3.3 Lecture 3: The equivalence principle (EP)

When Einstein developed his theory of gravity in the second decade of the 20th century he could not have followed the logic applied in the first lecture of this course. Instead his main goal was (most likely) to generalise special relativity away from inertial systems to any kind of motion an observer may have. This meant to abandon the restriction to Lorentz transformations and develop a theory where any kind of coordinate transformation can occur and have a physical interpretation, or in other words, to implement the equivalence principle (EP). After stating this principle in words we will try to find out how to describe it mathematically. This will result in the principle of general coordinate invariance.

The equivalence principle:
In any physical situation with gravitational fields present one may choose a "locally inertial", or "freely falling", frame (coordinate system) such that there are no gravitational forces sufficiently close to a given point. In that "local inertial" system the laws of physics are those of special relativity.

Comment: There are at least three possible versions of the EP:
Weak EP (WEP): applies only to falling massive bodies.
Medium strong EP (or EEP[4]): applies to all of physics except gravitational phenomena.
Strong EP (SEP): applies to all of physics including gravity.

The version that is implemented in Einstein's theory of gravity is the strong EP.
The observational status of the different EPs is very different. While the WEP is extremely well established the other versions are much less so, in particular the SEP[5].

Example 1: Light bending. Consider an elevator at rest in a gravitational field $\mathbf{g}$ pointing downwards towards the earth. What happens if a light ray is emitted perpendicularly from one wall of the elevator towards the other side? Does it hit that wall at the same height over the floor or not?

To answer this question we use the equivalence principle (EP) to replace the gravitational field of the earth by an acceleration upwards, i.e., by $-\mathbf{g}$. There are now two different systems that can be considered in which the answers must be the same! These are 1) the external non-accelerated (inertial) system and 2) the system of the elevator.

From system 1) it is clear that since the elevator is moving a distance $L = \frac{1}{2}gT^2$ upwards during the time $T = \frac{L_e}{c}$ it takes the light to travel the distance L_e across the elevator it must hit the other wall a distance $\Delta h = L = \frac{1}{2}\frac{g}{c^2}(L_e)^2$ lower than the point from which it was emitted. From the point of view of system 2) where the person only feels the gravitational field and does not know about any accelaration the conclusion must be that the light ray has been bent by the gravity field an amount Δh.

[4]EEP stands for the Einstein EP.
[5]See the Wikipedia overview "Equivalence principle" (not included in the course).

Exercise: If the elevator is instead falling freely in the field of the earth, how is the above argument altered?

If a light ray is moving L=1.0 km horizontally across the surface of the earth how must has it "fallen"? The answer is

$$\Delta h = \tfrac{1}{2}\frac{g}{c^2}(L)^2 = \frac{1}{2} \times 9,81 \times (3.0 \times 10^8)^{-2} \times (1.0 \times 10^3)^2 \mathrm{m} = 5 \times 10^{-11}\mathrm{m}. \tag{3.99}$$

The bending by a star or galaxy is much bigger but it is a bit more tricky to derive a formula that can be checked against observations. We will do this later!

Example 2: Red/blue-shift. The equivalence principle also implies that if a light pulse is sent upwards in the gravitational field of the Earth (or any other massive body) if will be red-shifted, i.e., loose energy. This is similar to the loss of kinetic energy of a massive body thrown upwards in the gravitational field. The way to argue here is to consider an elevator in free fall. Then the inside observer concludes that there is no gravitational effects at all and the frequencies of the light emitted ν_{em} (from the floor) and the light observed ν_{obs} (at the ceiling) are the same. However, an observer fixed in the outside system feeling the gravitational force will instead say that there are two competing effects which must cancel out: the relativistic Doppler effect and the gravitational effect. The relativistic Doppler effect is here for emitter and receiver moving towards each other with relative velocity $v_{rel} = -v$ where the velocity $v = gT$ is the extra velocity the elevator has picked up during the time T it took the light to travel from the floor to the ceiling. Thus

$$\frac{\nu_{obs}}{\nu_{em}} = \frac{\sqrt{1-\frac{v^2}{c^2}}}{1+v_{rel}/c} = \frac{\sqrt{1-\frac{v^2}{c^2}}}{1-v/c} = \sqrt{\frac{1+v/c}{1-v/c}} \approx 1 + v/c, \tag{3.100}$$

that is, a blue-shift (the observed energy $E = h\nu_{obs}$ is more energetic than the emitted light. From this one concludes that the gravitational effect is the opposite, i.e., a red-shift given by $\frac{\nu_{obs}}{\nu_{em}} \approx 1 - v/c$, i.e., light is loosing energy traveling upwards in a gravitational field!

To see in detail how the gravitational field can be eliminated by a change of frame we perform a coordinate transformation from the system of observer $\mathcal{O}$, i.e., from coordinates x^μ to a system $\mathcal{O}'$, with coordinates x'^μ which is in an accelerated motion relative $\mathcal{O}$ with a constant and homogeneous $\mathbf{a}$.

$$x'^\mu = (t', \mathbf{r}') : \;\; t' = t, \;\; \mathbf{r}' = \mathbf{r} - \frac{1}{2}\mathbf{a}t^2 \;\Rightarrow\; \ddot{\mathbf{r}} = \ddot{\mathbf{r}}' + \mathbf{a}. \tag{3.101}$$

This result means that one can eliminate any constant and homogeneous gravitational field completely from all dynamical equations. An example is the equation of motion for particle n in a system with N particles experiencing an external gravitational field $\mathbf{g}$: setting $\mathbf{a} = \mathbf{g}$ gives

$$m_n\ddot{\mathbf{r}} = m_n\mathbf{g} + \Sigma_{i\neq n}\mathbf{F}(\mathbf{r}_n - \mathbf{r}_i) \;\Rightarrow\; m_n\ddot{\mathbf{r}}'_n = \Sigma_{i\neq n}\mathbf{F}(\mathbf{r}'_n - \mathbf{r}'_i) \tag{3.102}$$

where the force $\mathbf{F}(\mathbf{r}_n - \mathbf{r}_i)$ from the other particles can be gravitational or of any other kind. Clearly in the primed system observer $\mathcal{O}'$ will experience no external gravitational force at all. If $\mathbf{g}$ is not homogeneous, like the field from the earth, the primed system is still given in terms of a homogeneous acceleration $\mathbf{a}$ (e.g., of a freely falling elevator) and thus the cancelation is only perfect at one single point, the center of mass of the elevator. At any other point close to it there are still small external gravitational forces present in the system called tidal forces.

From EP to General Covariance. The idea here is to use the equivalence principle to derive the effects of gravity by transforming various equations invariant under Lorentz transformations and expressed in terms of coordinates ξ^α valid in a freely falling frame to an arbitrary frame associated to some new coordinates x^μ. Here we consider three Lorentz covariant/invariant equations:

$$\text{Proper time}: \; d\tau^2 = -\eta_{\alpha\beta} d\xi^\alpha d\xi^\beta, \tag{3.103}$$

$$\text{Motion of a free particle}: \; \frac{d^2\xi^\alpha}{d\tau^2} = 0, \tag{3.104}$$

$$\text{Maxwell's equations}: \; \partial_\alpha F^{\alpha\beta} = -j^\beta, \; \partial_{[\alpha} F_{\beta\gamma]} = 0. \tag{3.105}$$

In transforming these equations to a general frame related to x^μ we use $\mu, \nu, \rho, ...$ to indicate objects transforming under **general coordinate transformations (diffs**[6]**)** as the **prototypes** dx^μ and ∂_μ. Thus we may define a **tensor under diffs** as an object whose upper and lower indices behave under general coordinate transformations as the prototypes:

$$x^\mu \;\to\; \tilde{x}^\mu = \tilde{x}^\mu(x) \;\Rightarrow\; d\tilde{x}^\mu = \frac{\partial \tilde{x}^\mu}{\partial x^\nu} dx^\nu, \; \tilde{\partial}_\mu = \frac{\partial x^\nu}{\partial \tilde{x}^\mu}\partial_\nu, \tag{3.106}$$

where we assume that the matrices formed from all partial derivatives $\frac{\partial \xi^\alpha}{\partial x^\mu}$ (or $\frac{\partial \tilde{x}^\mu}{\partial x^\nu}$) and its matrix inverse are well-behaved, in particular that their determinants are non-zero. An important consequence of the above is that the "exterior derivative" $d := d\tilde{x}^\mu \tilde{\partial}_\mu = dx^\mu \partial_\mu$, i.e., it is invariant under general coordinate transformations. This is reflected in the following formulae, which will be used frequently,

$$\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial x^\nu}{\partial \xi^\alpha} = \delta^\nu_\mu, \; \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial x^\mu}{\partial \xi^\beta} = \delta^\alpha_\beta. \tag{3.107}$$

First we basically just repeat what was done in lecture 2 namely change coordinates in the proper time:

$$d\tau^2 = -\eta_{\alpha\beta} d\xi^\alpha d\xi^\beta = -\left(\eta_{\alpha\beta}\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\nu}\right) dx^\mu dx^\nu := -g_{\mu\nu} dx^\mu dx^\nu, \tag{3.108}$$

which identifies the metric $g_{\mu\nu}$ as the expression in the bracket.
Comment: When Dirac fields (spinors) are used to describe how, e.g., electrons behave in a gravitational field one must define a so called **vierbein** ${e_\mu}^\alpha(x)$ and write $g_{\mu\nu} :=$

[6] "Diffs" is here short for "diffeomorphisms".

${e_\mu}^\alpha(x){e_\nu}^\beta(x)\eta_{\alpha\beta}$. (See Cartan's formulation.) This basically identifies ${e_\mu}^\alpha(x)$ with $\frac{\partial\xi^\alpha}{\partial x^\mu}(x)$.

A much more interesting computation is to perform this transformation on the equation of motion for a free particle moving on a trajectory given by $x^\mu(\tau)$. For this we need the following result, which is a direct consequence of the chain rule applied to $\xi^\alpha(x^\mu(\tau))$,

$$\frac{d\xi^\alpha}{d\tau} = \frac{dx^\mu}{d\tau}\frac{\partial\xi^\alpha}{\partial x^\mu}. \tag{3.109}$$

Then acting on this equation with a second τ-derivative we find that the free particle equation becomes

$$\frac{d^2\xi^\alpha}{d\tau^2} = \frac{d^2x^\mu}{d\tau^2}\frac{\partial\xi^\alpha}{\partial x^\mu} + \frac{dx^\mu}{d\tau}\frac{d}{d\tau}\frac{\partial\xi^\alpha}{\partial x^\mu} = \frac{d^2x^\mu}{d\tau^2}\frac{\partial\xi^\alpha}{\partial x^\mu} + \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\frac{\partial^2\xi^\alpha}{\partial x^\nu\partial x^\mu} = 0. \tag{3.110}$$

Thus if we can get rid of the factor multiplying $\frac{d^2x^\mu}{d\tau^2}$ we get an equation for this object which is just the acceleration in terms of the path parameter τ. This is easy to do by multiplying the above equation by $\frac{\partial x^\rho}{\partial\xi^\alpha}$ which, using the first eq. in (3.107), gives δ^ρ_μ. Then the equation for a particle in a general gravitational field reads

$$\frac{d^2x^\rho}{d\tau^2} + \left(\frac{\partial x^\rho}{\partial\xi^\alpha}\frac{\partial^2\xi^\alpha}{\partial x^\mu\partial x^\nu}\right)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0, \tag{3.111}$$

where the gravitational field is represented by the expression in the bracket of the second term and hence we find that it multiplies two velocity factors. Recall (see SW sections 2.4 and 2.7) the analogous force equation in EM $f^\alpha = m\frac{d^2x^\alpha}{d\tau^2} = e{F^\alpha}_\beta\frac{dx^\beta}{d\tau}$ in Minkowski space which only contains one velocity factor. It is standard at this point to define the object in the bracket as the **affine connection** $\Gamma^\mu_{\nu\rho}$, that is

$$\Gamma^\mu_{\nu\rho} := \frac{\partial x^\mu}{\partial\xi^\alpha}\frac{\partial^2\xi^\alpha}{\partial x^\nu\partial x^\rho}, \tag{3.112}$$

and thus the standard form of the **geodesic equation** reads

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0. \tag{3.113}$$

Note that the affine connection is symmetric in the two lower indices. The name "geodesic equation" comes from the fact that this equation can be obtained from a variational principle extremising the length of a path of a particle. This fact will be demonstrated below.

The comparison with EM is extremely interesting and provides a lot of information about the mathematics as well as the physics of the geodesic equation. While also the force equation in EM can be written in a general frame (as we will do later) it will keep its structure whatever coordinates we use. This is due to the fact that $F_{\alpha\beta}$ becomes $F_{\mu\nu}$ in a general coordinate system and is thus a tensor under general coordinate transformations. The affine connection on the other hand is **NOT** a tensor as we will show shortly. This is why it is called "connection". In fact, the affine connection is in many ways an object similar to the four-vector potential in EM which is also often called "connection" (standard in mathematics) but in this case under $U(1)$ gauge transformations.

Exercise: The EM vector potential A_μ is both a connection and a tensor. Explain this fact! Is the same true for the affine connection?

To understand how the affine connection, which represents the presence of a gravitational force in the geodesic equation, and this entire theory of gravity is related to Newton's theory of gravity we must first express $\Gamma^\mu_{\nu\rho}$ in terms of the metric without having to involve any free falling coordinates like ξ^α. This is done in two steps as follows.

For the first step we recall the definition of the metric

$$g_{\mu\nu} = \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\nu}\eta_{\alpha\beta}. \tag{3.114}$$

Hitting this equation with a derivative ∂_ρ gives

$$\partial_\rho g_{\mu\nu} = \frac{\partial^2 \xi^\alpha}{\partial x^\rho \partial x^\mu}\frac{\partial \xi^\beta}{\partial x^\nu}\eta_{\alpha\beta} + \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{\partial^2 \xi^\beta}{\partial x^\rho \partial x^\nu}\eta_{\alpha\beta}. \tag{3.115}$$

Using the fact that eq. (3.112) can be rewritten as $\frac{\partial \xi^\alpha}{\partial x^\sigma}\Gamma^\sigma_{\nu\rho} = \frac{\partial^2 \xi^\alpha}{\partial x^\nu \partial x^\rho}$ the second derivatives in the previous equation can be replaced by affine connections and the equation becomes

$$\partial_\rho g_{\mu\nu} = \Gamma^\sigma_{\rho\mu} g_{\sigma\nu} + \Gamma^\sigma_{\rho\nu} g_{\sigma\mu}. \tag{3.116}$$

Note that this equation does not involve ξ^α and that $\Gamma^\mu_{\nu\rho} = 0$ implies $\partial_\rho g_{\mu\nu} = 0$.

The second step is to solve the last equation and express the affine connection directly in terms of the metric and a derivative of it. To do this one needs a trick. Compute the following (funny) combination of terms (note the position of the indices)

$$\partial_\rho g_{\mu\nu} + \partial_\mu g_{\rho\nu} - \partial_\nu g_{\mu\rho} = 2\Gamma^\sigma_{\rho\mu} g_{\sigma\nu}, \tag{3.117}$$

where the RHS follows directly by inserting the final result of the first step above. The last equation is better written in the standard form

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\sigma}(\partial_\nu g_{\rho\sigma} + \partial_\rho g_{\nu\sigma} - \partial_\sigma g_{\nu\rho}). \tag{3.118}$$

Thus $\partial_\rho g_{\mu\nu} = 0$ implies $\Gamma^\mu_{\nu\rho} = 0$ which means that these two statements are actually equivalent. One therefore should ask if any similar statements can be made about the second derivatives of the metric in a freely falling frame. This question, which is of fundamental importance for the geometric interpretation of GR, will be discussed and answered shortly.

Exercise: Perform the last step above leading to eq. (3.118).

Comment: The concept of **manifold**, used here rather carelessly to represent curved (or flat) surfaces, has, however, the for us very useful property that very close to any point it looks flat. This is quite similar to how we think of the EP when applied to the spacetime manifold. This similarity is perhaps the first argument in our development of GR that really suggests that gravity is related to curvature.

Question: One may still wonder: Is such an interpretation in terms geometry necessary or only one of convenience? After all we could, although extremely inconvenient, expand the metric as we did before, i.e., as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, and express all equations involving gravity in terms of $h_{\mu\nu}$. This would give a theory entirely living in Minkowski space.

3.3.1 The geodesic equation

Geodesics are defined via a variational principle as the extremal of the proper time (or length of the world line) here viewed as an action functional

$$S[x] = \int_A^B d\tau = \int_A^B \sqrt{-g_{\mu\nu}dx^\mu dx^\nu} = \int_A^B d\sigma \sqrt{-g_{\mu\nu}(x)\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}} := \int_A^B d\sigma\, L(x(\sigma)), \tag{3.119}$$

where we used $dx^\mu = \frac{dx^\mu}{d\sigma}d\sigma$ in the third equality. The variation of the action is defined by

$$\delta S[x] = (S[x+\delta x] - S[x])|_{(\text{linear in } \delta x)} = 0, \tag{3.120}$$

under variations δx^μ which are arbitrary except that they vanish at the end-points. The integral is between two spacetime points A and B with the path parametrised either by τ or σ (where $\tau = \tau(\sigma)$ is a monotonic function, i.e., $\frac{\partial \tau}{\partial \sigma} > 0$). Setting the variation equal to zero will imply the geodesic equation as we now show.

The variation, in terms of the Lagrangian L (using $\delta L = \frac{\partial L}{\partial x^\mu}\delta x^\mu$), is then

$$\delta L = -\frac{1}{2}\frac{1}{\sqrt{-g_{\rho\sigma}(x)\frac{dx^\rho}{d\sigma}\frac{dx^\sigma}{d\sigma}}}\delta(g_{\mu\nu}(x)\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}). \tag{3.121}$$

At this point we have to take some care since the variation δx^μ appears under a derivative in σ and such an expression must be integrated by parts so that the variation can be pulled out of the whole expression for δL. In doing this we will find that the inverse square root is useful since

$$d\sigma\frac{1}{\sqrt{-g_{\mu\nu}(x)\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}}} = d\sigma\frac{1}{\frac{d\tau}{d\sigma}} = d\tau(\frac{d\sigma}{d\tau})^2, \tag{3.122}$$

and can thus be used to convert the two σ derivatives to τ derivatives: $\frac{d}{d\tau} = (\frac{d\sigma}{d\tau})\frac{d}{d\sigma}$. Thus

$$\delta L = -\frac{1}{2}\delta(g_{\mu\nu}(x)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}) = -\frac{1}{2}(\delta g_{\mu\nu}(x)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + 2g_{\mu\nu}(x)\frac{dx^\mu}{d\tau}\frac{d\delta x^\nu}{d\tau}), \tag{3.123}$$

which is now easy to integrate by parts[7] since the square root factor has disappeared. Thus

$$\delta L = -\frac{1}{2}(\delta x^\rho\frac{\partial g_{\mu\nu}(x)}{\partial x^\rho}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} - 2\frac{dg_{\mu\nu}(x)}{d\tau}\frac{dx^\mu}{d\tau}\delta x^\nu - 2g_{\mu\nu}(x)\frac{d^2x^\mu}{d\tau^2}\delta x^\nu), \tag{3.124}$$

Rewriting the second term using $\frac{dg_{\mu\nu}}{d\tau} = \frac{\partial g_{\mu\nu}}{\partial x^\rho}\frac{dx^\rho}{d\tau}$ we get exactly

$$\delta L = (\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau})g_{\mu\sigma}\delta x^\sigma. \tag{3.125}$$

Thus the variational principle $\delta S[x] = 0$ implies, since δx^σ is an arbitrary variation, that the expression in the bracket vanishes, i.e., the geodesic equation.

[7] Note that the integration by parts should be done in the integral. However, this is understood here although we express it in terms of the Lagrangian. This is very often done in the literature.

3.3.2 Newtonian limit

To make contact with Newtonian gravity one may consider the following limits of the geodesic equation:
1. Slowly moving particles, i.e., with $v \ll c$. Since we are here only interested in the acceleration we can set $v = 0$.
2. Stationary gravitational fields, i.e., $g_{\mu\nu}$ is time independent.
3. Weak field limit, i.e., we set $g_{\mu} = \eta_{\mu\nu} + h_{\mu\nu}$ where $h_{\mu\nu}$ is a small perturbation of the Minkowski space metric, and we then expand the equations to first order in $h_{\mu\nu}$. Note that $h_{\mu\nu}$ is now a field in Minkowski space so its indices are raised and lowered by $\eta_{\mu\nu}$.

Implementing these limits the geodesic equation

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0 \tag{3.126}$$

simplifies as follows

$$1. \Rightarrow \ \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\frac{dx^0}{d\tau}\frac{dx^0}{d\tau} = 0, \tag{3.127}$$

$$2. \Rightarrow \ \Gamma^\mu_{00} = \frac{1}{2}g^{\mu\sigma}(2\partial_0 g_{0\sigma} - \partial_\sigma g_{00}) = -\frac{1}{2}g^{\mu i}\partial_i g_{00}, \tag{3.128}$$

where the index i is the space part of σ. To first order in $h_{\mu\nu}$ this becomes

$$3. \Rightarrow \ \Gamma^\mu_{00} = -\frac{1}{2}\eta^{\mu i}\partial_i h_{00} \ \Rightarrow \ \frac{d^2x^\mu}{d\tau^2} = \frac{1}{2}\eta^{\mu i}\partial_i h_{00}. \tag{3.129}$$

Thus splitting the μ index of the geodesic equation into time and space we get (with $x^0 := t$)

$$\frac{d^2t}{d\tau^2} = 0, \ \ \frac{d^2x^i}{d\tau^2} = \frac{1}{2}\eta^{ij}\partial_j h_{00}(\frac{dt}{d\tau})^2, \text{ or, } \frac{d^2\mathbf{r}}{d\tau^2} = \frac{1}{2}\nabla h_{00}(\frac{dt}{d\tau})^2. \tag{3.130}$$

The first of these equations implies that $\frac{dt}{d\tau}$=constant so we use $(\frac{dt}{d\tau})^2$ to convert the derivatives in the second equation to t-derivatives. It thus reads

$$\ddot{\mathbf{r}} = \frac{1}{2}\nabla h_{00}. \tag{3.131}$$

This is just Newton's second law $\ddot{\mathbf{r}} = -\nabla\phi(\mathbf{r})$ in terms of the gravitational potential which implies the identification

$$h_{00} = -2\phi \ \Rightarrow \ g_{00} = -(1 + 2\phi). \tag{3.132}$$

This fact will be used below for the potential from the earth, i.e., $\phi(r) = -\frac{GM}{r}$. Note that $g_{00} = 0$ for $r_0 = 2GM$ which may appear very strange at this point but will be fully analysed in the context of black holes later in the course.

3.3.3 Time dilatations

The key point in this section is the utilisation of a single coordinate system to argue that the proper time, or proper length, is different at different spacetime points due to the x^μ dependence of the metric $g_{\mu\nu}(x)$ and the state of motion at each point, i.e., the path $x^\mu(\tau)$ through[8] $v^\mu = \frac{dx^\mu}{dt}$. The final formula, when comparing $d\tau$ at two points A and B and eliminating dt, reads

$$\frac{d\tau_A}{d\tau_B} = \frac{\sqrt{-g_{\mu\nu}(x_A)v_A^\mu v_A^\nu}}{\sqrt{-g_{\mu\nu}(x_B)v_B^\mu v_B^\nu}} = \frac{\nu_B}{\nu_A}, \tag{3.133}$$

where the last equality gives the relation between the frequencies observed or emitted at the two points for a light signal sent between them.

A good example illustrating the use of this formula is to compare two observers at rest in a fixed gravitational field from a point source. Then $\sqrt{-g_{\mu\nu}(x)v^\mu v^\nu} = \sqrt{-g_{00}(r)}$ where $g_{00} = -(1+2\phi(r)) = -(1-\frac{2GM}{r})$ and the formula above becomes

$$\frac{d\tau_A}{d\tau_B} = \frac{\sqrt{1-\frac{2GM}{r_A}}}{\sqrt{1-\frac{2GM}{r_B}}} = \frac{\nu_B}{\nu_A}. \tag{3.134}$$

If $r_A > r_B$ then $1-\frac{2GM}{r_A} > 1-\frac{2GM}{r_B}$ which implies $d\tau_A > d\tau_B$ and $\nu_A < \nu_B$. The physical interpretation of these two inequalities is that time flows slower and the energy, $E = h\nu$, is larger for the point closest to the source. E.g., for a light signal sent upwards in a gravitational field this means that it looses energy, i.e., is red-shifted. An extreme situation seems to arise at the radius $r_0 = 2GM$ where time stops. For the earth the value of this radius is a about two centimeters and thus far inside the earth. This particular implication for how time is affected therefore becomes irrelevant. Objects having the radius $r_0 = 2GM$ outside its "surface" are called black holes which will be discussed later in the course.

Comment: Note the change in attitude when going from special to general relativity towards the role played by dt and $d\tau$.

[8] Compare to $u^\mu = \frac{dx^\mu}{d\tau}$.