Consider the wave equation

$$\begin{cases} \text{(DE)} & \ddot{u} - \Delta u = 0 \ \left(\int f \neq 0\right), & \text{in } \Omega, \\ \text{(BC)} & u = 0, & \text{on } \Gamma := \partial\Omega \\ \text{(I.C.)} & (u = u_0) \wedge (\dot{u} = v_0), & \text{in } \Omega \text{ and for } t = 0. \end{cases} \qquad \left(\ddot{u} = \frac{\partial^2 u}{\partial t^2}\right)$$

Conservation of Energy

Multiply the equation by $\dot{u}$ and integrate over Ω to get

$$\int_\Omega \ddot{u}\dot{u} - \underbrace{\int_\Omega (\Delta u)\dot{u}}_{= \int_\Omega \nabla u \cdot \nabla \dot{u} \ \leftarrow (\text{Green's} + (\text{BC}))} = 0 \iff \int_\Omega \frac{1}{2}(\dot{u})^2_t + \int_\Omega \frac{1}{2}(|\nabla u|^2)_t = 0$$

i.e.

$$\frac{1}{2}\frac{d}{dt}\left(\|\dot{u}\|^2 + \|\nabla u\|^2\right) = 0 \implies$$

$$\frac{1}{2}\|\dot{u}\|^2 + \frac{1}{2}\|\nabla u\|^2 = \text{Constant, independent of } t. \quad (*)$$

Therefore the total energy is conserved. $\boxtimes$

Remark $(*)$ can also be expressed as

$$\underbrace{\frac{1}{2}\|\dot{u}\|}_{=\text{Kinetic energy}} + \underbrace{\frac{1}{2}\|\nabla u\|^2}_{=\text{Potential energy}} = \frac{1}{2}\|v_0\|^2 + \frac{1}{2}\|\nabla u_0\|^2$$

Exercise 1. Show that $\|\nabla \dot{u}\|^2 + \|\Delta u\|^2 = $ Constant independent of t.

Hint. Multiply (DE) by $-\Delta\dot{u}$ and integrate over Ω.

Alternative: Differentiate w.r.t. x and multiply by $\dot{u}$ ....

Exercise 2. Derive a total conservation of energy relation using the Robin type (B.C.): $\frac{\partial u}{\partial n} + u = 0$, on $\partial\Omega$.

Weak formulation: Multiply the equation $\ddot{u} - \Delta u = f$ by a test function $v \in H_0^1(\Omega)$ and integrate over Ω to obtain

$$\int_\Omega f v\,dx = \int_\Omega \ddot{u} v\,dx - \int_\Omega (\Delta u) v\,dx = \{\text{Green's}\} =$$

$$= \int_\Omega \ddot{u} v\,dx + \int_\Omega \nabla u \cdot \nabla v\,dx - \cancel{\int_{\partial\Omega} (n \cdot \nabla u) \overset{0}{v}\,ds}$$

Variational formulation: Find $u \in H_0^1(\Omega)$ such that

$$(VF)_W \quad \int_\Omega \ddot{u} v\,dx + \int_\Omega (\nabla u \cdot \nabla v)\,dx = \int_\Omega f v\,dx, \quad \forall v \in H_0^1(\Omega).$$

The semi-discrete problem (SDP): Let T_h be a partition of Ω and $\{x_j\}_{j=1}^M$ internal nodes in this partition, $\{\varphi_j\}_{j=1}^M$ the basis functions for the discrete function space $\overset{\circ}{V}_h$:

$$\overset{\circ}{V}_h := \{v : v \text{ is continuous piecewise linear on } T_h \text{ and } v = 0 \text{ on } \partial\Omega\}.$$

The we have the following semi-discrete (discretized w.r.t. x) problem: Find $u_h \in \overset{\circ}{V}_h$ such that

$$(FEM)_W: \quad \int_\Omega \ddot{u}_h v\,dx + \int_\Omega \nabla u_h \cdot \nabla v\,dx = \int_\Omega f v\,dx, \quad \forall v \in \overset{\circ}{V}_h.$$

Next we make the ansatz: $u_h(x,t) = \sum_{j=1}^M \xi_j(t) \varphi_j(x)$, (where $\xi_j(t)$ are M time-dependent unknown coefficients) and choose $v = \varphi_i$, $i = 1, 2, \dots, M$. Then (FEM) can be rewritten as

$$\sum_{j=1}^M \ddot{\xi}(t) \int_\Omega \varphi_i \varphi_j\,dx + \sum_{j=1}^M \xi_j(t) \int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j\,dx = \int_\Omega f \varphi_i\,dx, \quad i = 1, 2, \dots, M$$

$$\Longleftrightarrow \boxed{A\ddot{\xi}(t) + S\xi(t) = b(t),} \quad t \in I = (0, T]$$

Thm (SDE 1). The spatial discrete solution u_h of the wave equation given by $(FEM)_W$ satisfies the a priori error estimate

$$\|u(t)-u_h(t)\| \leq Ch^2\left(\|Du(\cdot,t)\| + \int_0^t \|\ddot{u}(\cdot,s)\|\,ds\right)$$

The pf follows the general path of the one for the heat equation.

Thm (SDE 2) Let u and u_h be the solution for $(VF)_W$ and $(FEM)_W$, respectively. Then, there is a, non-increasing, time dependent constant $C(t)$ such that for $t \geqq 0$,

$$\|u(t)-u_h(t)\| + h|u(t)-u_h(t)|_1 + \|u_t(t)-u_{h,t}(t)\| \leq$$

$$\leq c\left(|u_{0,h}-R_h u_0|_1 + \|v_h - R_h v\|\right) +$$

$$C(t)h^2\left[\|u(t)\|_2 + \|u_t(t)\|_2 + \left(\int_0^t \|u_{tt}\|_2^2\,ds\right)^{1/2}\right].$$

Pf. We start with Ritz projection split:

$$u-u_h = (u-R_h u) + (R_h u - u_h) = \rho + \theta.$$

We may bound ρ & ρ_t as in the case of heat equation:

$$\|\rho(t)\| + h|\rho(t)|_1 \leq Ch^2\|u(t)\|_2 \quad \& \quad \|\rho_t(t)\| \leq Ch^2\|u_t(t)\|_2.$$

As for the θ, we note that

$$(\theta_{tt},\chi) + a(\theta,\chi) = -(\rho_{tt},\chi) \qquad \forall \chi \in \overset{\circ}{V}_h,\ t>0$$

To separate the effects of initial data and the source term, we let $\theta = \eta + \zeta$ where

$$\begin{cases} (\eta_{tt},\psi) + a(\eta,\psi) = 0, & \forall \psi \in \overset{\circ}{V}_h \\ \eta(0) = \theta(0), \quad \eta_t(0) = \theta_t(0). \end{cases}$$

Then η and η_t are bounded as in the conservation of energy.
As for the contribution from ξ, we note that by the initial data for η; ξ satisfies $\xi(0) = \xi_t(0) = 0$. Hence with $\psi = \xi_t$ we end up with

$$\frac{1}{2}\frac{d}{dt}\left(\|\xi_t\|^2 + |\xi|_1^2\right) = -(\rho_{tt}, \xi_t) \le \frac{1}{2}\|\rho_{tt}\|^2 + \frac{1}{2}\|\xi_t\|^2.$$

Then integrating w.r.t. t $\Rightarrow$

$$\|\xi_t(t)\|^2 + |\xi(t)|_1^2 \le \cancel{|\xi_t(0)|^2} + \cancel{|\xi(0)|^2} + \int_0^t \|\rho_{tt}\|^2 ds + \int_0^t \|\xi_t\|^2 ds.$$

Then using Gronwall's lemma (with $C(t) = e^t$) $\Rightarrow$

$$\|\xi_t(t)\|^2 + |\xi(t)|_1^2 \le C(t)\int_0^t \|\rho_{tt}\|^2 ds \le C(t)\, h^4 \int_0^t \|u_{tt}\|_2^2 ds$$

$\Rightarrow$ H. $\boxtimes$