Problem 27

Prove that $-\Delta u = 0$, if for $\mathbf{x} \neq 0$ $u: \mathbb{R}^2 \to \mathbb{R}$ is given by $u(\mathbf{x}) = \log\left(\frac{1}{|\mathbf{x}|}\right)$, $\mathbf{x} = (x,y)$

Solution: Alt 1. By direct calculus: $\Delta u = u_{xx} + u_{yy} \Rightarrow -\Delta u = 0$

Alt 2. Recall the definition of the "rot" functions for vectors & scalars:

if $u(x,y) = (u_1, u_2)^T$, i.e., u a vector, then $\text{rot}\, u := \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}$ i.e. a scalar

if $u(x,y)$ is a scalar, then $\text{rot}\, u := \left(\frac{\partial u}{\partial y}, -\frac{\partial u}{\partial x}\right)$ which is a vector

Now as in problem 26: $-\Delta u = \text{rot}\,\text{rot}\, u$.

We show that starting with the scalar function $u(\mathbf{x}) = \log\left(\frac{1}{|\mathbf{x}|}\right)$ we get $\text{rot}(\text{rot}\, u) \equiv 0$.

Now $u(x,y) = \log \frac{1}{|\mathbf{x}|} = \log \frac{1}{\sqrt{x^2+y^2}} \equiv \log w$, $w = (x^2+y^2)^{-1/2}$

$\Rightarrow u'_x = \frac{w'_x}{w}$ & $u'_y = \frac{w'_y}{w}$.

$$w = (x^2+y^2)^{-1/2} \Rightarrow \begin{cases} w'_x = -x(x^2+y^2)^{-3/2} \Rightarrow \frac{w'_x}{w} = \frac{-x(x^2+y^2)^{-3/2}}{(x^2+y^2)^{-1/2}} = \frac{-x}{x^2+y^2} \\ w'_y = -y(x^2+y^2)^{-3/2} \Rightarrow \frac{w'_y}{w} = \frac{-y}{x^2+y^2} \end{cases}$$

Hence $\text{rot}\, u = (u'_y, -u'_x) = \left(\frac{w'_y}{w}, -\frac{w'_x}{w}\right) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right) \equiv (\xi, \zeta)$

$$\Rightarrow \text{rot}(\text{rot}\, u) = \frac{\partial \zeta}{\partial x} - \frac{\partial \xi}{\partial y} = \frac{x^2+y^2-2x(x)}{(x^2+y^2)^2} + \frac{(x^2+y^2)-2y(y)}{(x^2+y^2)^2} \equiv 0.$$

$\boxtimes$

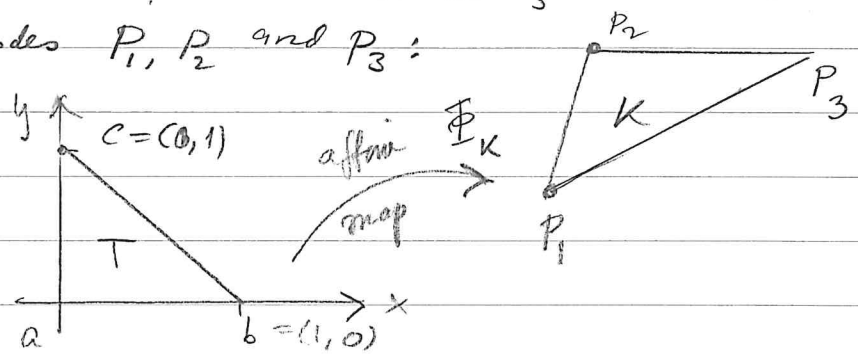
Thus once again: $\Delta\left(\log \frac{1}{|\mathbf{x}|}\right) = 0$, $(\mathbf{x} \neq 0)$

That is: $\log\left(\frac{1}{|\mathbf{x}|}\right)$ is a harmonic function, $(\mathbf{x} \neq 0)$.

Interpolation error in 2d, in L_∞-norm
[Through problems 8–11, Chapter 10] -2-

MA.
Lecture 14/2-

10.8 Express the tent functions $\lambda_1, \lambda_2, \lambda_3$ on a triangle K with nodes P_1, P_2 and P_3:



Consider the standard element T with $a=(0,0)$, $b=(1,0)$ and $c=(0,1)$

Then, the standard bases functions are
$$\begin{cases} \phi_1(\mathbf{x}) = 1-x-y \\ \phi_2(\mathbf{x}) = x \\ \phi_3(\mathbf{x}) = y \end{cases}$$
wher $\mathbf{x}=(x,y) \in T$.

Let now P_1, P_2 and P_3 be the nodes of a triangle K, and define the affine map

$$\Phi_K : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \text{ such that } \quad \mathbf{x} \xmapsto{\Phi_K} \underbrace{(\vec{P_2}-\vec{P_1}, \vec{P_3}-\vec{P_1})}_{=:A}\mathbf{x} + \vec{P_1}$$

($\vec{P_i}$ is the vector $\vec{OP_i}$, $i=1,2,3$)

Then we have
$$\begin{cases} \Phi_K(a) = P_1 \\ \Phi_K(b) = P_2 \\ \Phi_K(c) = P_3 \end{cases}$$

Let now $B := A^{-1}$. Then $\Phi_K^{-1}(\mathbf{x}) = B(\mathbf{x}-P_1)$ and the tent functions $\lambda_1, \lambda_2, \lambda_3$ on K are given by

$$\lambda_i(\mathbf{x}) = \phi_i(\Phi_K^{-1}(\mathbf{x})).$$

We have the following identities; Let $P_i = \begin{pmatrix} (P_i)_1 \\ (P_i)_2 \end{pmatrix}$

a) $\sum_{i=1}^{3} \lambda_i(x) = 1$

b) $\sum_{i=1}^{3} \left((P_i)_j - x_j\right) \lambda_i(x) = 0 \quad j=1,2$ (10.5.2)

c) $\sum_{i=1}^{3} \frac{\partial}{\partial x_k} \lambda_i(x) = 0$

d) $\sum_{i=1}^{3} \left((P_i)_j - x_j\right) \frac{\partial \lambda_i}{\partial x_k}(x) = \delta_{jk}, \quad j,k=1,2$ (10.5.3)

Solution: First we observe that

$$\Phi_k^{-1}(x) = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix} \begin{pmatrix} x_1 - (P_1)_1 \\ x_2 - (P_1)_2 \end{pmatrix} = \begin{pmatrix} \beta_{11}(x_1 - (P_1)_1) + \beta_{12}(x_2 - (P_1)_2) \\ \beta_{21}(x_1 - (P_1)_1) + \beta_{22}(x_2 - (P_1)_2) \end{pmatrix} =: \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix}$$

$$\Rightarrow \frac{\partial (\Phi_k^{-1})_j}{\partial x_k} = b_{jk}, \quad j,k=1,2.$$

$\Rightarrow$ a): $\sum_{i=1}^{3} \lambda_i(x) = \sum_{i=1}^{3} \phi_i(\tilde{x}) = 1 - (\tilde{x}_1 + \tilde{x}_2) - \tilde{x}_1 - \tilde{x}_2 = 1.$

b): Consider the function $f_j(\bar{x}) = x_j$, $j=1,2$ (Component function).

Taylor formula $\Rightarrow$ $f_j(P_i) = f_j(x) + \underbrace{\nabla f_j(x) \begin{pmatrix} (P_i)_1 - x_1 \\ (P_i)_2 - x_2 \end{pmatrix}}_{(P_i)_j - x_j}.$

(obs! no remainder term because the function is linear)

Linear Interpolation of f_j:

$$(\Pi f_j)(x) = \sum_{i=1}^{3} f_j(P_i)\lambda_i(x) = \sum_{i=1}^{3} \left[f_j(x) + \left((P_i)_j - x_j\right)\right]\lambda_i(x)$$

$$= f_j(x) \underbrace{\sum_{i=1}^{3} \lambda_i(x)}_{=1 \Leftarrow (a)} + \sum_{i=1}^{3} \left((P_i)_j - x_j\right)\lambda_i(x). \quad (*)$$

Note that $\Pi f_j = f_j$ $\leftarrow$ (f_j is linear) $\Rightarrow$

$$f_j(x) = (\Pi f_j)(x) \overset{(*)}{=} f_j(x) + \sum_{i=1}^{3} \left((P_i)_j - x_j\right)\lambda_i(x)$$

$$\Rightarrow \sum_{i=1}^{3} \left((P_i)_j - x_j\right)\lambda_i(x) = 0. \quad \Rightarrow (b).$$

10.9 c)

$$\sum_{i=1}^{3} \frac{\partial \lambda_i}{\partial x_{j_0}}(x) = \sum_{i=1}^{3} \Bigg(\underbrace{\partial_1 \phi_i(\Phi_K^{-1}(x))}_{\substack{\partial_1\phi_1 = -1 \\ \partial_1\phi_2 = 1 \\ \partial_1\phi_3 = 0}} \underbrace{\frac{\partial(\Phi_K^{-1})_1}{\partial x_j}}_{= b_{1j}} + \underbrace{\partial_2 \phi_i(\Phi_K^{-1}(x))}_{\substack{\partial_2\phi_1 = -1 \\ \partial_2\phi_2 = 0 \\ \partial_2\phi_3 = 1}} \underbrace{\frac{(\partial \Phi_K^{-1})_2}{\partial x_j}}_{= b_{2j}} \Bigg)$$

$$= \dots = 0$$

10.9 d) Once again we consider $f_j(x) = x_j$, $j = 1,2$.

$$f_j(x) = \{f_j \text{ linear}\} = (\Pi f_j)(x) = \sum_{i=1}^{3} f_j(P_i)\lambda_i(x)$$

$$\Rightarrow \underbrace{\frac{\partial f_j(x)}{\partial x_k}}_{= \delta_{jk}} = \sum_{i=1}^{3} \underbrace{f_j(P_i)}_{\underline{\underline{\text{Taylor}}}\ f_j(x) + \underbrace{\nabla f_j(x) \cdot \begin{pmatrix} (P_i)_1 - x_1 \\ (P_i)_2 - x_2 \end{pmatrix}}_{= (P_i)_j - x_j}} \frac{\partial \lambda_i}{\partial x_k}(x)$$

$$\Rightarrow \delta_{jk} = \underbrace{\sum_{i=1}^{3} f_j(x) \frac{\partial \lambda_i}{\partial x_k}(x)}_{= f_j(x) \underbrace{\sum_{i=1}^{3} \frac{\partial \lambda_i}{\partial x_k}(x)}_{=0}} + \sum_{i=1}^{3} \left((P_i)_j - x_j\right) \frac{\partial \lambda_i}{\partial x_k}(x)$$

$$\Rightarrow \sum_{i=1}^{3} \left((P_i)_j - x_j\right) \frac{\partial \lambda_i}{\partial x_k}(x) = \delta_{jk}. \quad \boxtimes$$

Chapter 10 / problem 10

Interpolation in 2D (Solution) -5-

M. A.

Lecture 14/ -5-

Consider the interpolation error in 2-dimensions

$$f(x) - \Pi_h f(x) = -\sum_{i=1}^{3} r_i(x)\,\lambda_i(x).$$

Here λ_i are the basis functions and $r_i(x)$ is the reminder, show that

$$|r_i(x)| \le \frac{1}{2} h_K \|D^2 f\|_{L_\infty(K)}, \quad i = 1,2,3.$$

<u>Solution</u>: Let $f \in C^2(\mathbb{R})$. By Taylor's formula we have

$$f(P_i) = f(x) + \underbrace{Df(x)\cdot\begin{pmatrix}(P_i)_1 - x_1\\ (P_i)_2 - x_2\end{pmatrix}}_{=:\, l_i(x)} + \underbrace{\sum_{j,k=1}^{2} \frac{1}{2}\partial_j\partial_k f(x)\left((P_i)_j - \xi_j\right)\left((P_i)_k - \xi_k\right)}_{=:\, r_i(x)},$$

where ξ_j, ξ_k, $j,k = 1,2$ are on segments $\subset K$, connecting P_i & x.

We plug in this expression into the definition of the linear interpolation:

$$(\Pi f)(x) = \sum_{i=1}^{3} f(P_i)\lambda_i(x) \overset{!!}{=} \sum_{i=1}^{3}\left(f(x) + l_i(x) + r_i(x)\right)\lambda_i(x).$$

We have that

- $\sum_{i=1}^{3} f(x)\lambda_i(x) = f(x)\sum_{i=1}^{3}\lambda_i(x) = \{10.5.2\} = f(x)$
- $\sum_{i=1}^{3} l_i(x)\lambda_i(x) = \sum_{i=1}^{3}\left\{(\partial_1 f)(x)\left((P_i)_1 - x_1\right)\lambda_i(x) + (\partial_2 f)(x)\left((P_i)_2 - x_2\right)\lambda_i(x)\right\}$
  $= 0 \quad \{10.5.2 \text{ (second term)}\}.$

$$\Rightarrow f(x) - (\Pi f)(x) = -\sum_{i=1}^{3} r_i(x)\lambda_i(x)$$

Now using the def of $r_i(x)$:

$$|r_i(x)| \le \frac{1}{2}\underbrace{\sum_{j,k=1}^{2}|\partial_j\partial_k f(x)|}_{\le \|D^2 f\|_\infty}\overbrace{|(P_i)_j - \xi_j|}^{\le h_K}\overbrace{|(P_i)_k - \xi_k|}^{\le h_K} \le \frac{1}{2} h_K^2 \|D_2 f\|_\infty \quad \blacksquare$$

$\rightsquigarrow$ As for the gradient we get

$$\text{(I)} \quad \left(\nabla(\Pi f)\right)(x) = \sum_{i=1}^{3} f(P_i)\nabla\lambda_i(x) = \{\text{Taylor}\}$$

$$= \sum_{i=1}^{3} \left(f(x) + l_i(x) + r_i(x)\right)\nabla\lambda_i(x).$$

using the first relation of (10.5.3) we have

$$\text{(II)} \quad \sum_{i=1}^{3} f(x)\nabla\lambda_i(x) = f(x)\sum_{i=1}^{3}\nabla\lambda_i(x) = 0.$$

As for the $l_i(x)$-term, using the second relation of (10.5.3)

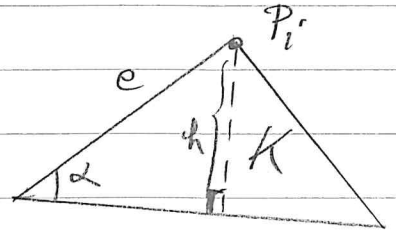
$$\sum_{i=1}^{3} l_i(x)\nabla\lambda_i(x) := \sum_{i=1}^{3}\left(\left[(\partial_1 f)(x)\underbrace{\cancelto{1}{((P_i)_1 - x_1)}}_{\to 0} + (\partial_2 f)(x)\underbrace{\cancelto{0}{((P_i)_2 - x_2)}}_{\to 1}\right]\begin{pmatrix}\partial_1\lambda_i(x)\\ \partial_2\lambda_i(x)\end{pmatrix}\right)$$

$$\text{(III)} \quad = \begin{pmatrix}\partial_1 f(x)\\ \partial_2 f(x)\end{pmatrix} = \nabla f(x).$$

Hence inserting (II) and (III) in (I) we get

$$\underline{\underline{\left(\nabla(f - \Pi f)\right)(x) = -\sum_{i=1}^{3} r_i(x)\nabla\lambda_i(x).}}$$

<u>10.11</u> Estimating $|\nabla\lambda_i(x)|$:



$$|\nabla\lambda_i| = \frac{1}{|h|} = \frac{1}{(\sin\alpha)\,e} \quad \left\{e \geq \frac{h_K}{2}\right\}$$

$$\leq \frac{2}{h_K \sin(\tau_K)}.$$

Smallest angle in the triangle K.