4 Problems

For all problems where it is relevant, state your convention concerning the sign of the Riemann tensor: Is the curvature scalar for S^2 positive (as in the lectures) or negative (as in Weinberg's book)?

4.1 Special relativity

1. Show that the proper time $d\tau$ is invariant under Lorentz transformations.

2. Write out the Lorentz transformation matrix $\Lambda^{\alpha}{}_{\beta}$ for a standard Lorentz boost in the x-direction. Do this also when the velocity is in an arbitrary direction. Find also $\Lambda^{\alpha\beta}$ and $\Lambda_{\alpha\beta}$.

3. Suppose that a particle is moving with velocity $\mathbf{u}$ at an angle θ from the x-axis in the xy-plane of a system S. What is the corresponding angle θ' in the system S' moving with velocity $\mathbf{v} = (v, 0, 0)$ relative the system S?

4. Consider two rockets, A and B, moving in the system S with velocities $\mathbf{u}$ and $\mathbf{v}$, respectively. Find the relative velocity of B in the reference system of A.

5. Consider the Maxwell's equations in Lorentz covariant form

$$\partial_{\alpha} F^{\alpha\beta} = -J^{\beta}, \;\; \partial_{[\alpha} F_{\beta\gamma]} = 0. \tag{4.1}$$

a) Show that the above equations are equivalent to the four Maxwell equations expressed in terms of the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$.
b) Give $F_{\alpha\beta}$ in terms of the 4-vector potential and show that the above equations are gauge invariant, and that the second equation above is an identity, the so called *Bianchi identity.*
d) Design a derivative, that you may denote D_{α}, that is covariant under $U(1)$, i.e., local phase, transformations of a complex scalar field.
e) Show that the current in Maxwell's equations above is divergence free and that its associated charge is conserved.

6. Derive the stress tensor (energy-momentum tensor) for electromagnetism. That is, perform a constant space-time translation $x^{\alpha} \to \tilde{x}^{\alpha} = x^{\alpha} + \xi^{\alpha}$ on the Maxwell lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta}, \tag{4.2}$$

and identify the conserved current. This procedure does not produce a symmetric stress tensor which can be remedied by adding a divergence free (on α) term $\partial_{\gamma} K^{\gamma\alpha\beta}$ where $K^{\gamma\alpha\beta}$ is anti-symmetric in its first two indices. Show that by setting

$$K^{\gamma\alpha\beta} = F^{\alpha\gamma} A^{\beta}, \tag{4.3}$$

one arrives at a stress tensor that is symmetric, traceless and divergence free on-shell. This is the essence of Noether's theorem.

7. Consider the stress tensor for a charged particle

$$T^{\alpha\beta}(x) = \frac{1}{m} \int d\tau p^\alpha p^\beta \delta^4(x - x(\tau)). \tag{4.4}$$

a) Verify that this stress tensor is not conserved by itself if coupled to an electromagnetic field.
b) Add the stress tensor for the electromagnetic field and show that the sum of these two stress tensors is conserved.
c)Why are the two pieces of the stress tensor not conserved separately?

4.2 The equivalence principle

1. The strength of forces.
a) Compare the electrostatic and gravitational forces between two protons 2 Fermi apart.
b) What should the mass of the proton be for these forces to cancel exactly?
c) Compute the Newtonian gravitational potential at the surface of the sun, the earth, the moon and the proton. Note that the gravitational potential is dimensionless, i.e., the answer is just a number.

2. Laika is orbiting the earth in a spaceship. She has chosen the altitude of the circular orbit carefully so that her onboard clock ticks at the same rate as a clock at rest on the surface of the earth. What is this altitude? You can neglect the rotation of the earth.
a) Derive the ordinary time dilatation effect in special relativity. Is the clock on earth or in the spaceship faster? What exactly does this mean in terms of ticks of the clocks?
b) Compare the previous result to the one you get by using a single coordinate system (neglecting the presence of the earth) as usually done in GR. Are the physical results the same?
c) Derive the pure gravity effect (i.e., assume Laika is at rest) by comparing time on earth with Laika's time. Use the following notation: Earth radius = R_0 and for Laika use $R_L = R_0 + h$ where h is the altitude.
d) Determine Laika's altitude (give R_L) by comparing the results in a) (or b)) and c).
e) Find the altitude (give R_L) by comparing $d\tau$ at the two locations, the orbit and the surface of the earth, in the same coordinate system. You may assume that all metric components except g_{00} are the same as in Minkowski space.

4.3 Geometry

1. Consider the manifold defined as the hypersurface $x^2 + y^2 - z^2 = -R^2$ (with $R^2 > 0$).
a) Derive the metric by defining $x^2 + y^2 = r^2$ and eliminating z.
b) Draw the surface and determine its signature (i.e., is it euclidean or lorentzian?).
c) What happens if we start from $x^2 + y^2 - z^2 = R^2$ instead?