QUESTIONS ON TENSOR ANALYSIS

–GRAVITATION AND COSMOLOGY (FFM071)–

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Here are some questions on tensors that might come up on the oral exam.

Problem 1. Indicate whether the following statements are true or false.

- The Minkowski metric $\eta_{\alpha\beta}$ is invariant under Lorentz transformations.

- An arbitrary spacetime 4-vector x^α is invariant under Lorentz transformations.

- The first postulate of special relativity states that the outcome of a physical experiment will depend on the coordinate system you use.

- Your worldline lies inside of the lightcone.

- An observer at rest measures the lifetime of a muon in motion to be longer than that of a muon at rest.

- Maxwell's equations are Lorentz invariant but Newton's laws are not.

- The affine connection $\Gamma^\lambda_{\mu\nu}$ is a mixed $(1,2)$-tensor with respect to general coordinate transformations.

- The object $\partial_\alpha V^\beta$ does not transform as a tensor under Lorentz transformations.

- The components of the Kronecker delta symbol δ^μ_ν are the same in all coordinate systems.

- The epsilon-symbol $\epsilon^{\mu\nu\rho\sigma}$ is invariant under general coordinate transformations.

Problem 2. Do any of the following equations fail to make sense for generic[1] coordinate systems? If yes, indicate which ones.

$$\eta_{\alpha\beta}\Phi^{\beta} = \Phi_{\alpha} \tag{1}$$

$$\Lambda^{\alpha_1}{}_{\beta_1}\Lambda^{\alpha_2}{}_{\beta_2}\eta_{\alpha_1\alpha_2}\Lambda^{\beta_1}{}_{\gamma_1}\Lambda^{\beta_2}{}_{\gamma_2}\mathrm{d}x^{\gamma_1}\mathrm{d}x^{\gamma_2} = \eta_{\delta_1\delta_2}\mathrm{d}x^{\delta_1}\mathrm{d}x^{\delta_2} \tag{2}$$

$$V^{\alpha\alpha} = \mathrm{Tr}V \tag{3}$$

$$V_i V^i = V_\alpha V^\alpha \tag{4}$$

$$\mathrm{d}\tau^2 = 0.12 \tag{5}$$

$$T^{\mu\nu} = 196883 \tag{6}$$

$$\partial_\alpha F_{\alpha\beta} = -J_\beta \tag{7}$$

$$D_\nu V_\mu = \partial_\nu V_\mu - \Gamma^{\sigma}_{\mu\nu}V_\rho \tag{8}$$

$$g^{\mu\nu}g_{\nu\rho} = \delta^{\mu}_{\rho} \tag{9}$$

$$D_\mu V_\nu - D_\nu V_\mu = \partial_\mu V_\nu - \partial_\nu V_\mu \tag{10}$$

Problem 3.

(a) Let $g_{\mu\nu}$ be the metric tensor in D dimensions and $g^{\mu\nu}$ its inverse. Calculate $g^{\mu\nu}g_{\mu\nu}$.

(b) Let $T_\mu{}^\nu$ be a tensor. Write down its transformation under general coordinate transformations $x^\mu \to x'^\mu$. Suggest at least one way (but more if you can) in which you can construct something coordinate *invariant* from $T_\mu{}^\nu$.

(c) Show that $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu$ is invariant under general coordinate transformations.

(d) Let V_μ be a covariant vector. Construct $V^\mu \equiv g^{\mu\nu}V_\nu$ and show that it transforms correctly as a contravariant vector.

(e) Let V_μ be a covariant vector. Compute the general coordinate transformation of $\partial_\mu V_\nu$. Is $\partial_\mu V_\nu$ a tensor?

(f) The affine connection is defined by

$$\Gamma^{\lambda}_{\mu\nu} = \frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu} \tag{11}$$

[1] "generic" implies that the equations should be sensible for an arbitrary coordinate frame of the type indicated by the index structure. For instance, the components of a 4-vector are generically non-vanishing.

where x^μ is an arbitrary coordinate frame and ξ^α is an inertial frame (flat). Derive the transformation of the affine connection under general coordinate transformations $x^\mu \to x'^\mu$.

Problem 4.

(a) Define a *Lorentz tensor.*

(b) Define an *arbitrary tensor.* (under general coord. transformation).

(c) Define a *tensor density.*

Problem 5. Let $T^{\alpha\beta}{}_{\gamma\delta}$ be a tensor, and define the combination $\widetilde{T}^{\alpha\beta}{}_{\gamma\delta} \equiv \frac{1}{2}(T^{\alpha\beta}{}_{\gamma\delta} + T^{\alpha\beta}{}_{\delta\gamma})$. Evaluate the expression:

(12) $$\widetilde{T}^{\alpha\beta}{}_{\gamma\delta}\,\epsilon^{\gamma\delta\varepsilon\eta}$$

and identify the resulting object. Is it a tensor?

Problem 6. Let $T^{\mu\nu}$ be a tensor and define

(13) $$S^{\mu\nu} \equiv \frac{1}{2}(T^{\mu\nu} + T^{\nu\mu}), \qquad A^{\mu\nu} \equiv \frac{1}{2}(T^{\mu\nu} - T^{\nu\mu}).$$

Let furthermore $g_{\mu\nu}$ be the metric tensor and $\epsilon_{\mu\nu\rho\sigma}$ the Levi-Civita symbol in 4d. Show that:

$$\begin{aligned} g_{\mu\nu}T^{\mu\nu} &= S^{\mu\nu}g_{\mu\nu} \\ \epsilon_{\mu\nu\rho\sigma}T^{\rho\sigma} &= \epsilon_{\mu\nu\rho\sigma}A^{\rho\sigma} \\ S_{\mu\nu}A^{\mu\nu} &= 0. \end{aligned} \qquad (14)$$
