2. Derive the metric on S^2 (the two-sphere) from its definition $x^2 + y^2 + z^2 = a^2$:
a) in (standard) polar coordinates (θ, ϕ),
b) and by eliminating z (set $x^2 + y^2 = r^2$).
c) Compute the affine connection in both of these sets of coordinates from its definition.

3. Find the metric of a two-dimensional flat surface in polar coordinates and compute the affine connection, Riemann tensor, Ricci tensor and curvature scalar.

4. Consider the two-sphere with unit radius. Embed it into $\mathbf{R}^3$ and use a stereographic projection from the south pole onto the plane through the equator to derive the metric.
a) Show that the answer is the Fubini-Study metric on the Riemann sphere:

$$ds^2 = 4\frac{dx^2 + dy^2}{(1 + x^2 + y^2)^2}, \tag{4.5}$$

b) Compute the Ricci tensor and show that it is an Einstein metric i.e., that it satisfies the equation $R_{ij} = \alpha g_{ij}$ for some parameter α.
c) Change the sign in the denominator and recalculate the Ricci tensor. What is α now?

5. Prove that the two metrics on the unit radius S^2

$$ds^2 = 4\frac{dx^2 + dy^2}{(1 + x^2 + y^2)^2}, \tag{4.6}$$

and

$$ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2, \tag{4.7}$$

are the same by giving the transformation between the two coordinate systems.

6. Consider the variation of the path length between A and B, i.e., $S[x] = \int_A^B d\tau$.
a) Show that the terms with a derivative on the metric in $\delta S[x] = 0$ gives $\Gamma^\mu_{\nu\rho}$.
b) Show that the equations obtained from the variation of S' where (here $\dot{x}^\mu = \frac{dx^\mu}{d\tau}$)

$$S'[x] = \int d\tau L' = \int_A^B d\tau(-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu), \tag{4.8}$$

(note that there is *no* square root) are the same as those coming from $S[x] = \int_A^B d\tau$.
c) What is the basic property that is possessed by S but not by S'?
d) The geodesic equations obtained in a) and b) arise as the so called Euler-Lagrange equations (EL eqs). The EL eqs are usually expressed in terms of a Lagrangian L as

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) - \frac{\partial L}{\partial x^\mu} = 0. \tag{4.9}$$

Construct a Lagrangian (like the one in b) above) by turning the metric $ds^2(x^i, dx^i)$ into a Langrangian $L(x^i, \dot{x}^i)$ by replacing dx^i by $\dot{x}^i$ and derive the affine connection for the metric on the 2-sphere in both coordinate systems obtained in Problem 4.3.2. above.

7. Derive the relation between the affine connections in two different coordinate systems. Is the affine connection a tensor?

8. Use the covariant derivative D_i in EM as a guide for constructing a covariant derivative ∇_μ in GR as follows. Consider a vector V_ν and design its derivative $\nabla_\mu V_\nu$ so that it transforms as a two-indexed tensor, i.e.,

$$\tilde{\nabla}_\mu \tilde{V}_\nu = \frac{\partial x^\rho}{\partial \tilde{x}^\mu} \frac{\partial x^\sigma}{\partial \tilde{x}^\nu} \nabla_\rho V_\sigma. \tag{4.10}$$

9. Write out explicitly the Laplacian acting on a scalar field, i.e.,

$$\Box \phi = \nabla_\mu \nabla^\mu \phi, \tag{4.11}$$

on a flat two-dimensional space in polar coordinates. This operator can also be written $\nabla^\mu \nabla_\mu \phi$ where you should note the change in the position of the upper and lower indices. Why are these two expressions for the $\Box$ operator equivalent?

10. The metric outside a straight, infinitely long cosmic string along the z-axis is

$$d\tau^2 = dt^2 - dr^2 - (1 - 8mG)r^2 d\alpha^2 - dz^2, \tag{4.12}$$

in cylindrical coordinates (t, r, α, z) $(0 \leq \alpha \leq 2\pi)$. Here m is the mass per unit length of the string. Show that the metric is flat and that a distant object, situated behind the string, gives rise to two images. Draw a picture to illustrate the lensing effect. **11.** Write down the equations of motion for a free particle on a flat two-dimensional surface expressed in polar coordinates.

12. Find all geodesics on a 2-sphere of radius a embedded in euclidean $\mathbf{R}^3$.

13. Find all time-like geodesics of the two-dimensional metric

$$d\tau^2 = \frac{1}{t^2} dt^2 - \frac{1}{t^2} dx^2. \tag{4.13}$$

14. Find all time-like and light-like geodesics for the two-dimensional metric

$$d\tau^2 = t^4 dt^2 - t^2 dx^2. \tag{4.14}$$

15. Use the usual coordinates (θ, ϕ) on the two-sphere and perform a parallel transport of a contravariant vector A^μ around a latitude circle ($\theta = \theta_0$, a constant). Start from $(A^\theta, A^\phi) = (1, 0)$ at $\phi = 0$ and give the result as a function of ϕ. Is there any special values of θ_0? What happens to the square $A^2 := A^\mu A_\mu$ when transported around the circle?

4.4 Curvature and symmetries

1. Consider the two-dimensional sphere with radius a. Compute the affine connection, Riemann tensor, Ricci tensor and curvature scalar for this two-sphere in polar coordinates (θ, ϕ).

2. Consider the metric for the unit two-sphere in polar coordinates (θ, ϕ).
a) Find all Killing vectors.
b) Show that the Killing vector fields generate the $so(3)$ Lie algebra.

3. Consider the metrics

$$ds^2 = \frac{dr^2}{1-k\frac{r^2}{L^2}} + r^2 d\phi^2, \;\; k = 1, 0, -1. \tag{4.15}$$

a) Compute the Riemann tensor, the Ricci tensor and the curvature scalar. b) Do the curvature scalars, R, come out as expected (their dependence on L and their sign)? c) What is the geometry of the manifold in each case? Note that $r \leq L$ in the case $k = +1$. Why is this condition necessary?

4. Consider the metrics for $k = 1, 0, -1$ in the previous problem again. Note that $0 \leq \phi \leq 2\pi$.
a) Compute the length of origin-centered circles as a function of r for the three cases in the previous problem.
b) Then compute the path lengths $s(r)$ for fixed ϕ between the origin and the point with coordinates (r, ϕ).
c) Find the circumferences $\mathcal{O}(s)$ of the circles, that is, as functions of the proper radius s.
d) Are the final results sensible?

5. Consider a space-time whose Riemann tensor is

$$R_{\mu\nu\rho\sigma} = f(x)(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}). \tag{4.16}$$

a) Show that this tensor has the correct symmetry properties to be a Riemann tensor.
b) Show that the function has to be constant in dimension $D \geq 3$.
c) Find the relation between the cosmological constant Λ and f by solving Einstein's equations in an empty spacetime.

6. Consider the metric defined by

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 - 4\cosh(\tfrac{x}{2})[\cosh(\tfrac{x}{2})(dt + dx) - \sinh(\tfrac{x}{2})dy]dx. \tag{4.17}$$

a) Write out the metric in matrix form.
b) Does this metric describe a maximally symmetric spacetime? Find the answer by computing the Riemann tensor.
c) Find a coordinate transformation that makes the previous result obvious.