The Heat equation: Stability

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Consider the initial boundary value problem

$$\begin{cases} (DE) & \dot{u} - \Delta u = 0 \quad \text{in } \Omega \subset \mathbb{R}^2 \ (\text{or } \mathbb{R}^d,\ d=1,2,3) \\ (BV) & u = 0 \quad \text{on } \Gamma := \partial\Omega \\ (IV) & u(x,0) = u_0(x), \quad \text{for } x \in \Omega \end{cases}$$

Where $\dot{u} = \frac{\partial u}{\partial t}$.

Stability Estimates

Multiply the (DE) by u and integrate over Ω:

$$(1) \quad \int_\Omega \dot{u}\, u \, dx - \int_\Omega (\Delta u) u \, dx = 0.$$

Note that $\dot{u}\,u = \frac{1}{2}\frac{d}{dt}(u^2)$ and by Green's formula $-\int_\Omega (\Delta u) u = \int_\Omega \nabla u \cdot \nabla u$, due to the fact that $u = 0$ on $\partial\Omega$. Thus (1) is rewritten as:

$$(2) \quad \frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + \int_\Omega |\nabla u|^2 dx = 0 \iff \frac{1}{2}\frac{d}{dt}\|u\|^2 + \|\nabla u\|^2 = 0,$$

where $\|\cdot\|$ denotes the $L_2(\Omega)$-norm: Integrating in t $\xrightarrow{\int_0^t \dots ds}$

$$\frac{1}{2}\|u\|^2(t) - \frac{1}{2}\|u\|^2(0) + \int_0^t \|\nabla u\|^2 ds = 0,$$

i.e.

$$\|u\|^2(t) + 2\int_0^t \|\nabla u\|^2 ds = \|u_0\|^2.$$

In particular we have

$$(3a) \quad \boxed{\|u\|(t) \le \|u_0\|} \quad \& \quad (3b) \quad \boxed{\int_0^t \|\nabla u\|^2 ds \le \frac{1}{2}\|u_0\|^2}$$

Exercise 1. Derive the corresp. estimates with BC. $\frac{\partial u}{\partial n} = 0$.

Exercise 2. Show that $\|\nabla u\|(t) \le \|\nabla u_0\|$, (mult. by $-\Delta u$ & $\int$). $\xrightarrow{(*)}$ is (*) valid for $u_0 = \text{constant}$?

To obtain further stability estimates, we multiply the DE by $-t\Delta u$ and integrate over Ω:

$$(4)\quad -t\int_\Omega \dot{u}\,\Delta u\,dx + t\int_\Omega (\Delta u)^2\,dx = 0.$$

using Green's formula we have $\int_\Omega \dot{u}\Delta u = -\int_\Omega \nabla\dot{u}\cdot\nabla u = -\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2$,

so that (4) $\Leftrightarrow$

$$t\,\tfrac{1}{2}\|\nabla u\|^2 + t\|\Delta u\|^2 = 0 \iff \frac{d}{dt}\left(t\|\nabla u\|^2\right) + 2t\|\Delta u\|^2 = \|\nabla u\|^2.$$

Relabeling $t\to s$ & integrating over $s\in(0,t)$ yields

$$\int_0^t \frac{d}{ds}\left(s\|\nabla u\|^2_{(s)}\right)ds + 2\int_0^t s\|\Delta u\|^2_{(s)}\,ds = \int_0^t \|\nabla u\|^2 ds \underset{(3b)}{\leq} \frac{1}{2}\|u_0\|^2.$$

i.e.

$$t\|\nabla u\|^2(t) + 2\int_0^t s\|\Delta u\|^2_{(s)}\,ds \leq \frac{1}{2}\|u_0\|^2.$$

In particular we have

$$(5a)\quad \boxed{\|\nabla u\|(t) \leq \frac{1}{\sqrt{2t}}\|u_0\|}, \qquad (5b)\quad \boxed{\left(\int_0^t s\|\Delta u\|^2_{(s)}\,ds\right)^{1/2} \leq \frac{1}{2}\|u_0\|}$$

Analogously

$$(6)\quad \|\Delta u\|(t) \leq \frac{1}{\sqrt{2}\,t}\|u_0\|$$

Exercise 3: Prove (6).

Hint: Multiply the (DE) by $t^2\Delta^2 u$ and note $\Delta u = \dot{u} = 0$ on $\partial\Omega$.

Alternative: Differentiate $\dot{u} - \Delta u$ w.r.t. t and multiply by $t^2\dot{u}$.

Cont

The Heat equation. Stability

FEM for semi-discrete problem

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Evidently (5a) and (6) below up for $t \downarrow 0$. The following stability estimate is obtained avoiding a small neighbourhood of $t=0$:

$$\int_\varepsilon^t \|\dot{u}\|(s)\,ds = \int_\varepsilon^t \|\Delta u\|(s)\,ds = \int_\varepsilon^t \frac{1}{\sqrt{s}}\cdot\sqrt{s}\,\|\Delta u\|(s)\,ds \quad \le \{(5a)\}$$

$$\le \left(\int_\varepsilon^t \frac{1}{s}\,ds\right)^{1/2} \cdot \left(\int_\varepsilon^t s\,\|\Delta u\|^2(s)\,ds\right)^{1/2} \le \{(5b)\}$$

(7)

$$\le \frac{1}{2}\sqrt{\ln\frac{t}{\varepsilon}}\,\|u_0\|, \qquad (\varepsilon > 0 \text{ is small}).$$

Non-homogeneous heat equation:

$$\begin{cases} \text{(DE)} & \dot{u}(x,t) - \Delta u(x,t) = f(x,t), \quad x \in \Omega \subset \mathbb{R}^2,\ t>0 \\ \text{(BV)} & u(x,0) = 0, \quad x \in \Gamma = \partial\Omega \\ \text{(IV)} & u(x,0) = u_0(x), \quad x \in \Omega \end{cases}$$

Weak formulation: Using Green's formula we can write

$$\text{(VF)} \quad (u_t, \varphi) + (\nabla u, \nabla\varphi) = (f,\varphi), \qquad \forall\, \varphi \in H_0^1(\Omega),\ t>0.$$

Let now $\mathcal{T}_h$ be a triangulation of Ω with interior nodes $\{x_j\}_{j=1}^M$. Define

$$V_h^0 := \{\chi : \chi \text{ continuous, piecewise linear on } \mathcal{T}_h \text{ and } \chi = 0 \text{ on } \partial\Omega\}.$$

Further let $\{\chi_j\}_{j=1}^M$ be the standard basis for V_h^0, corresp. to nodes $\{x_j\}_{j=1}^M$.

Piecewise linear F.E approximation: For each t, find $u_h(\cdot,t) \in V_h^0$ s.t.

$$\begin{cases} \text{(FEM)} & (u_{h,t}, \chi) + (\nabla u_h, \nabla\chi) = (f,\chi), \quad \forall\, \chi \in V_h^0,\ t>0 \\ & u_h(x,0) = u_{0,h}(x). \end{cases}$$

The (FEM) is equivalent to find the time-dependent coefficient $\xi_j(t)$ in

$$u_h(x,t) = \sum_{j=1}^{M} \xi_j(t)\,\psi_j(t)$$

such that

$$\begin{cases} \sum_{j=1}^{M} \xi_j'(t)\,(\psi_j, \psi_i) + \sum_{j=1}^{M} \xi_j(t)\,(\nabla\psi_j, \nabla\psi_i) = (f(t), \psi_i), & i=1,2,\dots,M. \\ \xi_j(0) = \eta_j, \quad j=1,\dots,M \qquad (\eta_j \text{ are nodal values of } u_{0,h}). \end{cases}$$

In the matrix form this can be written as

$$(8)\quad M\xi'(t) + S\xi(t) = b(t), \quad t>0, \qquad \xi(0)=\eta,$$

where $M = (m_{ij}) = ((\psi_j, \psi_i))$ is the mass matrix

$S = (s_{ij}) = ((\nabla\psi_j, \nabla\psi_i))$ is the stiffness matrix

and $b = (b_i) = ((f, \psi_i))$ is the load vector.

Both M and S are symmetric & positive definite. In particular M is invertible. Thus (8) can be rewritten as the following system of ODE:

$$(9)\quad \xi'(t) + M^{-1}S\xi(t) = M^{-1}b(t), \quad t>0 \text{ & } \xi(0)=\eta$$

which has a unique solution for $t>0$.

Semi-discrete Stability: Let $\chi = u_h$ in (FEM) then

$$(u_{h,t}, u_h) + (\nabla u_h, \nabla u_h) = (f, u_h) \qquad u_h(0) = u_{0,h}$$

The same procedure as above yields the stability estimate.

$$(10)\quad \|u_h(t)\| \le \|u_{0,h}\| + \int_0^t \|f\|\,ds.$$

We skip repeating the already outlined proof. □

To proceed we define the discrete Laplacian: $\Delta_h : \mathring{V}_h \to \mathring{V}_h$ as in a discrete analogue of Green's formula:

$$(-\Delta_h \Psi, \chi) = (\nabla\Psi, \nabla\chi), \qquad \forall \chi \in \mathring{V}_h$$

Δ_h is self-adjoint and $-\Delta_h$ is positive definite w.r.t. L_2 inner product. Let P_h be the L_2 projection onto $\mathring{V}_h$; $P_h f \in \mathring{V}_h \Rightarrow$

(L_2-projection): $(f, \chi) = (P_h f, \chi) \Leftrightarrow (f - P_h f, \chi) = 0 \quad \forall \chi \in \mathring{V}_h$.

Then (VF) can be written as

$$(u_{h,t} - \Delta_h u_h - P_h f, \chi) = 0 \qquad \forall \chi \in \mathring{V}_h.$$

Now since $(u_{h,t} - \Delta_h u_h - P_h f) \in \mathring{V}_h$ we can take $\chi = u_{h,t} - \Delta_h u_h - P_h f$

and conclude that

semi discrete problem

$$u_{h,t} - \Delta_h u_h = P_h f, \quad t > 0, \quad u_h(0) = u_{0,h}$$

Now consider $E_h(t)$ as the solution operator of the homogeneous ($f = 0$) problem. Then $u_h(t) = E_h(t) u_{0,h}$, and recalling Duhamel's

$$u_h(t) = E_h(t) u_{0,h} + \int_0^t E_h(t-s) P_h f(s)\, ds$$

Compare with the analytic solution of the IVP.

Recalling the interpolant $I_h : C_0(\bar{\Omega}) \to \mathring{V}_h$ and the $L_2(\Omega)$ version of the error bound (stated in thm 10.2) we can prove the semi-discrete error estimate below: Applying

Thm u and u_h are the solutions of the heat equation at its FE approx. respectively. Then we have the following $L_2(\Omega)$-error bound:

$$\|u_h(t)-u(t)\| \le \|u_{0,h}-u_0\| + Ch^2\left(\|u_0\|_2 + \int_0^t \|u_t\|_2\, ds\right).$$

Pf. Recall the common estimates for interpolation I_h, L_2-projection P_h and Ritz projection R_h: $\boxed{a(R_h v - v, \varphi) = 0, \ \forall \varphi \in V_h^0, v \in H_0^1}$

$$\|v - v_h\| \le Ch^2\|v\|_2, \text{ where } v_h = I_h v, \text{ or } v_h = P_h v \text{ or } v_h = R_h v.$$

Below we use the Ritz projection and write

$$(11)\quad u - u_h = (u - R_h u) + (R_h u - u_h) =: \rho + \theta$$

For the Ritz projection we have the following estimate:

$$(12)\quad \|u - R_h u\| + h\|u - R_h u\|_1 \le Ch^s\|u\|_s, \qquad \forall u \in H^s \cap H_0^1. \Rightarrow$$

$$(13)\quad \|\rho(t)\| \le Ch^2\|u(t)\|_2 = Ch^2\|u_0 + \int_0^t u_s\, ds\| \le Ch^2\left(\|u_0\|^2 + \int_0^t \|u_t\|_2\, ds\right).$$

As for the θ; we may write using the notation $a(w, \xi) = (\nabla w, \nabla \xi)$

$$(\theta_t, \varphi) + a(\theta, \varphi) = -(u_{h,t}, \varphi) - a(u_h, \varphi) + (R_h u_{h,t}, \varphi) + a(R_h u_{h,t}, \varphi)$$
$$= -(f, \varphi) + (R_h u_t, \varphi) + a(u, \varphi) = (-u_t + R_h u_t, \varphi).$$

Now we use the fact that R_h commutes with the time derivative:

$$R_h u_t = (R_h u)_t \Rightarrow$$

$$(14)\quad (\theta_t, \varphi) + a(\theta, \varphi) = -(\rho_t, \varphi)$$

Applying (10) to (14) we get

$$\|\theta(t)\| \le \|\theta(0)\| + \int_0^t \|\rho_s\|\, ds.$$

Note that

$$(15)\quad \|\theta(0)\| \le \|R_h u_0 - u_{0,h}\| \le \underbrace{\|R_h u_0 - u_0\|}_{(11)} + \|u_0 - u_{0,h}\| \le \|u_0 - u_{0,h}\| + \underbrace{Ch^2\|u_0\|_2}$$

$$(16)\quad \|\rho_t\| = \|u_t - R_h u_t\| \le Ch^2\|u_t\|_2.$$

Now (11) – (16) yield the desired result. ∎