16. The concept of world-lines can rather directly be generalised to world-surfaces. Consider the Lagrangian $S[X] = \int d\tau$ whose Euler-Lagrange equations are just the geodesic equation showing that geodesics extremise the distance between two spacetime points.
a) Construct the action that generalises the Lagrangian for geodesics to two-dimensional world-surfaces.
b) Compute the Euler-Lagrange equations from the world-surface Lagrangian in a).

4.4 Curvature and symmetries

1. Consider the two-dimensional sphere with radius a. Compute the affine connection, Riemann tensor, Ricci tensor and curvature scalar for this two-sphere in polar coordinates (θ, ϕ).

2. Consider the metric for the unit two-sphere in polar coordinates (θ, ϕ).
a) Find all Killing vectors.
b) Show that the Killing vector fields generate the $so(3)$ Lie algebra.

3. Consider the metrics

$$ds^2 = \frac{dr^2}{1-k\frac{r^2}{L^2}} + r^2 d\phi^2, \quad k = 1, 0, -1. \tag{4.15}$$

a) Compute the Riemann tensor, the Ricci tensor and the curvature scalar.
b) Do the curvature scalars, R, come out as expected (dependence on L and signature)?
c) What is the geometry of the manifold in each case? Note that $r \leq L$ in the case $k = +1$. Why is this condition necessary?

4. Consider the metrics for $k = 1, 0, -1$ in the previous problem again ($0 \leq \phi \leq 2\pi$).
a) Compute the length of origin-centered circles as a function of r for the three cases in the previous problem.
b) Then compute the path lengths $s(r)$ for fixed ϕ between the origin and the point with coordinates (r, ϕ).
c) Find the circumferences $\mathcal{O}(s)$ of the circles, that is, as functions of the proper radius s.
d) Are the final results sensible?

5. Consider a space-time whose Riemann tensor is

$$R_{\mu\nu\rho\sigma} = f(x)(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}). \tag{4.16}$$

a) Show that this tensor has the correct symmetry properties to be a Riemann tensor.
b) Show that the function has to be constant in dimension $D \geq 3$.
c) Find the relation between the cosmological constant Λ and f by solving Einstein's equations in an empty spacetime.

6. Consider the metric defined by

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 - 4\cosh(\tfrac{x}{2})[\cosh(\tfrac{x}{2})(dt + dx) - \sinh(\tfrac{x}{2})dy]dx. \qquad (4.17)$$

a) Write out the metric in matrix form.
b) Does this metric describe a maximally symmetric spacetime? Find the answer by computing the Riemann tensor.
c) Find a coordinate transformation that makes the previous result obvious.

7. Three-dimensional anti-de Sitter space (AdS_3) is given by the metric

$$ds^2 = -du^2 - dv^2 + dx^2 + dy^2, \qquad (4.18)$$

where the coordinates are constrained by

$$-u^2 - v^2 + x^2 + y^2 = -L^2. \qquad (4.19)$$

This constraint can be solved by

$$\begin{aligned} u &= \sqrt{L^2 + r^2}\cos\tfrac{t}{L}, \\ v &= \sqrt{L^2 + r^2}\sin\tfrac{t}{L}, \\ x &= r\cos\phi, \\ y &= r\sin\phi. \end{aligned} \qquad (4.20)$$

a) Find the metric in the coordinates (t, r, ϕ).
b) Compute the proper distance from the origin $r = 0$ to spacial infinity $r \to \infty$.
c) Find the time it takes for a photon to travel the distance in b).
d) Repeat the calculations in b) and c) for the Schwarzschild metric but now between $r = \infty$ and the event horizon.
e) What is the (maximal) symmetry of AdS_3?

8. Consider the action for a coupled Einstein-Maxwell system (in the conventions used in lectures)

$$S[g_{\mu\nu}, A_\mu] = \tfrac{1}{16\pi G}\int d^4x\sqrt{g}(R - 2\Lambda) - \tfrac{1}{4}\int d^4x\sqrt{g}F_{\mu\nu}F^{\mu\nu}. \qquad (4.21)$$

Derive the Einstein equations including the expression for the stress tensor on the RHS by varying the action with respect to the metric. Give the details of the calculations including the variations of $g^{\mu\nu}$, g and $R_{\mu\nu}$.

9. Compute the curvature scalar for the following two metrics:
a) $d\tau^2 = e^{\frac{z}{a}}(dt^2 - dx^2 - dy^2 - dz^2)$,
b) $d\tau^2 = \frac{a^2}{z^2}(dt^2 - dx^2 - dy^2 - dz^2)$.

10. Consider a (non-flat) torus (i.e., the surface of a donut) with angular coordinates ϕ_1, ϕ_2. You may think of the torus as a cylinder that has been bent and whose end-circles have been glued together. Let ϕ_1 be the angle going around the hole of radius r_1 (to the center of the bent cylinder) and ϕ_2 the one going around the cylinder with radius r_2. These coordinates describe the embedding of the torus in euclidean $\mathbf{R}^3$ given by

$$x = (r_1 + r_2 \cos\phi_2)\cos\phi_1, \tag{4.22}$$
$$y = (r_1 + r_2 \cos\phi_2)\sin\phi_1, \tag{4.23}$$
$$z = r_2 \sin\phi_2. \tag{4.24}$$

a) Find the metric of the torus using the embedding above.
b) Compute the Ricci scalar.
c) Find all Killing vectors for this metric.

11. A Killing vector is a vector K_μ that satisfies the equation

$$\nabla_{(\mu}K_{\nu)} = 0, \tag{4.25}$$

where ∇_μ is the covariant derivative. Suppose space-time has an arbitrary metric $g_{\mu\nu}$ admitting N Killing vectors $K^{(I)}_\mu$, $I = 1, 2, .., N$. Let $x^\mu(\tau)$ be a geodesic of a freely falling particle in this space-time parametrised by τ. The 4-velocity is $u^\mu = \frac{dx^\mu(\tau)}{d\tau}$. Show that the N vectors

$$V^{(I)} = K^{(I)}_\mu u^\mu, \tag{4.26}$$

are constants of the motion for the freely falling particle.

12. A beacon radiating at a fixed frequency ν_0 is released at time $t = 0$ towards a black hole of mass M by an observer situated very far away from the black hole. The observer stays at constant distance while the probe is falling. Show that the frequency of the beacon (when it is close to the event horizon) as measured by the observer can be written as $\nu \propto e^{-t/K}$ for some constant K and relate K to the mass of the black hole.

13. Consider a spherically symmetric and static solution of Einstein's equations in three space-time dimensions, describing a point particle with mass M. Give the explicit metric for the solution. Is this space-time flat? Will there be gravitational lensing in this space-time?

14. The Weyl tensor $C_{\mu\nu\rho\sigma}$ is defined by

$$R_{\mu\nu}{}^{\rho\sigma} = \tfrac{4}{D-2}\delta^{[\rho}_{[\mu}S_{\nu]}{}^{\sigma]} + C_{\mu\nu}{}^{\rho\sigma}, \tag{4.27}$$

where $S_{\mu\nu}$ is the symmetric Schouten tensor

$$S_{\mu\nu} = R_{\mu\nu} - \tfrac{1}{2(D-1)}g_{\mu\nu}R. \tag{4.28}$$

a) If two metrics are conformally related via

$$\tilde{g}_{\mu\nu} = e^{2\phi(x)} g_{\mu\nu}, \tag{4.29}$$

what is the relation between the corresponding Weyl tensors?
b) Calculate the Weyl tensor for a gravitational plane wave.

15. Consider the **Poincaré half-plane** which is defined on the upper half-plane $(x, y > 0)$ by the metric:

$$ds^2 = \frac{1}{y^2}(dx^2 + dy^2). \tag{4.30}$$

a) Compute the Riemann tensor, Ricci tensor and curvature scalar. Is it an Einstein space?
b) Compute the distance between the two points (x_0, y_1) and (x_0, y_2) along the line x_0 =constant. Is the x axis part of the manifold? Is r=infinity part of the manifold?
c) Show that geodesics are given by $(x - x_0)^2 + y^2 = r_0^2$ for some constants x_0 and r_0 and draw them on the upper half-plane.

16. Consider the two-dimensional integral $\int \sqrt{g} R \, d^2x$.
a) Show that $\sqrt{g}R$ is a total derivative.
b) The result in a) means that the integral $\int \sqrt{g} R \, d^2x$ is a topological invariant. This is related to the **Gauss-Bonnet theorem** which is expressed in terms of the Euler characteristic $\chi(\mathcal{M}) = \frac{1}{4\pi} \int \sqrt{g} R \, d^2x$. For manifolds without boundaries the theorem reads

$$\chi(\mathcal{M}) = 2(1 - g), \text{ where g is the genus.} \tag{4.31}$$

Compute $\chi(S^2)$ for the two-sphere of radius a using the result in a).
c) This topological invariant can also be computed by "triangulation", in terms of the number of corners (b_0), edges (b_1) and sides (b_2), by

$$\chi(\mathcal{M}) = b_0 - b_1 + b_2. \tag{4.32}$$

Check this for the sphere, e.g., a cube, and the torus.

4.5 Schwarzschild and wormhole geometries

1. Find in a systematic way all the Killing vectors of the Schwarzschild metric

$$ds^2 = -(1 - \tfrac{2MG}{r})dt^2 + (1 - \tfrac{2MG}{r})^{-1} dr^2 + r^2 d\Omega^2. \tag{4.33}$$

2. Consider the space-time metric

$$ds^2 = -dt^2 + dr^2 + (r^2 + a^2) d\Omega^2, \tag{4.34}$$

where $d\Omega^2$ is the metric on the round two-sphere of unit radius. Metrics of this kind have been proposed to describe "wormholes", tunnels between different regions of space-time. The "radial coordinate" r takes values from minus to plus ∞.

a) Write out the explicit form of the geodesic equation.
b) Check if there are time-like geodesics traversing the wormhole (i.e., that go from large negative to large positive values of r or vice versa).
c) Compute the Riemann tensor.
d) Compute the Einstein tensor and check if there are any problems to find a stress tensor satisfying the Einstein equations.
e) Find all isometries of this wormhole.

3. Find the equations of motion for a massive particle in the Schwarzschild geometry.

4. Calculate the deflection of a light ray gracing the sun's surface.

5. Using the Schwarzschild metric, find the proper length of the curves
a) $r = const$, $\theta = const$, $0 \leq \phi \leq 2\pi$,
b) $\theta = const$, $\phi = const$, $r_1 \leq r \leq r_2$.

6. Consider a massive particle moving in the geometry of the Schwarzschild metric.
a) Derive an equation for the radial coordinate $r(\tau)$ that resembles Newton's second law $m\frac{d^2r}{d\tau^2} = F(r)$ where $F(r) = -\frac{dV(r)}{dr}$ is the force due to the potential $V(r)$. Determine the so obtained effective potential $V_{eff}(r)$.
b) Are there any values of $J = r^2\dot{\phi}$ for which it is possible for the particle to be in a stable circular orbit? If so, what is radius of the orbit?
c) Specialise to the case of a photon in a circular orbit. Are there any stable or unstable such orbits? If so, what is the radius?

7. The death-defying spaceman Spiff lands on the hypothetical neutron star Buster. His spacecraft measures 20 meters in height. Buster's mass is $M = 2 \cdot 10^{30}$ kg and its radius is $R = 5$ km. How far can Spiff walk away from the ship and still see it? According to Earthly standards, Spiff is a short man, being only 1 meter tall. Compare with the classical result (i.e., without gravity). Any approximations must be motivated.

8. A galaxy acts as a gravitational lens for a very distant quasar. The galaxy is 100 Mpcs away, and the distance to the quasar can be assumed to be much greater. The image is a ring with radius 1.0 arcsec. What is the mass of the galaxy? Answer in solar masses.

4.6 Gravitational waves

1. A binary system consists of two neutron stars orbiting their common center of mass in circular orbits. Both stars have masses equal to 1.4 solar mass, and the distance between their center of mass equals the diameter of our sun. Due to emission of gravitational waves the system will lose energy and the stars will slowly spiral towards each other while the orbital frequency increases. What is the power emitted in gravitational radiation from the two stars and how fast will the orbital period decrease in time? You need to justify any approximation used.