U_2, \ldots, U_6$ satisfying the corresponding discrete variational formulation.

Here is an example:

Example: Let $\Omega = \{(x,y) : 0<x<4,\ 0<y<3\}$ and make a FEM discretization of

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega . \end{cases}$$

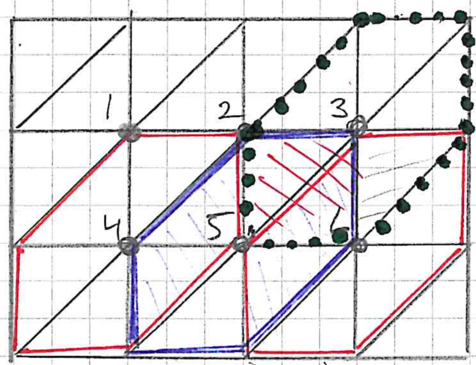
Using Green's formula the variational formulation is:

Find a solution u vanishing at the boundary such that

$$\iint_\Omega (\nabla u \cdot \nabla v)\, dx\, dy = \iint_\Omega f v \, dx\, dy \quad \text{for all } v \in C^1(\Omega),$$
with $v=0$ on $\partial\Omega$.

Next, we make a test function space of piecewise linears.

Ω:



We triangulate Ω as in the figure, and let

$\overset{\circ}{V}_h$ = "Space of all continuous functions which are linear on each subtriangle and are zero on the boundary $\partial\Omega$."

Since such a function is uniquely determined by its value at the vertices of the triangles and 0 on the boundary, so indeed in our example we have only 6 inner vertices of interest. Now, as in the "1-D" case we construct basis functions (6 of them in this particular case), with values 1 at one of the nodes and zero at the others. Then we get the two-dimensional test functions as above.