3.8 Lectures 13 - 15: Gravitational waves

There are three key aspects of gravitational waves that one needs to study in order to understand how they can observed on Earth:
1. How are they generated?
2. How do they propagate?
3. How are they detected on Earth?

Before analysing these aspects in detail let us describe them schematically.
1) Here we must solve Einstein's equation with a stress tensor that describes the source, e.g., a pair of black holes or neutron stars spiralling closer and closer towards each other. This is done using the linear approximation and retarded Green's function.
2) Once the gravitational field is obtained from the source it will propagate in empty space. We need to analyse the degrees of freedom of the wave and to compute the energy transported away by the wave. The latter requires the gravitational stress tensor restricted to second order terms in the weak gravitational field of the wave.
3) When the wave arrives at Earth it can be picked up due to its effect on physical distances that start to oscillate according to the metric wave when it passes by the "observatory".

There exist two kinds of observations supporting the existence of gravitational waves:
1) Indirect ones due to the loss of energy in a gravitationally bound system, e.g., of binary pulsars[15] spiralling towards each other (Nobel prize 1993 to Hulse and Taylor).
2) Direct ones by LIGO and others; first observation in 2015 known as *GW150914* (Nobel prize 2017 to Thorne, Weiss and Barish)[16].

An important property of the wave is that the gravitational field, the metric, is extremely weak which implies that the linear field approximation is very good. This will help us solve the equations but also make their detection on Earth extremely challenging as we will discuss later. Einstein himself was well aware of this kind of linear analysis but failed to find an exact wave solution to his equations. Towards the end of his life he actually believed that there did not exist any wave solutions to the full non-linear equations. However, as it turns out, he was wrong and wave solutions were found by other relativists soon after he died in 1955[17]. Strangely enough, the mathematician H.W. Brinkmann found wave solutions, $pp-waves$, already in 1924 but this was not known to the physicists.

Having established the existence of gravitational waves one wonders if they should be quantised much in the same way as done in electromagnetism where photons, the quantum of the wave, can be observed experimentally. The analogue in gravity, the graviton, is an entirely different question and may never be observed even if they in principle do exist. Quantum gravity is one of the deep issues in physics since it is known that applying QFT to Einstein's theory of gravity does not work. Thus one of the two theories must be modified!

[15] Rapidly rotating neutron stars emitting pulses of light.
[16] See K. Riles ArXiv hep-ex/1209.0667.
[17] See sect. 3 of the nice review by Hill and Nurowski, ArXiv physics.hist-ph/1608.08673.

3.8.1 The weak field approximation

Our first task is to expand Einstein's equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}, \tag{3.222}$$

to first order in the weak field perturbation around Minkowski space, that is we write the metric as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{3.223}$$

where $h_{\mu\nu}$ is small (recall that the metric is dimensionless). Then using the following Riemann tensor expression derived before

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}(\partial_\sigma\partial_\mu g_{\nu\rho} + \partial_\nu\partial_\rho g_{\mu\sigma} - \partial_\rho\partial_\mu g_{\nu\sigma} - \partial_\sigma\partial_\nu g_{\mu\rho}) + g_{\tau\epsilon}(\Gamma^\tau_{\mu\sigma}\Gamma^\epsilon_{\nu\rho} - \Gamma^\tau_{\nu\sigma}\Gamma^\epsilon_{\mu\rho}), \tag{3.224}$$

where the $\Gamma\Gamma$ terms start at second order in $h_{\mu\nu}$, it is clear that to first order the Ricci is

$$R^{(1)}_{\mu\rho} = \eta^{\nu\sigma}R^{(1)}_{\mu\nu\rho\sigma} = -\frac{1}{2}(\Box h_{\mu\rho} - \partial_\mu\partial^\sigma h_{\sigma\rho} - \partial_\rho\partial^\sigma h_{\sigma\mu} + \partial_\mu\partial_\rho h), \tag{3.225}$$

where one should note that from now on indices are raised and lowered with the Minkowski metric $\eta_{\mu\nu}$ (and its inverse) and the trace is $h = \eta^{\mu\nu}h_{\mu\nu}$. This result is very promising in view of what was discussed in lecture 1 and the $\Box$ equation for gravity we found there.

There is, however, a problem since the expression for $R^{(1)}_{\mu\rho}$ above contains additional two-derivative terms. These terms imply that the Einstein equation cannot be solved since the operator acting on $h_{\mu\nu}$ is not invertible, i.e., this operator has no Green's function. The way to see this is to show that the operator has an eigenvector with zero eigenvalue. Its existence is a consequence of gauge invariance as we now show which, in fact, will also complete the connection to the story developped in lecture 1.

To this end, let us consider the inverse metric and its transformation under general coordinate transformations

$$g'^{\mu\nu}(x') = \frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x'^\nu}{\partial x^\sigma}g^{\rho\sigma}(x). \tag{3.226}$$

Then if we consider only infinitesimal coordinate transformations we get, with $g^\mu \approx \eta^{\mu\nu} - h^{\mu\nu}$,

$$x^\mu \to x'^\mu = f^\mu(x) \approx x^\mu + \epsilon^\mu(x) \Rightarrow h'^{\mu\nu}(x') \approx h^{\mu\nu} - (\partial^\mu\epsilon^\nu + \partial^\nu\epsilon^\mu), \tag{3.227}$$

where we note the x' as the argument on the LHS. To lowest order in both $h_{\mu\nu}$ and the parameter ϵ^μ we can drop the prime on x'. Usually the result is rewritten by lowering the indices with the Minkowski metric which gives the infinitesimal transformation rule[18]

$$\delta h_{\mu\nu}(x) \approx -(\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu). \tag{3.228}$$

[18]This looks like a gauge transformation if compared to EM but, as seen here, it is just a result of the tensor property of the metric.

With this transformation rule we see immediately that

$$\delta R^{(1)}_{\mu\nu} = 0, \tag{3.229}$$

and hence that $\delta h_{\mu\nu}(x) \approx -(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu)$ has zero eigenvalue. The solution to this problem then suggests itself, namely we should try to find a gauge condition that eliminates the extra terms and leaves only the $\Box$ term in $\delta R^{(1)}_{\mu\nu}$. The situation has then become identical to the one in Maxwell theory where the Lorentz gauge condition solves the problem. In GR the analogues gauge condition is called the *harmonic gauge* condition:

$$\Gamma^\mu := g^{\nu\rho}\Gamma^\mu_{\nu\rho} = 0. \tag{3.230}$$

To first order in $h_{\mu\nu}$ is reads

$$\Gamma^\mu \approx \frac{1}{2}(2\partial_\nu h^{\mu\nu} - \partial^\mu h^\nu{}_\nu) = 0, \quad \text{that is} \;\; \partial_\nu h^{\mu\nu} = \frac{1}{2}\partial^\mu h^\nu{}_\nu, \;\; \text{at} \;\; \mathcal{O}(h). \tag{3.231}$$

At this point it is convenient to rewrite Einstein's equations so that the Ricci tensor appears by itself on the LHS. To find an expression for the curvature scalar we simple trace Einstein's equation which gives $R - 2R = 8\pi GT$ where $T = T^\mu{}_\mu$. Then the elimination of the R term in Einstein's equations gives

$$R_{\mu\nu} = 8\pi G(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T) := 8\pi G\, S_{\mu\nu}, \tag{3.232}$$

where we have also defined the "trace reduced" stress tensor $S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T$.

Finally, if we insert the linearised harmonic gauge condition $\partial_\nu h^{\mu\nu} = \frac{1}{2}\partial^\mu h^\nu{}_\nu$ into the linearised trace reduced Einstein equation we find that it simplifies directly to the $\Box$ equation (recall the first lecture)

$$\Box\, h_{\mu\nu} = -16\pi G\, S_{\mu\nu}. \tag{3.233}$$

This equation can now be solved for the metric perturbation $h_{\mu\nu}$ by means of the *retarded* Green's function $G(x;x')$ satisfying (proof is provided below)

$$\Box_x G(\mathbf{r},t;\mathbf{r}',t') = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')\delta(t-t'), \tag{3.234}$$

as follows

$$h^{ret}_{\mu\nu}(\mathbf{r},t) = 4G\int_V d^3x'\,\frac{S_{\mu\nu}(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|}{|\mathbf{r}-\mathbf{r}'|}, \tag{3.235}$$

where V is the space volume containing the source, i.e., where $S_{\mu\nu} \neq 0$.

Comment: The fact that the actual stress tensor $T_{\mu\nu}$ is divergence free implies that $S_{\mu\nu}$ satisfies

$$\partial_\nu S^{\mu\nu} = \frac{1}{2}\partial^\mu S^\nu{}_\nu, \tag{3.236}$$

which in fact it must since $h_{\mu\nu}$ also satisfies this (gauge) condition due to eq. (3.233).

3.8.2 The retarded Green's function

When deriving the metric perturbation $h_{\mu\nu}(x)$ from a source $T_{\mu\nu}(x)$, e.g., due to two black holes or neutron stars orbiting each other, we need to start from the linearised Einstein equations in the harmonic gauge

$$\Box h_{\mu\nu}(x) = -16\pi G S_{\mu\nu}(x), \text{ where } S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T_\rho{}^\rho, \tag{3.237}$$

and via the corresponding Green's function find the retarded solution (i.e., the solution that respects causality)

$$h^{ret}_{\mu\nu}(t,\mathbf{r}) = 4G\int_V d^3r'\,\frac{S_{\mu\nu}(t-|\mathbf{r}-\mathbf{r}'|,\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}, \tag{3.238}$$

where V is the spatial volume taken up by the source.

To show this we first derive the Green's function which is defined through the equation

$$\Box_x G(t,\mathbf{r};t',\mathbf{r}') = -4\pi\delta^3(\mathbf{r}-\mathbf{r}')\delta(t-t'). \tag{3.239}$$

Below we will prove that the retarded Green's function is given by

$$G^{ret}(t,\mathbf{r};t',\mathbf{r}') = \frac{\delta(t'-(t-|\mathbf{r}-\mathbf{r}'|))}{|\mathbf{r}-\mathbf{r}'|}, \tag{3.240}$$

which when integrated over the source $S_{\mu\nu}(t',\mathbf{r}')$ leads directly to the answer for $h^{ret}_{\mu\nu}(t,\mathbf{r})$ given above after performing the time integral.

To derive the expression for the Green's function given above we start by doing a Fourier transformation of the equation for the Green's function using the fact that

$$\delta^3(\mathbf{r}-\mathbf{r}')\delta(t-t') = \frac{1}{(2\pi)^4}\int d^3k\int d\omega\, e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')-i\omega(t-t')}, \tag{3.241}$$

and the definition of the Fourier transform $g(\omega,\mathbf{k})$

$$G(t,\mathbf{r};t',\mathbf{r}') = \int d^3k\int d\omega\, g(\omega,\mathbf{k})\, e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')-i\omega(t-t')}. \tag{3.242}$$

These definition leads directly to the solution

$$g(\omega,\mathbf{k}) = \frac{1}{4\pi^3}\frac{1}{k^2-\omega^2}, \tag{3.243}$$

which, however, is ill-defined since when $k^2=\omega^2$ this expression blows up.

The way out of this dilemma is to let ω be complex and introduce a small parameter ϵ to shift the poles in ω, i.e., $\omega=\pm k$ off the real ω axis. The integral that needs to be performed is then

$$G^{ret}(t,\mathbf{r};t',\mathbf{r}') = \frac{1}{4\pi^3}\int d^3k\int_C d\omega\,\frac{1}{k^2-(\omega+i\epsilon)^2}\,e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')-i\omega(t-t')}. \tag{3.244}$$

The crucial point here is that the two poles in the complex ω plane are now both under the real ω axis: $\omega = k-i\epsilon$ and $\omega=-k-i\epsilon$. Thus by closing the contour C in the upper half-plane we get zero for $t<t'$ (i.e., causality) since there are no poles and the half-circle part

goes to zero when its radius goes to infinity ($e^{-i\omega(t-t')} = e^{-iRe(\omega)(t-t')-i^2 Im(\omega)(t-t')} \to 0$ as $Im(\omega) \to \infty$).

On the other hand, choosing to close the contour in the lower half-plane gives a zero contribution form the half-circle for $t > t'$ (i.e., causality) so the residue theorem gives the final result from the poles as follows. We first reverse the direction of the loop around the poles which are defined by the factorisation $k^2 - (\omega + i\epsilon)^2 = -(\omega - k + i\epsilon)(\omega + k + i\epsilon)$ and rewrite the integral as

$$G^{ret}(t, \mathbf{r}; t', \mathbf{r}') = \frac{(-1)^2}{4\pi^3} \int d^3k e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \oint_C d\omega \, \frac{e^{-i\omega(t-t')}}{(\omega - k + i\epsilon)(\omega + k + i\epsilon)}. \tag{3.245}$$

The residue theorem then implies that the ω integral is, after taking the limit $\epsilon \to 0$,

$$\oint_C d\omega \, \frac{e^{-i\omega(t-t')}}{(\omega - k + i\epsilon)(\omega + k + i\epsilon)} = 2\pi i (\frac{e^{-ik(t-t')}}{2k} + \frac{e^{ik(t-t')}}{-2k}) = 2\pi \frac{\sin k(t-t')}{k}. \tag{3.246}$$

The Green's function then becomes

$$G^{ret}(t, \mathbf{r}; t', \mathbf{r}') = \frac{1}{2\pi^2} \int d^3k \, e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \frac{\sin k(t-t')}{k}, \tag{3.247}$$

where the momentum integral is easily done using $d^3k = k^2 dk \sin\theta d\theta d\phi$ and orienting the momentum axis so that $\mathbf{k} \cdot (\mathbf{r} - \mathbf{r}') = k(|\mathbf{r} - \mathbf{r}'|)\cos\theta$. The angular part of the integral is

$$\int_0^\pi e^{ik|\mathbf{r}-\mathbf{r}'|\cos\theta} \sin\theta d\theta \int_0^{2\pi} d\phi = 2\pi \frac{e^{ik|\mathbf{r}-\mathbf{r}'|} - e^{-ik|\mathbf{r}-\mathbf{r}'|}}{ik|\mathbf{r}-\mathbf{r}'|} = \frac{4\pi}{k|\mathbf{r}-\mathbf{r}'|} \sin(k|\mathbf{r}-\mathbf{r}'|), \tag{3.248}$$

which leads to

$$G^{ret}(t, \mathbf{r}; t', \mathbf{r}') = \frac{2}{\pi|\mathbf{r}-\mathbf{r}'|} \int_0^\infty dk \, \sin k(t-t') \sin k|\mathbf{r}-\mathbf{r}'|. \tag{3.249}$$

The last integral is then done by rewriting the sin functions as sums of two exponentials, using that the integral is symmetric under $k \to -k$ to get $\int_0^\infty dk = \frac{1}{2}\int_{-\infty}^\infty dk$, and flipping $k \to -k$ in two of the four terms to get

$$\int_0^\infty dk \, \sin k(t-t') \sin k|\mathbf{r}-\mathbf{r}'| = \frac{1}{2}\int_{-\infty}^\infty dk \sin k(t-t') \sin k|\mathbf{r}-\mathbf{r}'| = \frac{\pi}{2}\delta((t-t') - |\mathbf{r}-\mathbf{r}'|), \tag{3.250}$$

where we have also used the fact that $t > t'$ (which means that $\delta((t-t') + |\mathbf{r}-\mathbf{r}'|) = 0$).

So finally we find the Green's function to be just

$$G^{ret}(t, \mathbf{r}; t', \mathbf{r}') = \frac{\delta((t-t') - |\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|} = \frac{\delta(t' - (t - |\mathbf{r}-\mathbf{r}'|))}{|\mathbf{r}-\mathbf{r}'|}, \tag{3.251}$$

where we also give the result in the form (last expression) that will be used below. In fact, integrating this Green's function, which provides the response from a delta-function source, over the whole spread-out source $S_{\mu\nu}(\mathbf{r}', t')$ and doing the t'-integral, we get the total response as given in the beginning, i.e.,

$$h^{ret}(\mathbf{r}, t) = 4G \int dt' d^3r' \, G^{ret}(t, \mathbf{r}; t', \mathbf{r}') \, S_{\mu\nu}(t', \mathbf{r}') = 4G \int_V d^3r' \, \frac{S_{\mu\nu}(t - |\mathbf{r}-\mathbf{r}'|, \mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}. \tag{3.252}$$

Note that the original equation $\Box h_{\mu\nu}(x) = -16\pi G S_{\mu\nu}(x)$ follows trivially from this expression for $h^{ret}(\mathbf{r}, t)$ and the definition of the Green's function.

3.8.3 Plane waves

When moving away from the source the wave looks more and more like a plane wave: thus for a given momentum k^μ the metric fluctuations are

$$h^{(k)}_{\mu\nu}(x) = e^{(k)}_{\mu\nu} e^{ik_\sigma x^\sigma} + e^{\star(k)}_{\mu\nu} e^{-ik_\sigma x^\sigma}, \tag{3.253}$$

where the x-independent $e^{(k)}_{\mu\nu}$ is the metric *polarisation tensor*. This wave is traveling in vacuum so the mode function must satisfy

$$\Box h_{\mu\nu} = 0, \;\; \partial_\nu h^{\mu\nu} = \frac{1}{2}\partial^\mu h^\nu{}_\nu, \tag{3.254}$$

which after Fourier transformation to momentum space implies that

$$k^2 = 0, \;\; k^\mu e_{\mu\nu} = \frac{1}{2} k_\nu e^\mu{}_\mu. \tag{3.255}$$

The task is now to understand what kind of information is stored in this wave, i.e., we should try to find the physical degrees of freedom (dof) and see what kind of physics they are related to. The physical dof are given by the gauge independent components of $\epsilon^{(k)}_{\mu\nu}$ which is a 4x4 symmetric tensor and thus has 10 matrix components. However, they satisfy the four conditions $k^\mu e_{\mu\nu} = \frac{1}{2} k_\nu e^\mu{}_\mu$ which leaves six independent components.

Now we must also consider the gauge transformations remaining (if any) after imposing the harmonic gauge condition. Recall that $\Gamma^\mu = 0$ implied $\Box x'^\mu(x) = 0$ which at the linear level reads $\Box \epsilon^\mu(x) = 0$. Thus both the metric $h_{\mu\nu}(x)$ and the parameters $\epsilon^\mu(x)$ satisfy the $\Box = 0$ equation and therefore it is still possible to consider gauge transformations of the plane wave type (the i is just for convenience)

$$\epsilon_\mu(x) = i\epsilon^{(k)}_\mu e^{ik\cdot x} - i\epsilon^{\star(k)}_\mu e^{-ik\cdot x}. \tag{3.256}$$

These remaining gauge transformations read in momentum space

$$e'_{\mu\nu} = e_{\mu\nu} + k_\mu \epsilon_\nu + k_\nu \epsilon_\mu, \tag{3.257}$$

which means that we can eliminate (gauge fix) another four of the six independent dof leaving two physical dof in the gravitational wave.

The final conclusion is, as for the photon in EM, that the graviton (assuming that the field can be quantised as in QFT) is a massless particle with two degrees of freedom, or polarisation states, moving with the speed of light (c).

To make this discussion completely explicit let us consider a plane wave moving in the z-direction, i.e., $\mathbf{k} = (0, 0, k)$ with $k > 0$. Then $k^2 = 0$ implies $k^\mu = (k, 0, 0, k)$ that is $k^1 = k^2 = 0$ and $k^0 = k^3 = k$. As the second step we must solve $k^\mu e_{\mu\nu} = \frac{1}{2} k_\nu e^\mu{}_\mu$. Inserting $k^\mu = (k, 0, 0, k)$ gives

$$k(e_{3\nu} + e_{0\nu}) = \frac{1}{2} k_\nu (e_{11} + e_{22} + e_{33} - e_{00}). \tag{3.258}$$

Writing out these four equation for $\nu = 0, 1, 2, 3$ gives four equations that are solved by

$$e_{01} = -e_{31}, \;\; e_{02} = -e_{32}, \;\; e_{03} = -\frac{1}{2}(e_{00} + e_{33}), \;\; e_{22} = -e_{11}. \tag{3.259}$$

Finally, we can use the gauge transformations of the six so far independent $e_{\mu\nu}$ components, chosen to be $e_{00}, e_{11}, e_{33}, e_{12}, e_{23}, e_{31}$. These transformations read, with $k_\mu = (-k, 0, 0, k)$,

$$e'_{00} = e_{00} - 2k\epsilon_0, \;\; e'_{11} = e_{11}, \tag{3.260}$$

$$e'_{33} = e_{33} + 2k\epsilon_3, \;\; e'_{12} = e_{12}, \tag{3.261}$$

$$e'_{23} = e_{23} + k\epsilon_2, \;\;\; e'_{31} = e_{31} + k\epsilon_1. \tag{3.262}$$

So we now see explicitly that the four parameters ϵ_μ can be used to set all the remaining components of $e_{\mu\nu}$ to zero except e_{11} and e_{12} which then are the independent physical degrees of freedom in this wave. Recall however, that we still have the relation $e_{22} = -e_{11}$.

The result above is nicely summarised by giving the polarisation tensor in matrix form (with ordering $\mu = (0123)$)

$$e_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & e_{11} & e_{12} & 0 \\ 0 & e_{12} & -e_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.263}$$

One can now read off the spin by applying a rotation in the xy-plane, with $x^i = (x, y)$,

$$R_i{}^j = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \tag{3.264}$$

to the matrix

$$e_{ij} = \begin{pmatrix} e_{11} & e_{12} \\ e_{12} & -e_{11} \end{pmatrix}. \tag{3.265}$$

This computation is in matrix form $e' = ReR^T$ which is easily computed: $e'_{11} = \cos^2\theta e_{11} + 2\cos\theta\sin\theta - \sin^2\theta$ etc. The spin, or rather helicity since the speed of the wave is c, is identified by forming complex combinations of the two non-zero components of $e_{\mu\nu}$, i.e., $e_\pm = e_{11} \mp ie_{12}$. Then the rotations become

$$e'_\pm = e^{\pm 2i\theta} e_\pm, \; \Rightarrow \text{helicity } h = \pm 2. \tag{3.266}$$

In a similar way the four dof that were gauged away have helicity 1 and 0:

$$h \pm 1 : \;\; f_\pm = e_{13} \mp e_{23}, \tag{3.267}$$

$$h = 0 : \;\; e_{33} \text{ and } e_{00}. \tag{3.268}$$

In the notation that is most common in the literature one renames e_{11}, e_{12} as $h_+, h_\times$ and the plane wave is thus written

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e^{i(kz - \omega t)}. \tag{3.269}$$

The new notation tells you that h_+ gives oscillations in the x and y directions while $h_\times$ is related to oscillations in the $x \pm y$ directions. The metric is

$$d\tau^2 = dt^2 - (1+h_+e^{i(kz-\omega t)})dx^2 - (1-h_+e^{i(kz-\omega t)})dy^2 - 2(h_\times e^{i(kz-\omega t)})dxdy - dz^2. \quad (3.270)$$

This result means that the physical distance between two particles located at fixed coordinate values will oscillate when the wave is passing them. We will compute the size of this effect later when we discuss the observations of gravitational waves made by LIGO.

3.8.4 Energy transported by a gravitational wave

In order to compute how fast the distance between two black holes or neutron stars in orbit around each other shrink we must find out how much energy is transported away by the gravitational waves generated by this system.

The transport of gravitational energy is given by the space components of the energy-momentum 4-vector p^μ which in turn is the zero component of the gravitational stress-tensor $t^{\mu\nu}$ (which is the corresponding current) integrated over space: recall the definition

$$t_{\mu\nu} = (R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) - (R^{(1)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^{(1)}), \quad (3.271)$$

whose lowest order terms in $h_{\mu\nu}$ are quadratic. These quadratic terms will be sufficient for our purposes here:

$$t^{(2)}_{\mu\nu} = -\frac{1}{8\pi G}(-\frac{1}{2}h_{\mu\nu}R^{(1)} + \frac{1}{2}\eta_{\mu\nu}h^{\rho\sigma}R^{(1)}_{\rho\sigma} + R^{(2)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^{(2)}), \quad (3.272)$$

where (derived in SW eq. 7.6.2)

$$R^{(1)}_{\mu\nu} = -\frac{1}{2}(\Box h_{\mu\nu} - \partial_\mu\partial^\rho h_{\nu\rho} - \partial_\nu\partial^\rho h_{\mu\rho} + \partial_\mu\partial_\nu h), \quad (3.273)$$

and (the messy derivation is not given by SW but the result is quoted in eq. 7.6.15)

$$\begin{aligned} R^{(2)}_{\mu\nu} &= \frac{1}{2}h^{\rho\sigma}(\partial_\mu\partial_\nu h_{\rho\sigma} - \partial_\mu\partial_\rho h_{\nu\sigma} - \partial_\nu\partial_\rho h_{\mu\sigma} + \partial_\rho\partial_\sigma h_{\mu\nu}) \\ &+ \frac{1}{4}(2\partial^\rho h_{\rho\sigma} - \partial_\sigma h)(\partial_\mu h_\nu{}^\sigma + \partial_\nu h_\mu{}^\sigma - \partial^\sigma h_{\mu\nu}) \\ &- \frac{1}{4}(\partial_\rho h_{\mu\sigma} + \partial_\mu h_{\rho\sigma} - \partial_\sigma h_{\mu\rho})(\partial^\rho h_\nu{}^\sigma + \partial_\nu h^{\rho\sigma} - \partial^\sigma h_\nu{}^\rho). \end{aligned} \quad (3.274)$$

The expression for the gravitational stress tensor $t_{\mu\nu}$ is thus rather complicated even when restricted to the second order $h_{\mu\nu}$ terms. However, we are here only interested in $t^{(2)}_{\mu\nu}$ for plane waves which means that $h_{\mu\nu}$ satisfies $\Box h_{\mu\nu} = 0,\ \partial_\sigma h^\sigma{}_\mu = \frac{1}{2}\partial_\mu h$. If inserted into the first order Ricci tensor we see directly that

$$R^{(1)}_{\mu\nu}(\text{plane wave}) = 0. \quad (3.275)$$

Thus we have

$$t^{(2)}_{\mu\nu}(\text{plane wave}) = -\frac{1}{8\pi G}(R^{(2)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^{(2)})(\text{plane wave}), \quad (3.276)$$

where the conditions on $h_{\mu\nu}$ implies that the second line in $R^{(2)}_{\mu\nu}$(plane wave) vanishes. After the Fourier transformation to momentum space we can use $k_\mu h^{\mu\nu} = \frac{1}{2}\partial^\nu h^\mu{}_\mu$ and $R^{(2)}_{\mu\nu}$(plane wave) will simplify further (see below).

The final step is to insert into $t^{(2)}_{\mu\nu}$ a mode from the plane wave Fourier expansion

$$h_{\mu\nu} = e_{\mu\nu}e^{ik\cdot x} + e^*_{\mu\nu}e^{-ik\cdot x}, \text{ satisfying } k^2 = 0,\ k_\sigma e^\sigma{}_\mu = \frac{1}{2}k_\mu e^\sigma{}_\sigma. \tag{3.277}$$

Since the $t^{(2)}_{\mu\nu}$ is bilinear in $h_{\mu\nu}$ we get terms proportional to

$$e^{2ik\cdot x},\ e^{-2ik\cdot x},\ e^{ik\cdot x}e^{-ik\cdot x} = 1. \tag{3.278}$$

In applications where we only observe how the binary systems of black holes or neutron stars loose energy by gravitational radiation, as done in 1974 by Hulse and Taylor (Nobel prize 1993), the wave itself is not investigated so we need only the wave averaged over many wave lengths. This means that of the three kinds of terms mentioned above only the constant one survives. Thus (Re refers to the real part)

$$\langle R^{(2)}_{\mu\nu}\rangle = -Re\,(e^{*\rho\sigma}(k_\mu k_\nu e_{\rho\sigma} - k_\mu k_\rho e_{\nu\sigma} - k_\nu k_\sigma e_{\mu\rho} + k_\rho k_\sigma e_{\mu\nu}) \tag{3.279}$$

$$-\ Re\,(\ldots)\,. \tag{3.280}$$

Here we note that the momenta are no longer associated with other polarisation tensors the derivatives were acting on and the condition $k_\sigma e^\sigma{}_\mu = \frac{1}{2}k_\mu e^\sigma{}_\sigma$ can be applied again. $\langle R^{(2)}_{\mu\nu}\rangle$ then becomes a lot simpler and reads

$$\langle R^{(2)}_{\mu\nu}\rangle = -\frac{1}{2}k_\mu k_\nu(e^{*\rho\sigma}e_{\rho\sigma} - \frac{1}{2}|e^\rho{}_\rho|^2). \tag{3.281}$$

Thus $\langle R^{(2)\mu}_\mu\rangle = 0$, since $k^2 = 0$, and we get

$$\langle t^{(2)}_{\mu\nu}\rangle = \frac{k_\mu k_\nu}{16\pi G}(e^{*\rho\sigma}e_{\rho\sigma} - \frac{1}{2}|e^\rho{}_\rho|^2) = \frac{k_\mu k_\nu}{8\pi G}(|e_{11}|^2 + |e_{12}|^2) = \frac{k_\mu k_\nu}{16\pi G}(|e_+|^2 + |e_-|^2), \tag{3.282}$$

where we have used the result in eq. (3.263) and that $e_\pm = e_{11} \mp ie_{12}$.

Finally, we can now obtain the power P emitted by the gravitational wave in the solid angle $d\Omega$. The energy transport is given by the energy current density, i.e., the space components of the 4-momentum $\langle p^\mu_{grav}\rangle = \langle t^{(2)0\mu}\rangle$ multiplied by the area of the solid angle at radius r:

$$dP = \langle p^i_{grav}\rangle\hat{x}^i r^2 d\Omega \Rightarrow \frac{dP}{d\Omega} = \langle t^{(2)0i}\rangle\hat{x}^i r^2. \tag{3.283}$$

Thus to compute $\frac{dP}{d\Omega}$ we need to find the relation between $\langle t^{(2)}_{\mu\nu}\rangle$ and the source generating the wave in the first place, i.e., from $T_{\mu\nu}$ for the source. This is rather easily done starting from eq. (3.282) above and using the retarded solution to $\Box h_{\mu\nu} = -16\pi G S_{\mu\nu}$ in harmonic coordinates:

$$h^{ret}_{\mu\nu}(t,\mathbf{r}) = 4G\int_V d^3r'\,\frac{S_{\mu\nu}(t-|\mathbf{r}-\mathbf{r}'|,\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|},\ \text{where } S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T^\rho{}_\rho, \tag{3.284}$$

together with

$$h_{\mu\nu} = e_{\mu\nu}e^{ik\cdot x} + e^*_{\mu\nu}e^{-ik\cdot x},\ \text{satisfying}\ k^2 = 0,\ k_\sigma e^\sigma{}_\mu = \frac{1}{2}k_\mu e^\sigma{}_\sigma. \tag{3.285}$$

To facilitate the steps discussed in the previous paragraph one needs an approximation known as *quadrupole radiation*. To this end we consider a Fourier transformation in the time coordinate, either continuous or discrete,

$$T_{\mu\nu}(t,\mathbf{r}) = \int_0^\infty T_{\mu\nu}(\omega,\mathbf{r})e^{-i\omega t} + c.c. \tag{3.286}$$

$$= \Sigma_\omega T_{\mu\nu}(\omega,\mathbf{r})e^{-i\omega t} + c.c. \tag{3.287}$$

Then restricting ourselves to one single ω component we get

$$h^{ret}_{\mu\nu}(t,\mathbf{r}) = 4G\int_{r'<R}\frac{d^3r'}{|\mathbf{r}-\mathbf{r}'|}\,S_{\mu\nu}(\omega,\mathbf{r}')e^{-i\omega(t-|\mathbf{r}-\mathbf{r}'|)} + c.c.. \tag{3.288}$$

The integral is of course only over the volume of space where the source is non-zero, i.e., for $r' < R$ where R is the maximal extension of the source. The idea is now to evaluate the integral in the *region far away from the source*, i.e., for $r >> R$. Then

$$|\mathbf{r}-\mathbf{r}'| \approx r - \mathbf{r}'\cdot\hat{\mathbf{r}} \Rightarrow \frac{1}{|\mathbf{r}-\mathbf{r}'|} \approx \frac{1}{r}(1+\mathcal{O}(\tfrac{r'}{r})), \tag{3.289}$$

which make it possible to write $h^{ret}_{\mu\nu}(t,\mathbf{r})$ as follows

$$h^{ret}_{\mu\nu}(t,\mathbf{r}) \approx \frac{4G}{r}\,e^{i(-\omega t+\omega r)}\int d^3r'\,S_{\mu\nu}(\omega,\mathbf{r}')e^{-i\omega\mathbf{r}'\cdot\hat{\mathbf{r}}} + c.c.. \tag{3.290}$$

The exponential factor in front of the integral is a plane wave factor $e^{ik\cdot x}$ with wave vector $k^\mu = (k^0,\mathbf{k}) = (\omega,\omega\hat{\mathbf{r}})$. Note that $r\omega \propto r/\lambda >> 1$ since the size of λ is related to the size of the source. Comparing this result for $h^{ret}_{\mu\nu}(t,\mathbf{r})$ to the expansion above $h^{ret}_{\mu\nu}(t,\mathbf{r}) = e_{\mu\nu}(\omega,\mathbf{k})e^{ik\cdot x} + c.c.$ gives

$$e_{\mu\nu}(\omega,\mathbf{k}) = \frac{4G}{r}\int d^3r'\,S_{\mu\nu}(\omega,\mathbf{r}')e^{-i\omega\mathbf{r}'\cdot\hat{\mathbf{r}}} = \frac{4G}{r}S_{\mu\nu}(\omega,\mathbf{k}). \tag{3.291}$$

Inserting this result for $e_{\mu\nu}(\omega,\mathbf{k})$ into the expression for $\langle t^{(2)}_{\mu\nu}\rangle$ above and then use this in the expression for $\frac{dP}{d\Omega}$ gives (recall that $k^\mu = (k^0,\mathbf{k}) = (\omega,\omega\hat{\mathbf{r}})$)

$$\frac{dP}{d\Omega} = \frac{k^0\mathbf{k}\cdot\hat{\mathbf{r}}r^2}{16\pi G}\cdot(\frac{4G}{r})^2(S^{*\rho\sigma}(\omega,\mathbf{k})S_{\rho\sigma}(\omega,\mathbf{k}) - \frac{1}{2}|S^\rho{}_\rho(\omega,\mathbf{k})|^2) \tag{3.292}$$

$$= \frac{G\omega^2}{\pi}(T^{*\rho\sigma}(\omega,\mathbf{k})T_{\rho\sigma}(\omega,\mathbf{k}) - \frac{1}{2}|T^\rho{}_\rho(\omega,\mathbf{k})|^2), \tag{3.293}$$

where in the last step we have just inserted the definition $S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T^\rho{}_\rho$ which happens to give the same result in terms of $T_{\mu\nu}$.

As a last simplification one can use the fact that $T_{\mu\nu}$ is divergence free $k_\mu T^{\mu\nu}(\omega,\mathbf{k}) = 0$ which implies, using $k^0 = \omega$,

$$T_{0i} = -\frac{k^j}{\omega}T_{ij}, \quad T_{00} = \frac{k^ik^j}{\omega^2}T_{ij}, \tag{3.294}$$

which gives the result

$$\frac{dP}{d\Omega} = \frac{G\omega^2}{\pi}\Lambda_{ij,kl}(\hat{\mathbf{k}})T^*_{ij}(\omega,\mathbf{k})T_{kl}(\omega,\mathbf{k}), \tag{3.295}$$

where

$$\Lambda_{ij,kl}(\hat{\mathbf{k}}) = \delta_{i(k}\delta_{l)j} - 2\delta_{(i(k}\hat{k}_{l)}\hat{k}_{j)} + \frac{1}{2}\hat{k}_i\hat{k}_j\hat{k}_k\hat{k}_l - \frac{1}{2}\delta_{ij}\delta_{kl} + \frac{1}{2}\delta_{ij}\hat{k}_k\hat{k}_l + \frac{1}{2}\delta_{kl}\hat{k}_i\hat{k}_j. \tag{3.296}$$

At this point it is useful to introduce the **quadrupole approximation**. Consider again the space Fourier transform

$$T_{ij}(\omega,\mathbf{k}) = \int d^3r'\, T_{ij}(\omega,\mathbf{r}')e^{-i\mathbf{k}\cdot\mathbf{r}'}. \tag{3.297}$$

Assume now that the source consists of massive bodies moving with typical velocities $v << 1$. Then the radiation produced by the system will have typical frequency $\omega = v/R$ so that a natural approximation is provided by $\omega R = kR << 1$ which means that to lowest order in kr' we have

$$T_{ij}(\omega,\mathbf{k}) = \int d^3r'\, T_{ij}(\omega,\mathbf{r}'). \tag{3.298}$$

From the above discussion of the divergence free condition on the stress tensor we found $\partial_i\partial_j T^{ij}(\omega,\mathbf{r}') = -\omega^2 T^{00}(\omega,\mathbf{r}')$ which we can integrate against $x'^k x'^l$ so that

$$\int d^3r'\, x'^k x'^l \partial'_i\partial'_j T^{ij}(\omega,\mathbf{r}') = -\omega^2\int d^3r'\, x'^k x'^l T^{00}(\omega,\mathbf{r}'). \tag{3.299}$$

Integrating the LHS by parts gives $2\int d^3r'\, T^{kl}(\omega,\mathbf{r}') = 2T^{kl}(\omega,\mathbf{k})$ where in the last equality we used the result in eq. (3.298). Thus we get the quadrupole approximation result

$$T^{ij}(\omega,\mathbf{k}) \approx -\frac{\omega^2}{2}D^{ij}(\omega), \ \text{ where } \ D^{ij}(\omega) := \int d^3r'\, x'^i x'^j T^{00}(\omega,\mathbf{r}'). \tag{3.300}$$

One should note here that in this approximation T^{ij} is momentum independent. With these results at hand the (differential) emitted power is finally given by

$$\frac{dP}{d\Omega} = \frac{G\omega^6}{4\pi}\Lambda_{ij,kl}(\hat{\mathbf{k}})D^*_{ij}(\omega)D_{kl}(\omega). \tag{3.301}$$

If we are interested only in the total power emitted $P = \int \frac{dP}{d\Omega}\,d\Omega$, as in the Hulse-Taylor case, the required integrals of $\Lambda_{ij,kl}(\hat{\mathbf{k}})$ are

$$\int d\Omega = 4\pi, \quad \int d\Omega\,\hat{k}_i\hat{k}_j = \frac{4\pi}{3}\delta_{ij}, \quad \int d\Omega\,\hat{k}_i\hat{k}_j\hat{k}_k\hat{k}_l = \frac{4\pi}{15}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{3.302}$$

giving the total power emitted at frequency ω as

$$P = \frac{2G\omega^6}{5}(D^{*ij}(\omega)D_{ij}(\omega) - \frac{1}{3}|D^i{}_i|^2(\omega)). \tag{3.303}$$

Note the very strong dependence on ω.

We end this discussion by deriving two results that are relevant for the observations made by LIGO[19] starting with the first ever detection of gravitational waves from a binary system of two black holes in 2015 (called "GW150914"). The first result is the power emitted at the merger of the black holes and the second is the "strain" in the wave detected at LIGO.

A system of two equally massive bodies (M) in orbit around each other, and a distance d apart, is described by (dropping primes on source coordinates here)

$$\text{Body 1}: \ \mathbf{r_1}(t) = \frac{d}{2}(\cos(\Omega t)\,\hat{x} + \sin(\Omega t)\,\hat{y}), \tag{3.304}$$

$$\text{Body 2}: \ \mathbf{r_2}(t) = -\mathbf{r_1}(t). \tag{3.305}$$

Thus

$$T^{00}(t,\mathbf{r}) = M\delta^3(\mathbf{r} - \mathbf{r_1}(t)) + M\delta^3(\mathbf{r} - \mathbf{r_2}(t)). \tag{3.306}$$

The components of $D_{ij} = \int d^3r\, x^i x^j T^{00}$ are then easily calculated:

$$D_{xx} = 2M\frac{d^2}{4}\cos^2(\Omega t) = \frac{1}{4}Md^2(1 + \cos(2\Omega t)), \tag{3.307}$$

$$D_{xy} = \frac{1}{4}Md^2\sin(2\Omega t), \tag{3.308}$$

$$D_{yy} = \frac{1}{4}Md^2(1 - \cos(2\Omega t)). \tag{3.309}$$

Then comparing this to $D_{ij}(t) = \Sigma_\omega(e^{-i\omega t}D_{ij}(\omega) + e^{i\omega t}D^*_{ij}(\omega))$ gives

$$D_{xx}(\omega = 0) = \frac{1}{8}Md^2, \ \ D_{xx}(\omega = 2\Omega) = \frac{1}{8}Md^2, \tag{3.310}$$

$$D_{xy}(\omega = 0) = 0, \ \ D_{xy}(\omega = 2\Omega) = \frac{i}{8}Md^2, \tag{3.311}$$

$$D_{yy}(\omega = 0) = \frac{1}{8}Md^2, \ \ D_{yy}(\omega = 2\Omega) = -\frac{1}{8}Md^2. \tag{3.312}$$

Summing the total power from the two modes with $\omega = 0$ and $\omega = 2\Omega$ gives ($D_{ii} = 0$)

$$P = \Sigma_\omega \frac{2G\omega^6}{5}(D^*_{ij}D_{ij} - \frac{1}{3}|D_{ii}|^2) = \frac{8}{5}GM^2d^4\Omega^6 \tag{3.313}$$

which must be complemented by some powers of c to have the correct dimension:

$$P = \frac{8}{5}\frac{GM^2d^4\Omega^6}{c^5}. \tag{3.314}$$

This formula can be taken one step further by finding the frequency Ω in terms of the other quantities. Assuming we can use classical equations, the motion of body 1 is governed by $M\ddot{\mathbf{r}}_1 = -\frac{MMG}{d^3}(\mathbf{r}_1 - \mathbf{r}_2)$ that is $M\frac{d}{2}\Omega^2 = \frac{M^2G}{d^3}d$ giving $\Omega^2 = \frac{2MG}{d^3}$. Thus

$$P = \frac{64}{5}\frac{G^4M^5}{d^5c^5}. \tag{3.315}$$

[19]Laser Interferometer Gravitational wave Observatory.

This formula for the total emitted power is really remarkable as is seen by assuming that just before merger $d = 2r_s$, where r_s is the Schwarzschild radius $r_s = \frac{2MG}{c^2}$ which gives[20]

$$\text{At merger}: \ P = \frac{c^5}{80G}. \tag{3.316}$$

This rather strange result (independence of the mass of the black holes) implies that the energy emitted per second is $P \approx 5 \cdot 10^{50}\, W$ which will be checked against real observations below (see Abbott et al, PRL, on Canvas). Note that the energy of the sun corresponding to its mass $M = 2 \cdot 10^{30}\, kg$ is $E = Mc^2 = 2 \cdot 10^{47}\, Nm$.

The energy radiated away calculated above corresponds to the energy loss of the binary system: $P = -\frac{dE}{dt}$. Here E is the total energy possessed by the two bodies of equal mass M and velocity $v = \frac{d}{2}\Omega$:

$$E(t) = E_k + E_p = 2 \cdot \tfrac{1}{2} M (\tfrac{d}{2}\Omega)^2 - \tfrac{M^2 G}{d} = -\tfrac{1}{2}\left(\frac{M^5 G^2 \Omega^2(t)}{2}\right)^{\frac{1}{3}}, \tag{3.317}$$

where to get the last expression we have used $d^3 = \frac{2MG}{\Omega^2}$ obtained above. This relation tells us immediately that if E decreases then Ω has to increase and hence d decreases leading to a spiral-motion inwards. Since the only t dependence is in Ω, taking the t derivative of E we see immediately that

$$P = -\frac{dE}{dt} = \tfrac{1}{2}\left(\frac{M^5 G^2}{2}\right)^{\frac{1}{3}} \frac{2}{3} \Omega^{-\frac{1}{3}}(t) \frac{d\Omega(t)}{dt}, \tag{3.318}$$

from which we deduce the result

$$\frac{d\Omega}{dt} = \frac{192\sqrt{2}}{5} (GM)^{\frac{7}{2}} (d(t))^{-\frac{11}{2}} c^{-5}. \tag{3.319}$$

Turning this into an equation for the change of the period T instead, using $T = \frac{2\pi}{\Omega}$, we get

$$\frac{dT}{dt} = -\frac{2\pi}{\Omega^2}\frac{d\Omega}{dt} = -\frac{192\sqrt{2}\pi}{5}\left(\frac{MG}{d(t)c^2}\right)^{\frac{5}{2}}. \tag{3.320}$$

To get a feeling for the size of the different quantities discussed above, let us insert numbers for two different physical situations.

1) A binary system of two neutron stars in circular orbits around each other. Suppose the masses are equal and 1.4 solar masses, and that the distance between them is the diameter of our sun. Thus $M = 2.8 \cdot 10^{30}\, kg$, $d = 7.0 \cdot 10^8\, m$. Then

$$P = 3.4 \cdot 10^{24}\, W, \quad T = 6 \cdot 10^3\, s, \quad dT = -5 \cdot 10^{-13}\, dt. \tag{3.321}$$

The velocity of the neutron stars is about $v = \frac{d\Omega}{2} = 5 \cdot 10^4\, m/s$.

2) If we instead consider two black holes with masses around 30 solar masses (as for

[20] $G = 6.7 \cdot 10^{-11} \frac{m^3}{kg \cdot s^2}$ and $c = 3.0 \cdot 10^8 m$.

GW150914) at the point of merger, i.e., then the distance between them is about two Schwarzschild radii $d = 2r_s \approx 180\,km$[21]. This gives

$$P = 5 \cdot 10^{50}\,W, \quad \Omega = \frac{c}{2\sqrt{2}\,r_s} = 10^3\,s^{-1}, \quad T = 6 \cdot 10^{-3}\,s, \quad dT = -\mathcal{O}(1)\,dt. \tag{3.322}$$

The velocity of the two black holes is about the same as in the previous example $v = \frac{d\Omega}{2} = 5 \cdot 10^4\,m/s$. From these results we see that the whole period of revolution is gone in one period so the final stage before merger is perhaps $10^{-3}\,s$ long in which time the energy loss is about $PT = 5 \cdot 10^{47}\,Nm$ corresponding to 2 to 3 solar masses. This is very close to the observations made for $GW150914$ at LIGO (see Abbott et al in PRL).

Note that close to the point of merger the usual weak field approximation of the gravitational field is no longer valid (e.g., $h_{00} = \frac{2GM}{r_s} = 1$) and the full non-linear Einstein equations must in principle be used. To get accurate answers in this situation one therefore has to resort to advanced numerical methods. On top of this technical issue it should be noted that the physics at merger, that is, of the actual transition from two black holes to a single one is not understood.

3.8.5 The "strain" observed at LIGO

The observations of gravitational waves at LIGO and other similar "observatories" (Virgo, and in the future LISA) are done by actually measuring the change in distance between two points 4 km apart using interferometry. The setup is similar to the one in the Michelson-Morley experiment which in 1887 checked the speed of light in different directions relative the motion of the Earth. For LIGO, however, the problem is that the expected signal is so extremely small (see below) that the possibilities of ever being able to detect any waves was minute.

The strain is defined as

$$\text{strain} := h = \frac{\Delta L}{L}, \tag{3.323}$$

where h is h_+ or $h_\times$ from the wave $h^{ret}_{\mu\nu}$ discussed in detail above, and L is the size of the observatory (for LIGO $L = 4\,km$) while ΔL is the change in size, i.e., the effect to be measured. Thus we must obtain the value of h from the information about the binary system.

The starting point to get a value for h is the formula

$$h^{ret}_{ij} = \frac{4G}{r} S_{ij}(\omega, \mathbf{k}) = \frac{4G}{r} T_{ij}(\omega, \mathbf{k}) = \frac{4G}{r}\frac{\omega^2}{2} D_{ij}(\omega) = \frac{4G}{r}\frac{\omega^2}{2} \int d^3r' x'^i x'^j T^{00}(\omega, \mathbf{r}'). \tag{3.324}$$

An estimate of the last expression above can be obtained if we note that its largest value is attained at merger. Then $x'^i x'^j$ is roughly $(2r_s)^2$ and taking this value outside the integral the rest of it corresponds to the total energy of the two black holes which is $2M$. Thus this approximation gives

$$h = 16 \frac{GM\omega^2 r_s^2}{rc^4}. \tag{3.325}$$

[21]For the sun $r_s = 3\,km$.

Using $r_s = \frac{2GM}{c^2}$ this becomes

$$h \approx 8\frac{\omega^2 r_s^3}{c^2 r} = \frac{r_s}{r}, \tag{3.326}$$

where we in the last step used $\omega^2 = \frac{2GM}{(2r_s)^3} = \frac{c^2}{8r_s^2}$ from Newton's equation of motion.

This is a most remarkable formula. Making a rough guess about likely distances r to some merger events and an estimate about their likely masses h was believed to end up 10^{-20} or smaller. The event observed $GW150914$ comes from a binary black hole system with each black hole mass about 30 solar masses giving $r_s \approx 10^5\, m$ and a distance from us of about[22] $440\, Mpc = 1.4\, G\, ly \approx 10^{25}\, m$. Thus its strain is

$$\text{GW150914}: \quad h \approx 10^{-20}. \tag{3.327}$$

At LIGO, having $L = 4\, km$, the strain formula above then gives the change in the distance ΔL between the mirrors at the ends of the vacuum tubes of the interferometer:

$$\text{LIGO for GW150914}: \quad \Delta L \approx 10^{-18}\, m, \tag{3.328}$$

corresponding to 1/1000 of the size of a proton.

3.8.6 Quantum gravity

If one tries to apply the rules of QFT to Einstein's theory of gravity one runs into serious problems. As we have discussed several times in this course already one can expand Einstein's equations in small fluctuations around Minkowski space using $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. Since the Ricci tensor contains two derivatives and an expansion will give rise to terms with $h_{\mu\nu}$ to any power (since the inverse metric appears in the Ricci tensor) the theory will consist a kinetic term linear in $h_{\mu\nu}$, i.e., $\Box\, h_{\mu\nu}$, plus an infinite set of interaction terms all with two derivatives. In QFT the kinetic term gives the propagator (=Green's function) as $1/k^2$ while the interaction terms give rise to n-point vertices $k^2\,(h_{\mu\nu})^n$. Schematically one can then check if e.g. the one loop 4-point amplitude diverges or not: this Feynman diagram has four 3-point vertices and four propagators giving the k-dependence $(k^2)^4/(k^2)^4$ which when integrated over all momenta up to the cut-off Λ behaves as Λ^4 when $\Lambda^4 \to \infty$. In other words, it is highly divergent.

Adding more internal propagators improves the behaviour and eventually makes it convergent. However, adding more external propagators does not change the divergent property and one ends up with an infinite number of divergent n-point amplitudes. This makes the theory non-renormalisable and it looses all predictive power at a fundamental level. The remedy may be string theory where all these infinities are eliminated and the corrections to Einstein's theory are well-defined and computable (but only in principle since the mathematics is not yet under control).

The reason Einstein's theory without these quantum corrections (higher powers of Riemann, Ricci and curvature scalar to arbitrary order) can make such remarkably accurate predictions (e.g., at LIGO) is that all these corrections are extremely small at ordinary energy scales E: they behave as $\frac{E}{E_{Planck}}$ to some positive power. Recall that $E_{Planck} = 1.2 \times 10^{19}\, GeV$ and CERN runs at $10^4\, GeV$. The mass of the proton is about $1\, GeV$.

[22] 1 parsec (pc) = 3.26 lightyears (ly). A year is about 10^7 seconds.