3.2 Lecture 2: General coordinates on flat and curved manifolds

Question: Below we will discuss both flat and curved manifolds and different coordinate systems on them. Why, in view of the discussion in the previous lecture, is it not enough to consider only flat manifolds in the context of a theory of gravity? This question is rather deep and we will in the coming lectures gradually develop our understanding to the point where this becomes clear. Please discuss it with the other students!

We start the discussion, which is largely a review, of the role of coordinates by looking at the distance d between two points in $\mathbf{R}^2$ is

$$d(\mathbf{r_1}, \mathbf{r_2}) = |\mathbf{r_2} - \mathbf{r_1}| = |\Delta\mathbf{r}|. \tag{3.57}$$

It is standard to call this distance s, or Δs, instead so we may write it as (after squaring)

$$\Delta s^2 := (\Delta s)^2 = (\Delta\mathbf{r})^2 = (\Delta x)^2 + (\Delta y)^2. \tag{3.58}$$

An even more common notation for infinitesimal distances is

$$ds^2 := (ds)^2 = (d\mathbf{r})^2 = (dx)^2 + (dy)^2 := dx^2 + dy^2 = \delta_{ij} dx^i\, dx^j \text{ where } x^1 = x,\, x^2 = y. \tag{3.59}$$

In polar coordinates (r, θ), $x = r\cos\theta$, $y = r\sin\theta$, this reads, with $\tilde{x}^1 = r$, $\tilde{x}^2 = \theta$,

$$ds^2 = dx^2 + dy^2 = dr^2 + r^2 d\theta^2 = (dr,\, d\theta) \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix} \begin{pmatrix} dr \\ d\theta \end{pmatrix} = \tilde{g}_{ij} d\tilde{x}^i d\tilde{x}^j, \tag{3.60}$$

where we in the 2nd equality have used the differentiated form of the relations above:

$$dx = dr\cos\theta - d\theta\, r\sin\theta, \tag{3.61}$$

$$dy = dr\sin\theta + d\theta\, r\cos\theta. \tag{3.62}$$

Thus we see that this change of coordinates has the following implication for the metric g

$$x^i = (x, y),\ g_{ij} = \delta_{ij} \Rightarrow \tilde{x}^i = (r, \theta),\ \tilde{g}_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}. \tag{3.63}$$

The above change of coordinates can be expressed in a general notation as

$$x^i = x^i(\tilde{x}) \Rightarrow dx^i = \frac{\partial x^i}{\partial \tilde{x}^j} d\tilde{x}^j, \tag{3.64}$$

which helps us construct the metric in the new coordinates $\tilde{x}^i$

$$ds^2 = \delta_{ij} dx^i dx^j = \delta_{ij} (\frac{\partial x^i}{\partial \tilde{x}^m} d\tilde{x}^m)(\frac{\partial x^j}{\partial \tilde{x}^n} d\tilde{x}^n) = d\tilde{x}^m d\tilde{x}^n (\frac{\partial x^i}{\partial \tilde{x}^m}\frac{\partial x^j}{\partial \tilde{x}^n}\delta_{ij}) := d\tilde{x}^m d\tilde{x}^n \tilde{g}_{mn}. \tag{3.65}$$

From this computation the metric in the new coordinates thus reads

$$\tilde{g}_{mn} = \frac{\partial x^i}{\partial \tilde{x}^m}\frac{\partial x^j}{\partial \tilde{x}^n}\delta_{ij}. \tag{3.66}$$

One can then check directly that the above results are obtained again:

$$\tilde{g}_{rr} = (\frac{\partial x}{\partial r})^2 + (\frac{\partial y}{\partial r})^2 = \cos^2\theta + \sin^2\theta = 1, \tilde{g}_{\theta\theta} = r^2,\ \tilde{g}_{r\theta} = 0. \tag{3.67}$$

An important fact about the factors $\frac{\partial x^i}{\partial \tilde{x}^j}$ appearing in the metric above is that they should be invertible as matrices. This is needed in order for the new coordinates to span all directions close to the point (called a patch) where it is defined. Assuming this property we have $x = x(\tilde{x}) = x(\tilde{x}(x))$ and hence

$$dx^i = \frac{\partial x^i}{\partial \tilde{x}^j}\frac{\partial \tilde{x}^j}{\partial x^k}dx^k \Rightarrow \frac{\partial x^i}{\partial \tilde{x}^j}\frac{\partial \tilde{x}^j}{\partial x^k} = \delta^i_k. \tag{3.68}$$

So if we use matrix notation and write $A_i{}^j := \frac{\partial x^i}{\partial \tilde{x}^j}$ then its inverse $A^{-1}{}_i{}^j := \frac{\partial \tilde{x}^i}{\partial x^j}$ exists at least in some patch where the transformations behave well (i.e., don't become zero or infinite). Are the polar coordinates above well behaved everywhere?

It is interesting to note that by taking the determinant of eq. (3.66)

$$\det(\tilde{g}_{mn}) = \det(\frac{\partial x^i}{\partial \tilde{x}^m}\frac{\partial x^j}{\partial \tilde{x}^n}\delta_{ij}) = \det(A\mathbf{1}A^T) = (\det A)^2 > 0, \tag{3.69}$$

where we have defined $A_m{}^i := \frac{\partial x^i}{\partial \tilde{x}^m}$.
So far we have only mentioned coordinate systems that are orthogonal so let us see how to deal with those that are not. Consider the new coordinates $(\tilde{x}, \tilde{y})$ on $\mathbf{R^2}$ defined by

$$\tilde{x} = x + y,\ \ \tilde{y} = y. \tag{3.70}$$

Drawing the new coordinate lines in an orthogonal system with (x, y) on the axes it is clear that the system $(\tilde{x}, \tilde{y})$ is not orthogonal. Computing the metric in the tilde system gives, since $x = \tilde{x} - \tilde{y}$ and $y = \tilde{y}$,

$$ds^2 = dx^2 + dy^2 = (d\tilde{x} - d\tilde{y})^2 + (d\tilde{y})^2 = d\tilde{x}^2 + 2d\tilde{y}^2 - 2d\tilde{x}d\tilde{y}, \tag{3.71}$$

which gives the metric

$$\tilde{g}_{ij} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}. \tag{3.72}$$

A crucial question is now whether or not the tilde coordinates are independent of each other: Is $\frac{\partial \tilde{x}}{\partial \tilde{y}}$ equal to zero or not? The correct way to answer this question is as follows

$$\frac{\partial \tilde{x}}{\partial \tilde{y}} = \frac{\partial}{\partial \tilde{y}}\tilde{x} = (\frac{\partial x}{\partial \tilde{y}}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \tilde{y}}\frac{\partial}{\partial y})(x + y) = (-\frac{\partial}{\partial x} + \frac{\partial}{\partial y})(x + y) = 0. \tag{3.73}$$

The conclusion is that any change of coordinates leads to new coordinates that must be considered independent. Note that it is not allowed to use the coordinate relations directly in $\frac{\partial \tilde{x}}{\partial \tilde{y}}$ (which would give a non-zero result).

We now turn to a simple example of coordinates and metric on a curved manifold, namely the two-sphere S^2. Since the easiest way to define S^2 is to embed it into flat three-space $\mathbf{R}^3$ we start from

$$ds^2(\mathbf{R}^3) = dx^2 + dy^2 + dz^2, \tag{3.74}$$

and rewrite it in terms of the angular coordinates θ, ϕ defined by

$$x = a \sin\theta \cos\phi,\;\; y = a \sin\theta \sin\phi,\;\; z = a \cos\theta, \tag{3.75}$$

which tells us that $x^2 + y^2 + z^2 = a^2$ defining a two-sphere of radius a. Using the notation $x^i = (x, y, z)$ and $\tilde{x}^a = (\tilde{x}^1, \tilde{x}^2) = (\theta, \phi)$ and hence $x^i = x^i(\theta, \phi) = x^i(\tilde{x}^a)$ we have

$$dx = \tfrac{\partial x}{\partial \tilde{x}^a} d\tilde{x}^a = \tfrac{\partial x}{\partial \theta} d\theta + \tfrac{\partial x}{\partial \phi} d\phi = a\, d\theta \cos\theta \cos\phi - a\, d\phi \sin\theta \sin\phi, \tag{3.76}$$

$$dy = a\, d\theta \cos\theta \sin\phi + a\, d\phi \sin\theta \cos\phi, \tag{3.77}$$

$$dz = -a\, d\theta \sin\theta. \tag{3.78}$$

Then

$$ds^2(S^2) = (dx(\theta, \phi))^2 + (dy(\theta, \phi))^2 + (dz(\theta, \phi))^2 = a^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{3.79}$$

Finally from

$$ds^2(S^2) = d\tilde{x}^a\, d\tilde{x}^b \tilde{g}_{ab}, \tag{3.80}$$

we can read off the metric:

$$\tilde{g}_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}. \tag{3.81}$$

Computing the tangent vectors to the angle coordinate lines on the 2-sphere we see that these coordinates are orthogonal, i.e., $\mathbf{e}_\theta \cdot \mathbf{e}_\phi = 0$, where

$$\mathbf{e}_\theta := \tfrac{\partial \mathbf{r}}{\partial \theta} = a(\cos\theta \cos\phi, \cos\theta \sin\phi, -\sin\theta), \tag{3.82}$$

$$\mathbf{e}_\phi := \tfrac{\partial \mathbf{r}}{\partial \phi} = a(-\sin\theta \sin\phi, \sin\theta \cos\phi, 0). \tag{3.83}$$

Comment: Dropping the $\sin^2\theta$ factor in the metric gives $ds^2(T^2) = a^2(d\theta^2 + d\phi^2)$, the metric on the two-torus T^2. Can this two-torus be embedded in $\mathbf{R}^3$ like e.g. a donut?

Note that $det(\tilde{g}_{ab}(S^2)) = a^4 \sin^2\theta \geq 0$ emphasising the fact that these angular coordinates behave badly at some points (where $\tilde{g}_{ab}(S^2)) = 0$). One should also note that the embedding is not invertible since $\partial x^i / \partial \tilde{x}^a$ is not a square matrix.

It is now easy to compute distances on S^2 using $d := \int |ds|$:
Ex 1: $d(N \to S) = a \int_0^\pi d\theta = a\pi$ ($\phi = \phi_0$ = fixed)

Ex 2: d(Circles // to equator at $\theta = \theta_0$ = fixed) $= a \int_0^{2\pi} \sin\theta_0 \, d\phi = 2\pi \, a \sin\theta_0$.

The area is $A(S^2) = \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \sqrt{det\tilde{g}} = a^2 \int_0^{\pi} d\theta \sin\theta \int_0^{2\pi} d\phi = 4\pi \, a^2$.

Exercise 1: A trivial example in one dimension is the following. The distance in $\mathbf{R}$ between $x = 0$ and $x = 1$ is given by the metric $ds^2 = g_{xx} dx^2$ with $g_{xx} = 1$. Thus

$$s = \int_0^1 ds = \int_0^1 dx = 1. \tag{3.84}$$

If we introduce the inverse coordinate $y = 1/x$ then $dx = -dy/y^2$ and the integral becomes

$$s = \int_{\infty}^1 (-\frac{1}{y^2}) dy. \tag{3.85}$$

What is the metric g_{yy} in the y-coordinate system? Is the minus sign in the integral above a problem?

The Fubini-Study metric: (For more details see the actual lecture.)
A quite different-looking metric on S^2 is obtained by stereographic projection onto the equatorial plane. Projecting a sphere of radius a onto the equatorial plane with coordinates (ξ, η) (along the (x, y) directions) we can use congruent triangles to conclude that

$$\xi = a\frac{x}{a+z}, \;\; \eta = a\frac{y}{a+z}. \tag{3.86}$$

These relations imply that $\xi^2 + \eta^2 = a^2 \frac{a-z}{a+z}$ from which one can solve for z as a function of (ξ, η). The final result is written in terms of a complex coordinate $z := \xi + i\eta$ on the equatorial plane if the projection is from the south pole (S)(with the north pole (N) mapping to the origin and the south pole to infinity) and $(w, \bar{w})$ if N and S are interchanged. The sphere is then called the Riemann sphere (RS) with the mapping between the upper and lower hemispheres given by $z = 1/w$. This construction gives the following form of the metric, where we have put $a = 1$,

$$ds^2(RS) = 4\tfrac{dz d\bar{z}}{(1+\bar{z}z)^2} \tag{3.87}$$

If we set $z = \xi + i\eta$ this metric can be rewritten as

$$ds^2(RS) = 4\frac{d\xi^2 + d\eta^2}{(1+\xi^2+\eta^2)^2}, \tag{3.88}$$

which in polar coordinates $(\xi = r\cos\theta, \eta = r\sin\theta)$ becomes

$$ds^2(RS) = 4\frac{dr^2 + r^2 d\theta^2}{(1+r^2)^2}. \tag{3.89}$$

This last form is quite useful and its spacetime version arises in some discussions in cosmology.

A way to arrive at still another form of the metric on S^2 is to start from $x^2+y^2+z^2=1$ and define polar coordinates in the (x,y) plane. Thus $x = r\cos\alpha$ and $y = r\sin\alpha$ which gives

$$ds^2(S^2) = dx^2+dy^2+dz^2 = dr^2+r^2d\alpha^2+dz^2 = dr^2+r^2d\alpha^2+\tfrac{r^2}{(1-r^2)}dr^2 = \tfrac{dr^2}{(1-r^2)}+r^2d\alpha^2. \tag{3.90}$$

where we in the 3rd equality have used that $r^2+z^2=1$ implies $dz=-\frac{rdr}{z}$ and $z^2=1-r^2$.

One may note that the last form looks somewhat similar to the previous one, the differences being that the denominator has different signs and how it appears in the two terms in the metric. A natural question to ask is if there is a simple coordinate transformation between them. The answer is found if we first assume that $\theta=\alpha$. Then equating the $d\alpha^2$ terms, renaming the radial coordinate in the last metric $\tilde{r}$, gives the relation

$$\frac{2r}{1+r^2}=\tilde{r}. \tag{3.91}$$

Equating the dr^2 terms implies

$$\frac{4}{(1+r^2)^2}(\frac{dr}{d\tilde{r}})^2=\frac{1}{(1-\tilde{r}^2)}. \tag{3.92}$$

The last two relations must be compatible which is easily checked by computing $\frac{d\tilde{r}}{dr}$ from the algebraic relation between r and $\tilde{r}$ above. It gives $\frac{d\tilde{r}}{dr}=2\frac{1-r^2}{(1+r^2)^2}$.
Inserting this into the last relation it is found to be satisfied.

Exercise 2: Check the above forms of the S^2 metric.

Exercise 3: Compute the metric $ds^2(RS)=4\frac{dzd\bar{z}}{(1+\bar{z}z)}$ in terms of $(w,\bar{w})$ where $w=1/z$. Are these two forms of the metric well-defined over the whole of S^2?

An interesting and extremely useful phenomenon will appear if we flip the sign in the denominator of the Fubini-Study metric above. This gives the metric for the so called *hyperbolic plane*

$$ds^2(GBL)=4\frac{dr^2+r^2d\theta^2}{(1-r^2)^2}. \tag{3.93}$$

This geometry was discovered in the beginning of the 19'th century about two thousand years after Euclid found a logical problem with one of his 5'th axiom when trying construct an axiomatic approach to geometry. The main reason (perhaps) why this geometry took so long to find, by Gauss, Bolyai and Lobachevski, becomes clear when we ask how one can derive it from an embedding space. The surprising answer is that ordinary three-space does not work. Instead one has to use a three-dimensional with Minkowski signature: Introduce the defining constraint

$$x^2+y^2-z^2=-1, \tag{3.94}$$

then setting $r^2 = x^2 + y^2$ we get $z^2 = 1 + r^2$ and $dz = \frac{rdr}{z}$. Thus

$$ds^2(GBL) = dx^2+dy^2-dz^2 = dr^2+r^2d\alpha^2-dz^2 = dr^2+r^2d\alpha^2-\tfrac{r^2}{(1+r^2)}dr^2 = \tfrac{1}{(1+r^2)}dr^2+r^2d\alpha^2. \tag{3.95}$$

By a coordinate transformations relating this radial coordinate to a different one similar to what we did above for S^2, we can prove that the GBL metric can also be written as above, i.e., $ds^2(GBL) = 4\frac{dr^2+r^2d\theta^2}{(1-r^2)^2}$.

Exercise 4: Prove the last statement.

Comment 1: Note that while the signature is the same as for S^2, i.e., the euclidean $(+,+)$, what differs is the the curvature. The curvature (we will define this once we the mathematics of Riemannian geometry under control) is positive and constant over the whole of S^2 while for GBL it is negative and constant over the whole space. The reason "constant of the whole space" is emphasised here is that sometimes the GBL space is said to be like the surface of a horse saddle which obviously is embeddable in $\mathbf{R}^3$. However, the saddle does have negative curvature but it is not constant over the whole saddle. This fact makes the GBL space impossible to embed in $\mathbf{R}^3$.

NOTE: In the conventions used in Weinberg's book the sign of the curvature is opposite to the one used in these notes and in the lectures!

Comment 2: The GBL metric is sometimes referred to as the hyperbolic space H^2. The three-dimensional version H^3 is in fact a well-known thing for a physicist. Recall the equation satisfied by the relativistic 4-momentum p^μ: $p^2 + m^2 = 0$. Explicitly this reads

$$(p^1)^2 + (p^2)^2 + (p^3)^2 - (p^0)^2 = -m^2, \tag{3.96}$$

which defines the hyperbolic nature of the surface of possible momenta satisfying the mass-square condition in special relativity.

Comment 3: It is now easy to write all three geometries discussed here in one formula by introducing a parameter $k = 0, \pm 1$ as follows

$$ds^2(k) = \frac{dr^2}{(1 - kr^2/a^2)} + r^2 d\alpha^2, \tag{3.97}$$

where $k = 0$ corresponds to flat euclidean space and $k = \pm$ to S^2 and H^2, respectively. Radial distances as a function of r are then given by

$$s(r) = \int_0^r \frac{dr'}{\sqrt{(1 - kr'^2/a^2)}} = \begin{cases} = a \text{ arcsin } \frac{r}{a}, & k = 1\,(r \leq a),\ S^2 \\ = r & k = 0, \qquad \mathbf{R}^2 \\ = a \text{ arcsinh } \frac{r}{a}, & k = -1, \qquad H^2. \end{cases} \tag{3.98}$$

On S^2 the distance from the north pole to the equator is $s = \frac{\pi}{2}a$. From the formula given above we get $s(a) = a \arcsin \frac{a}{a}$, i.e., $\sin \frac{s(a)}{a} = 1$ giving the same result.

Exercise 5: Compute the distance to $r = \infty$ in the hyperbolic case. The answer is probably the expected one but as we will see later in the context of cosmology there is a very strange and interesting twist to this result for manifolds with Lorentzian signature.

Exercise 6: The two manifolds S^2 and H^2 are called *maximally symmetric* manifolds (more later in the course) and can be characterised by their symmetry groups: S^2 has symmetry group $SO(3)$. What is the symmetry group of H^2?

Question 1: As we have seen above there are many, in fact infinitely many, coordinate choices on any manifold. Therefore it is natural to ask how one can obtain coordinate independent information about the geometry?

Question 2: A question related to the previous one is the following one. How does one describe a manifold without referring to how it is embedded in a bigger space?

Question 3 (to keep in mind for later): Why do we have to consider manifolds that are not flat in formulating a theory of gravity?