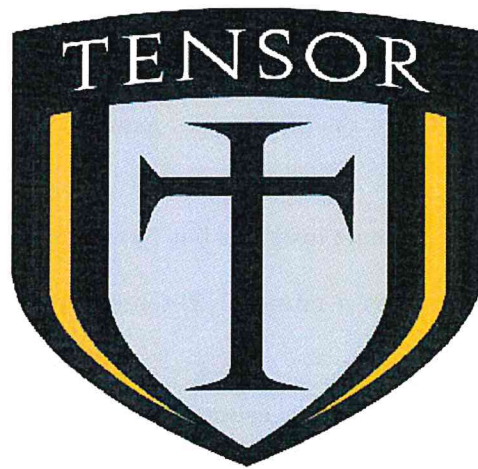


QUESTIONS ON TENSOR ANALYSIS
–GRAVITATION AND COSMOLOGY (FFM071)–

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Here are some questions on tensors that might come up on the oral exam.

Problem 1. Indicate whether the following statements are true or false.

- The Minkowski metric $\eta_{\alpha\beta}$ is invariant under Lorentz transformations.
 - An arbitrary spacetime 4-vector x^α is invariant under Lorentz transformations.
 - The first postulate of special relativity states that the outcome of a physical experiment will depend on the coordinate system you use.
 - Your worldline lies inside of the lightcone.
 - An observer at rest measures the lifetime of a muon in motion to be longer than that of a muon at rest.
 - Maxwell's equations are Lorentz invariant but Newton's laws are not.
 - The affine connection $\Gamma_{\mu\nu}^\lambda$ is a mixed $(1,2)$ -tensor with respect to general coordinate transformations.
 - The object $\partial_\alpha V^\beta$ does not transform as a tensor under Lorentz transformations.
 - The components of the Kronecker delta symbol δ_ν^μ are the same in all coordinate systems.
 - The epsilon-symbol $\epsilon^{\mu\nu\rho\sigma}$ is invariant under general coordinate transformations.
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Problem 2. Do any of the following equations fail to make sense for generic¹ coordinate systems? If yes, indicate which ones.

- (1) $\eta_{\alpha\beta}\Phi^\beta = \Phi_\alpha$
 - (2) $\Lambda^{\alpha_1}_{\beta_1}\Lambda^{\alpha_2}_{\beta_2}\eta_{\alpha_1\alpha_2}\Lambda^{\beta_1}_{\gamma_1}\Lambda^{\beta_2}_{\gamma_2}dx^{\gamma_1}dx^{\gamma_2} = \eta_{\delta_1\delta_2}dx^{\delta_1}dx^{\delta_2}$
 - (3) $V^{\alpha\alpha} = \text{Tr}V$
 - (4) $V_iV^i = V_\alpha V^\alpha$
 - (5) $d\tau^2 = 0.12$
 - (6) $T^{\mu\nu} = 196883$
 - (7) $\partial_\alpha F_{\alpha\beta} = -J_\beta$
 - (8) $D_\nu V_\mu = \partial_\nu V_\mu - \Gamma_{\mu\nu}^\sigma V_\rho$
 - (9) $g^{\mu\nu}g_{\nu\rho} = \delta_\rho^\mu$
 - (10) $D_\mu V_\nu - D_\nu V_\mu = \partial_\mu V_\nu - \partial_\nu V_\mu$
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Problem 3.

- (a) Let $g_{\mu\nu}$ be the metric tensor in D dimensions and $g^{\mu\nu}$ its inverse. Calculate $g^{\mu\nu}g_{\mu\nu}$.
- (b) Let $T_\mu{}^\nu$ be a tensor. Write down its transformation under general coordinate transformations $x^\mu \rightarrow x'^\mu$. Suggest at least one way (but more if you can) in which you can construct something coordinate *invariant* from $T_\mu{}^\nu$.
- (c) Show that $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ is invariant under general coordinate transformations.
- (d) Let V_μ be a covariant vector. Construct $V^\mu \equiv g^{\mu\nu}V_\nu$ and show that it transforms correctly as a contravariant vector.
- (e) Let V_μ be a covariant vector. Compute the general coordinate transformation of $\partial_\mu V_\nu$. Is $\partial_\mu V_\nu$ a tensor?
- (f) The affine connection is defined by

$$(11) \quad \Gamma_{\mu\nu}^\lambda = \frac{\partial x^\lambda}{\partial x'^\alpha} \frac{\partial^2 x'^\alpha}{\partial x^\mu \partial x^\nu}$$

¹“generic” implies that the equations should be sensible for an arbitrary coordinate frame of the type indicated by the index structure. For instance, the components of a 4-vector are generically non-vanishing.

where x^μ is an arbitrary coordinate frame and ξ^α is an inertial frame (flat). Derive the transformation of the affine connection under general coordinate transformations $x^\mu \rightarrow x'^\mu$.

Problem 4.

(a) Define a *Lorentz tensor*.

(b) Define an *arbitrary tensor*. (under general coord. transformations)

(c) Define a *tensor density*.

Problem 5. Let $T^{\alpha\beta}_{\gamma\delta}$ be a tensor, and define the combination $\tilde{T}^{\alpha\beta}_{\gamma\delta} \equiv \frac{1}{2}(T^{\alpha\beta}_{\gamma\delta} + T^{\alpha\beta}_{\delta\gamma})$. Evaluate the expression:

$$(12) \quad \tilde{T}^{\alpha\beta}_{\gamma\delta} \epsilon^{\gamma\delta\epsilon\eta}$$

and identify the resulting object. Is it a tensor?

Problem 6. Let $T^{\mu\nu}$ be a tensor and define

$$(13) \quad S^{\mu\nu} \equiv \frac{1}{2}(T^{\mu\nu} + T^{\nu\mu}), \quad A^{\mu\nu} \equiv \frac{1}{2}(T^{\mu\nu} - T^{\nu\mu}).$$

Let furthermore $g_{\mu\nu}$ be the metric tensor and $\epsilon_{\mu\nu\rho\sigma}$ the Levi-Civita symbol in 4d. Show that:

$$(14) \quad \begin{aligned} g_{\mu\nu} T^{\mu\nu} &= S^{\mu\nu} g_{\mu\nu} \\ \epsilon_{\mu\nu\rho\sigma} T^{\rho\sigma} &= \epsilon_{\mu\nu\rho\sigma} A^{\rho\sigma} \\ S_{\mu\nu} A^{\mu\nu} &= 0. \end{aligned}$$