3 Lecture notes

3.1 Lecture 1: Introduction to GR (General Relativity) with a review of EM

Nature: 4 forces (ordinary)+ matter(spin 1/2 fermions) + Higgs (also a kind of force)
Forces: EM, the weak and strong nuclear forces, and gravity
Theories: Standard model (SM) of elementary particles + GR/cosmological standard model
SM of elementary particles: the 3 non-gravity forces, electrons/positrons, neutrinos, three generations, Higgs and spontaneous symmetry breaking, confinement, etc
GR/SM of cosmology: GPS, light bending, perihelia precession, black holes, gravity waves, dark matter, dark energy, inflation, big bang, de Sitter, anti-de Sitter, etc

QFT: SM of elementary particles OK (QED fine to 1 part in 10^{12})!
Quantum gravity NOT OK $\Rightarrow$ String/M theory
(see the MSc courses: QFT, Standard Model (of elementary particles) and String/M theory)

Question: How do we get a classical theory of gravity?

Galileo (1564-1642): all bodies fall with the same speed
Newton (1642-1727), "Principia" (1686):
The "inertial mass" m is defined through Newtons 2nd law:

$$F = ma. \tag{3.1}$$

In "Principia" Newton discussed the force of gravity and found its "inverse square" behaviour (here the magnitude only)

$$F_g = G\frac{M_1 M_2}{r^2}, \tag{3.2}$$

where M is the "gravitational mass" (only positive, gravity is always attractive) and G is Newton's constant. The gravitational force has today been checked to great accuracy (see SW Chapter 1) but only for distances greater than a few microns ($\mu = 1.0 \times 10^{-6}$ m). This remarkable fact has been exploited in string theory (see the MSc course *String/M theory*).

With the two different notions of mass defined above Galileo's result can be formulated as *gravitational mass = inertial mass (m=M)*. See SW Sect. 1.2.

In the 2nd edition of "Principia" (1713) he also discussed some electric and magnetic phenomena, however, without any mathematical approach to them. The electrostatic force law was not put forward until the mid-1700 by people like Daniel Bernoulli, Alessandro Volta and Franz Aepinus, the latter in 1758. The electric force can be expressed as

$$F_e = \frac{Q_1 Q_2}{4\pi r^2}, \tag{3.3}$$

which makes the similarity to gravity very clear. Here Q is the charge (pos or neg: $Q = nq, n \in \mathbf{Z}$). Note that this particular form of the electric force is using the Heaviside-Lorentz units but that other definitions of the charge leads to slightly different expressions (i.e., different constants multiplying $\frac{Q_1 Q_2}{r^2}$).

Exercise 1: The gravitational force is much weaker than the electrostatic force! Check this for two protons $1,0 \times 10^{-10}$ meters apart. Is this a relevant comparison to make?

The aim of the following subsections is to start from the two force laws above and implement special relativity (i.e., Lorentz invariance) and thus the fact that the maximal speed is c, the speed of light, for each one. In the case of $\mathbf{F}_e$ we know the answer namely Maxwell's equations. This is reviewed in the next subsection where we emphasise certain aspects of Maxwell's theory that will be important in the context of gravity discussed in the second subsection. One such aspect is that the infinite range property of the two forces $\mathbf{F}_e$ and $\mathbf{F}_g$ means that in the field theory context the field has to be massless, as is the case for the photon. One way to understand this is that a massive force field will contain an exponentially damped factor e^{-mr} (making it "finite range" as for $Z^0, W^{\pm}$ in the standard model) which cannot exist in a theory with $m = 0$. In the subsection on gravity we try to apply the logic developed in the Maxwell case to the case of gravity. Noether's theorem is used in a crucial way so the last subsection is devoted to a brief review of this theorem and its application to a charged scalar field, both in the electromagnetic context and in the gravity one.

3.1.1 Electromagnetism and special relativity

The first point needed here is to express $\mathbf{F}_e$ in terms of a potential ϕ_e through

$$\mathbf{F}_e = q\,\mathbf{E} \;\text{ where } \mathbf{E} = -\nabla\phi_e. \tag{3.4}$$

Then recalling Gauss' law (see SW eq 2.7.1)

$$\nabla \cdot \mathbf{E} = \varepsilon (= \text{charge density often denoted } \rho) \tag{3.5}$$

$$= Q\delta(r) (\text{for a point charge at}\, r = 0), \tag{3.6}$$

we see that the charge inside a 3-volume V (containing the origin) is

$$Q = \int_V \varepsilon\, dV = \int_V \nabla \cdot \mathbf{E}\, dV = \int_{\partial V} \mathbf{E} \cdot d\mathbf{S} = \int_{S^2} \mathbf{E} \cdot \hat{\mathbf{r}}\, r^2 d\Omega, = \int_{S^2} E_r\, r^2 d\Omega \tag{3.7}$$

where ∂V is the boundary of the 3-volume V. The last two expressions above are obtained for a point charge at the origin. Using $\int_{S^2} d\Omega = 4\pi$, we finally get that the electric field from a point source is

$$\mathbf{E}(\mathbf{r}) = \frac{Q}{4\pi r^2}\,\hat{\mathbf{r}}. \tag{3.8}$$

In terms of the electric potential $\phi_e(r)$ this becomes

$$\mathbf{E}(\mathbf{r}) = -\nabla\phi_e \;\Rightarrow\; \phi_e(r) = \frac{Q}{4\pi r} \;\Rightarrow\; \nabla^2\phi_e = -\varepsilon = -Q\delta^3(\mathbf{r}). \tag{3.9}$$

Now we introduce the vector potential and Maxwell's equations (1864): (SW eq 2.7.5 and 2.7.11). The electromagnetic field strength tensor is (note that $\partial_\alpha := \frac{\partial}{\partial x^\alpha}$)

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha = -F_{\beta\alpha} \text{(i.e., it is an antisymmetric tensor)} \tag{3.10}$$

which is related to the electric and magnetic fields as follows
(the metric is diagonal: $\eta_{\alpha\beta} = diag(-1, +1, +1, +1)$)

$$F^{0i} = E_i = -F_{0i},\ F^{12} = B_3 \text{(and cyclic)} \tag{3.11}$$

The vector potential (in special relativity) is actually defined as

$$A^\alpha = (\phi_e, A_1, A_2, A_3),\ \ A_\alpha = (-\phi_e, A_1, A_2, A_3), \tag{3.12}$$

and hence we get the standard relations (with overdot := ∂_t and $B_1 = \partial_2 A_3 - \partial_3 A_2$ etc)

$$E_i = -\partial_i \phi_e - \dot{A}_i,\ B_i = \varepsilon_{ijk}\partial_j A_k, \tag{3.13}$$

or in vector notation

$$\mathbf{E} = -\nabla\phi_e - \dot{\mathbf{A}},\ \ \mathbf{B} = \nabla \times \mathbf{A}. \tag{3.14}$$

Introducing also the charge current density $\mathbf{j}$ in addition to the charge density ε, the four-vector current density reads

$$J^\alpha = (\varepsilon, j_1, j_2, j_3), \tag{3.15}$$

the Maxwell's equations take the form

$$\partial_\alpha F^{\alpha\beta} = -J^\beta,\ \varepsilon^{\alpha\beta\gamma\delta}\partial_\beta F_{\gamma\delta} = 0, \tag{3.16}$$

where, in the last set called the *Bianchi identities*, the Lorentz invariant Levi-Civita tensor $\varepsilon^{\alpha\beta\gamma\delta}$ is defined by (see SW Section 2.5)

$$\varepsilon^{\alpha\beta\gamma\delta}:\ \ \varepsilon^{0123} = +1 \text{ and totally antisymmetric.} \tag{3.17}$$

We note some important facts here:
1. Gauge invariance: $F_{\alpha\beta}$ is invariant under $A_\alpha \to A'_\alpha = A_\alpha + \partial_\alpha\Lambda(x)$, i.e., $F'_{\alpha\beta} = F_{\alpha\beta}$.
2. This invariance can be used to impose the Lorentz condition $\partial_\alpha A^\alpha = 0$ which reduces the number of independent components in the four-vector A_α to three. However, since the equation now obtained for the gauge parameter is $\Box\Lambda = 0$ the Lorentz condition does not eliminate the gauge freedom completely: $\Box\Lambda = (-\frac{1}{c^2}\partial_t^2 + \nabla^2)\Lambda = 0$ has wave solutions.
3. Imposing the Lorentz condition on Maxwell's equations without sources we find

$$\partial_\alpha F^{\alpha\beta} = \Box A^\beta - \partial^\beta\partial_\alpha A^\alpha = 0 \Rightarrow \Box A^\beta = 0 \tag{3.18}$$

so also A_α satisfy the wave equation in empty space. Thus these waves (light) propagate with the speed c. Both A_α and Λ now satisfy the free massless Klein-Gordon equation and hence the remaining gauge freedom is enough to eliminate one component of A_α completely. One can, e.g., impose the Coulomb gauge

$$A_0 = 0 \Rightarrow \nabla \cdot \mathbf{A} = 0, \tag{3.19}$$

which explains why the photon has only two degrees of freedom *on-shell*, i.e., when the equations of motion are satisfied. In particular, gauge invariance makes it possible to eliminate A_0 which is absolutely crucial since if kept when quantising the theory it would lead to negative normed physical states in the Hilbert space. This would make the whole theory non-unitary and thus useless in any physical context.

4. Maxwell's equations are invariant under inhomogeneous Lorentz (=Poincaré) transformations while Newtonian mechanics is invariant under Galileo transformations which do not mix space and time.

5. Both Newtonian mechanics and Maxwell's theory satisfy a principle of relativity related to their respective invariances (see SW sect. 1.3).

6. Maxwell's equations imply that the current J^α is "conserved", i.e., it is divergence free, $\partial_\alpha J^\alpha = 0$ which is just the continuity equation. This is extremely important since what is conserved is actually not the current but its associated charge. Using $\dot{J}^0 = -\nabla \cdot \mathbf{J}$ we find

$$Q := \int_{space} J^0 dV \Rightarrow \dot{Q} = -\int_{space} \nabla \cdot \mathbf{J} dV = 0, \tag{3.20}$$

where the integral is over all of space or over all space where the charge current density is non-zero so that the boundary term vanishes.

Recalling Noether's theorem (more later) a conserved charge exists as soon as there is a global symmetry, in this case a phase symmetry as in ordinary quantum mechanics (see SW section 4.10): the wave function is invariant under the phase transformation $\psi \to e^{i\alpha}\psi$ since only the absolut value of ψ is physical. This can be promoted to a local phase symmetry, i.e., a gauge invariance, under space dependent angles $\alpha(\mathbf{r})$ by introducing a vector potential $\mathbf{A}$ and a covariant derivative (SW eq. (4.10.3))

$$D_i = \partial_i - ieA_i. \tag{3.21}$$

Then *gauge covariance* means that

$$\psi'(\mathbf{r}) = e^{ie\alpha(\mathbf{r})}\psi(\mathbf{r}) \Rightarrow (D_i\psi)' = e^{ie\alpha(\mathbf{r})}D_i\psi \text{ provided } A_i'(\mathbf{r}) = A_i(\mathbf{r}) + \partial_i\alpha(\mathbf{r}). \tag{3.22}$$

This $A_i(\mathbf{r})$ is, in fact, just the electromagnetic vector-potential! Check the last equation above! (Note that $\alpha(\mathbf{r})$ was denoted $\Lambda(x)$ in Minkowski space above.)

3.1.2 Gravity in special relativity

Now we turn to the case of gravity. We will make the following crucial but natural assumption:

Gravity must be a relativistic (Lorentz covariant) field theory since

1. Galileo invariance is a low velocity approximation of Lorentz invariance so we should regard Lorentz invariance as the more basic one and demand it to hold also for gravity.
2. An oscillating massive body will produce wave-like force disturbances on other bodies (as for electric charges in EM) and since c is the maximal velocity these disturbances cannot propagate faster than light.
3. Since the electric and gravitational forces have the same long range feature $F \propto 1/r^2$ we may suspect that the gravitational field is massless (similar to the photon in EM) and should thus also satisfy the Lorentz invariant field equation $\Box(field) = source$, at least for weak fields (see below). This could mean that there is a massless particle, the *graviton*, similar to the massless photon! Thus we are looking for an equation of the form

$$\Box(\text{gravitational field}) = \text{gravitational source current}. \tag{3.23}$$

In particular this means that also for gravity the field and the current must be tensors of the same type!

Unfortunately, when looking for such an equation for gravity we are far worse off than Maxwell was when he found his equations. Maxwell knew about both ϕ_e and $\mathbf{A}$ as well as (almost all) their differential relations via $\mathbf{E}$ and $\mathbf{B}$, whereas in the gravity case we know at this point only ϕ_g and there is nothing similar to the three-vector potential $\mathbf{A}$ in Newtonian gravity. So at this point we can only introduce the analogues of $\mathbf{E}$ and ϕ_e in gravity, denoted $\mathbf{g}$ and ϕ_g:

$$\mathbf{F_g} = -G\frac{m_1 m_2}{r^2}\hat{\mathbf{r}} := m_1\mathbf{g}, \;\; \mathbf{g} = -\nabla\phi_g \; \Rightarrow \phi_g = -\frac{Gm_2}{r}. \tag{3.24}$$

The normalisation (and some signs) differs from EM since in gravity Gauss' law gives (here the mass density $\rho = m_2\delta(\mathbf{r})$, see SW page 152)

$$\int \mathbf{g}\cdot d\mathbf{S} = -4\pi G m_2 \;\; \Rightarrow \;\; \nabla^2\phi_g = 4\pi G\rho. \tag{3.25}$$

Note the important fact that the gravitational potential ϕ_g is dimensionless in natural units!

Exercise 2: The gravitational potential is in most situations extremely small. Check that it is dimensionless in natural units. Then define it so it becomes dimensionless in any units and compute its values at the "surface" of the sun, the earth and a proton.

Thus we need more input to make progress. There are at this point many options:
1. ϕ_g could be the whole story satisfying a Klein-Gordon equation $\Box\phi_g = \text{source}$,
2. ϕ_g could be the zero component of a gravitational 4-vector potential similar to in EM,
3. ϕ_g could be part of some bigger tensor but then we need to introduce many new tensor components:
a) the gravity tensor could be $B_{\alpha\beta} = -B_{\beta\alpha}$, i.e., an antisymmetric tensor or
b) the gravity tensor could be $h_{\alpha\beta} = h_{\beta\alpha}$, i.e., a symmetric tensor or
c) some even more horrible tensor with more indices!.

Comments on:
1. above: This is called "scalar gravity" (suggested by Nordström (1912)) and is ruled out by observations (massless scalar fields are ruled out by observations),
2. above: The "charge" in gravity has only one sign contradicting the fact that vector fields couple to charges with both signs (like for EM). (This is explained in the QFT course.)

Fortunately, it is not too difficult to pin down which of these cases is the most likely one to be correct:
From the expression for the gravitational force $F_g = Gm_1m_2/r^2$ it seems that the mass m plays the role of charge in gravity. However, m is NOT a conserved quantity in special relativity; the conserved quantity related to mass is energy: $E = mc^2$. If we want to mimic what happens in Maxwell's theory of EM, i.e., that $\partial_\alpha F^{\alpha\beta} = -J^\beta \Rightarrow \partial_\alpha J^\alpha = 0$, we need to find a divergence free current and the associated conserved charge in gravity.

The energy E is just the zero component of the four-momentum, i.e.,

$$p^\alpha = (E, p_1, p_2, p_3), \tag{3.26}$$

for which all four components are conserved. This is a consequence of Noether's theorem applied to time and space translation invariance of Minkowski space (see the last subsection below).

Thus we see that the conserved charge in gravity is a four-vector related to translation invariance in spacetime and hence the conservation equation $\dot{p}^\alpha = 0$ should come from a conserved current (i.e., divergence free) which has two vector indices, one is due to it being a current and one since its space integral of the zero component still has a vector index (the one on p^α): $p^\alpha = \int_{space} J^{0\alpha} dV$. However, this current with an extra index is a known quantity namely the stress, or energy-momentum, tensor, $T^{\alpha\beta}$ which is a conserved and symmetric tensor, see SW section 2.8 and the last subsection below on Noether's theorem (recall also SW sections 2.6 and 2.7).

The conclusion from the above rather simple arguments is that the most likely dynamical equation for gravity, corresponding to Maxwell's wave equation derived above (using the Lorentz gauge), should be in terms of a symmetric gravity field, from now on denoted $h_{\alpha\beta}$. The field equation must therefore be of the form (at least for weak gravity fields)

$$\Box h_{\alpha\beta} \propto T_{\alpha\beta}, \tag{3.27}$$

where $h_{00} = a\phi_g$ (since ϕ_g is a scalar in 3-space, that is, of $SO(3)$), where a is a constant to be determined by comparing to Newtionian physics, i.e.,

$$h_{\alpha\beta} = \begin{pmatrix} a\phi_g & h_{0j} \\ h_{i0} & h_{ij} \end{pmatrix}. \tag{3.28}$$

There are two very important facts here:
1. The above gravity equation is probably only valid after imposing some gauge condition. The reason for this is that there must exist a local symmetry, i.e., a gauge symmetry, so that the h_{0i} components can be gauge fixed away since if kept in the theory they would lead to negative norm states in the QFT Hilbert space of gravity (as for A_0 in EM, see discussion on the Coulomb gauge above). These new gauge transformations must be in terms of local parameters $\xi_\alpha(x)$ and read

$$\delta h_{\alpha\beta} = \partial_\alpha \xi_\beta + \partial_\beta \xi_\alpha, \tag{3.29}$$

where the parameters $\xi_\alpha(x)$ are connected to local (i.e., x-dependent) spacetime translations, i.e., coordinate transformations:

$$x^\alpha \to x'^\alpha = x^\alpha + \xi^\alpha(x). \tag{3.30}$$

2. Comparing again to EM, there is in gravity one extra feature that is the source of really serious complications namely the fact that if energy and momentum are the charges that source the gravity field then the theory must be non-linear in $h_{\alpha\beta}$ since the gravity field itself is charged (has energy and momentum)! This property makes gravity more similar to Yang-Mills than Maxwell's theory.

To summarize the situation this far: There must exist a (probably quite complicated) tensor involving two derivatives that is covariant under these gauge transformations (local translations in spacetime) similar to the field strength tensor in Yang-Mills theory. After imposing a proper gauge condition this tensor should reduce to $\Box h_{\alpha\beta}$ just like in the case of YM (and EM) after using the Lorentz gauge.

The final task is then to
1. find the, hopefully unique, gauge covariant field strength tensor in terms of $h_{\alpha\beta}$,
2. find the tensor with two derivatives formed from $h_{\alpha\beta}$ giving rise to the wave equation in the weak field limit (like $\partial_\alpha F^{\alpha\beta}$ in Yang-Mills theory),
3. find the gauge invariant non-linear field equations (like the Yang-Mills equations),
3. and find the physical interpretation of the field $h_{\alpha\beta}$.

There is, however, a good candidate for the gravity field namely the metric! In fact, if we already knew Riemannian geometry all the above issues would be rather easy to answer! So this is the purpose of the next few lectures. In the second lecture we discuss the metric in more general situations than we are used to from flat manifolds, e.g., on some simple curved manifolds (like the 2-dimensional sphere with angles θ, ϕ) and in cases where the coordinates are no longer orthogonal. The third lecture will deal with the equivalence principle (not used so far) which was of enormous importance to Einstein in trying to develop his theory of gravity, the general theory of relativity (GR). In the lectures after that we will discuss general tensors, Riemannian geometry and Einstein's equations which then will vindicate the results obtained above.

Question 1: Is the equivalence principle a consequence of the "derivation" of GR above or do we have to add it to the previous discussion in order to arrive at Einstein's theory? (You may return to this question after having studied SW Chapter 3.)

Question 2: Einstein presented his theory of gravity, general relativity, in 1915. Would it have been possible at that time to follow the logic given here? Why? (A similar version of this logic is due to R. Feynman.)

3.1.3 Noether's theorem for scalar fields: two different properties

Consider a massless relativistic *complex* scalar field $\phi(x)$ (and $\bar{\phi}$ its complex conjugate) with the action

$$S[\phi,\bar{\phi}] = \int \mathcal{L}(x)\, d^4x, \tag{3.31}$$

where the Lagrangian

$$\mathcal{L}(x) = -\eta^{\alpha\beta}\partial_\alpha\bar{\phi}\partial_\beta\phi. \tag{3.32}$$

Its variation under $\phi(x) \to \phi'(x) = \phi(x) + \delta\phi(x)$ is (to lowest order in $\delta\phi(x)$) in terms of $\mathcal{L}$

$$\delta\mathcal{L}(x) = (\mathcal{L}'(x) - \mathcal{L}(x))|_{\mathcal{O}(\delta\phi)} = -\eta^{\alpha\beta}\partial_\alpha\bar{\phi}\partial_\beta\delta\phi - \eta^{\alpha\beta}\partial_\alpha\delta\bar{\phi}\partial_\beta\phi \tag{3.33}$$

$$= (\eta^{\alpha\beta}\partial_\alpha\partial_\beta\bar{\phi})\delta\phi - \partial_\beta(\eta^{\alpha\beta}\partial_\alpha\bar{\phi}\delta\phi) + c.c., \tag{3.34}$$

after a partial integration which is OK since we actually consider the variation of the action. In the last line the first term is called "the bulk term" and the second "the boundary term" and similarly for the complex conjugated (c.c.) terms. The variation of the action is thus a functional $\delta S[\phi,\bar{\phi},\delta\phi,\delta\bar{\phi}]$ where the fields and their variations are independent and arbitrary functions at this point.

If we use a slightly more general theory with a potential $V(\bar{\phi},\phi)$ (may include a mass term $m^2\bar{\phi}\phi$) for the scalar field

$$\mathcal{L}(x) = -\eta^{\alpha\beta}\partial_\alpha\bar{\phi}\partial_\beta\phi - V(\phi,\bar{\phi}), \tag{3.35}$$

then the variation reads (this is the standard way to write the action, that is, with no terms in $\mathcal{L}(x)$ with two or more derivatives on a single field)

$$\begin{aligned}\delta S[\phi,\bar{\phi},\delta\phi,\delta\bar{\phi}] :&= (S[\phi+\delta\phi,\bar{\phi}+\delta\bar{\phi}] - S[\phi,\bar{\phi}])|_{\mathcal{O}(\delta\phi,\delta\bar{\phi})}\\ &= \int d^4x(\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi)}\delta(\partial_\alpha\phi)) + c.c.\\ &= \int d^4x(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\alpha\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi)})\delta\phi + \int d^4x\partial_\alpha(\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi)}\delta\phi) + c.c.,\end{aligned} \tag{3.36}$$

where, before doing the integration by parts in the last step, we have used the fact that $\delta(\partial_\alpha\phi) = \partial_\alpha(\delta\phi)$. Why is this true?

There are two fundamentally different things one can do with the above variation δS:

1. Hamiltons principle: By demanding $\delta\mathcal{L} = 0$ and $\delta\phi|_{time\ boundary} = 0$ (the same for the c.c.) we get the Euler-Lagrange (EL) equations from the bulk term and possible boundary conditions from the boundary terms in the space directions (often one of two kinds, Dirichlet and Neumann) since in the time direction the variation vanishes by definition. Thus we find

$$\text{Bulk term} \Rightarrow \frac{\partial\mathcal{L}}{\partial\phi} - \partial_\alpha \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi)} = 0, \tag{3.37}$$

and

$$\text{Boundary (space) term} \Rightarrow D.b.c.: \ \delta\phi|_{\text{space boundary}} = 0 \text{ or } N.b.c.: \ \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi)} = 0. \tag{3.38}$$

Note that $D.b.c.$ are usually given in the form $\phi|_{\text{space boundary}} = \text{constant}$.

In the case of the free complex scalar discussed above the EL equations are just the massless [1] Klein-Gordon equations

$$\Box\phi = 0, \ \ \Box\bar{\phi} = 0. \tag{3.39}$$

2. Noether's (first) theorem[2]: This theorem connects global symmetries to conserved (i.e., divergence free) currents and conserved charges. Global symmetries can be of two different kinds depending on if they affect spacetime (external) or not (internal):

a) Internal: U(1) phase transformations $\phi \to \phi' = e^{i\alpha}\phi$ and $\bar{\phi} \to \bar{\phi}' = e^{-i\alpha}\bar{\phi}$.
The action $S[\phi,\bar{\phi}] = \int(-\eta^{\alpha\beta}\partial_\alpha\bar{\phi}\partial_\beta\phi - V(\bar{\phi}\phi))\, d^4x$ is trivially invariant under these phase transformations when the parameter α is real and constant. For infinitesimal (α small) transformations the variation $\delta\phi(x) := (\phi(x') - \phi(x))|_{\mathcal{O}(\delta\phi)}$ gives $\delta\phi = i\alpha\phi$ and $\delta\bar{\phi} = -i\alpha\bar{\phi}$. Inserting this into the second line for $\delta\mathcal{L}(x)$ in eq.(3.36) shows of course that this is a symmetry, i.e., $\delta\mathcal{L}(x) = 0$. However, if we insert the field variations into the last line of $\delta\mathcal{L}(x)$ in eq. (3.36), that is after the integration by parts, we obtain Noether's result that on-shell (when the bulk term vanishes) $\delta\mathcal{L}(x) = 0$ gives rise to a conserved current $j^\alpha = (\partial^\alpha\bar{\phi})\delta\phi + (\partial^\alpha\phi)\delta\bar{\phi}$. The current is normally defined with the parameter removed, i.e.,

$$\partial_\alpha j^\alpha = 0, \text{ where } j^\alpha = i((\partial^\alpha\bar{\phi})\phi - (\partial^\alpha\phi)\bar{\phi}). \tag{3.40}$$

Note that the current itself is not charged!

Exercise 3: Show that the charge current j^α has itself no charge.

Exercise 4: Show that the current conserved on-shell.

[1] Free means that $V(\phi,\bar{\phi})$ has no cubic or higher terms and there are no couplings to any other fields.
[2] Her first theorem concerns global symmetries only. See the review by Bañados hep-th/1601.03616.

b) External: Coordinate transformations $x^\alpha \to x'^\alpha = x^\alpha + \xi^\alpha$.
Under such spacetime translations we have for scalar fields $\phi'(x') = \phi(x)$. Repeating the above discussion for phase transformations in this case, we rewrite the field transformation as $\phi'(x) = \phi(x) - (\phi'(x') - \phi'(x))$ from which we find the field variation

$$\delta\phi(x) := (\phi(x') - \phi(x))|_{\mathcal{O}(\xi^\alpha)} = -(\phi'(x') - \phi'(x))|_{\mathcal{O}(\xi^\alpha)} = -(\phi(x+\xi) - \phi(x))|_{\mathcal{O}(\xi^\alpha)} = -\xi^\alpha\partial_\alpha\phi(x). \tag{3.41}$$

Using these transformations, with ξ^α real and constant, in the first line of the variation of the Lagrangian in eq. (3.36) we now find that it does not vanish:

$$\delta_\xi\mathcal{L} = -\eta^{\alpha\beta}\partial_\alpha\bar{\phi}\partial_\beta\delta\phi - \eta^{\alpha\beta}\partial_\alpha\delta\bar{\phi}\partial_\beta\phi - (\partial_{(\bar{\phi}\phi)}V(\bar{\phi}\phi))(\delta\bar{\phi}\phi + \bar{\phi}\delta\phi) = -\xi^\alpha\partial_\alpha\mathcal{L}, \tag{3.42}$$

which, however, is expected since $\mathcal{L}$ is also a scalar field under coordinate transformations. Quite generally we write this as

$$\delta_\xi\mathcal{L} = \partial_\alpha J^\alpha. \tag{3.43}$$

Exercise 5: Fill in the details in the derivation of $\delta_\xi\mathcal{L} = -\xi^\alpha\partial_\alpha\mathcal{L}$ above.

Let us now derive the stress tensor for a free ($V = 0$) real scalar field. In this case we insert the field variation into the 3rd line in eq. (3.36), i.e., after performing the integration by parts, and require the bulk term to vanish (on-shell). This gives the result

$$\delta_\xi\mathcal{L} = \partial_\alpha\left(\frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi)}\delta\phi\right) \tag{3.44}$$

Subtracting these results for the variation we get the conserved current in this case

$$\partial_\alpha j^\alpha = 0 \text{ where } j^\alpha = \frac{\partial\mathcal{L}}{\partial(\partial_\alpha\phi)}\delta\phi - J^\alpha, \tag{3.45}$$

which is Noether's theorem now for external global symmetries.

Then we recall that $x'^\alpha = x^\alpha + \xi^\alpha \Rightarrow \delta\phi(x) = -\xi^\alpha\partial_\alpha\phi(x)$, and similarly for the Lagrangian itself, i.e., $\delta\mathcal{L}(x) = -\xi^\alpha\partial_\alpha\mathcal{L}(x) = \partial_\alpha(-\xi^\alpha\mathcal{L})$ which means that $J^\alpha = -\xi^\alpha\mathcal{L}$. Then in the case of a free real scalar field the conservation equation becomes $\partial_\alpha((\partial^\alpha\phi)\xi^\beta(\partial_\beta\phi) - \xi^\alpha\partial_\alpha(\frac{1}{2}\partial^\gamma\phi\partial_\gamma\phi) = 0$. Since the parameters ξ^α are constants this can rewritten as $\xi^\beta\partial_\alpha T^\alpha{}_\beta = 0$, where $T_{\alpha\beta}$ is the stress tensor. Dropping the parameters as usual we find

$$\partial_\alpha T^\alpha{}_\beta = 0, \text{ where } T_{\alpha\beta} = \partial_\alpha\phi\partial_\beta\phi - \tfrac{1}{2}\eta_{\alpha\beta}\partial^\gamma\phi\partial_\gamma\phi, \tag{3.46}$$

which is the standard form of the stress tensor for a free real scalar field[3]. This derivation does not automatically give a symmetric $T^{\alpha\beta}$ but this can always be corrected afterwards.

Exercise 6: Find the stress tensor for a complex scalar field with a potential $V(\bar{\phi}\phi)$. Is it conserved on-shell? Is it traceless?

[3] Note that this stress tensor is traceless which is due to the fact that the Lagrangian is scale invariant.