a) Write out the explicit form of the geodesic equation.
b) Check if there are time-like geodesics traversing the wormhole (i.e., that go from large negative to large positive values of r or vice versa).
c) Compute the Riemann tensor.
d) Compute the Einstein tensor and check if there are any problems to find a stress tensor satisfying the Einstein equations.
e) Find all isometries of this wormhole.

3. Find the equations of motion for a massive particle in the Schwarzschild geometry.

4. Calculate the deflection of a light ray gracing the sun's surface.

5. Using the Schwarzschild metric, find the proper length of the curves
a) $r = const$, $\theta = const$, $0 \leq \phi \leq 2\pi$,
b) $\theta = const$, $\phi = const$, $r_1 \leq r \leq r_2$.

6. Consider a massive particle moving in the geometry of the Schwarzschild metric.
a) Derive an equation for the radial coordinate $r(\tau)$ that resembles Newton's second law $m\frac{d^2r}{d\tau^2} = F(r)$ where $F(r) = -\frac{dV(r)}{dr}$ is the force due to the potential $V(r)$. Determine the so obtained effective potential $V_{eff}(r)$.
b) Are there any values of $J = r^2\dot{\phi}$ for which it is possible for the particle to be in a stable circular orbit? If so, what is radius of the orbit?
c) Specialise to the case of a photon in a circular orbit. Are there any stable or unstable such orbits? If so, what is the radius?

7. The death-defying spaceman Spiff lands on the hypothetical neutron star Buster. His spacecraft measures 20 meters in height. Buster's mass is $M = 2 \cdot 10^{30}$ kg and its radius is $R = 5$ km. How far can Spiff walk away from the ship and still see it? According to Earthly standards, Spiff is a short man, being only 1 meter tall. Compare with the classical result (i.e., without gravity). Any approximations must be motivated.

8. A galaxy acts as a gravitational lens for a very distant quasar. The galaxy is 100 Mpcs away, and the distance to the quasar can be assumed to be much greater. The image is a ring with radius 1.0 arcsec. What is the mass of the galaxy? Answer in solar masses.

4.6 Gravitational waves

1. A binary system consists of two neutron stars orbiting their common center of mass in circular orbits. Both stars have masses equal to 1.4 solar mass, and the distance between their center of mass equals the diameter of our sun. Due to emission of gravitational waves the system will lose energy and the stars will slowly spiral towards each other while the orbital frequency increases. What is the power emitted in gravitational radiation from the two stars and how fast will the orbital period decrease in time? You need to justify any approximation used.

2. A plane-fronted gravitational wave in the positive z-direction may be described by the metric

$$d\tau^2 = 2dudv - a^2(u)dx^2 - b^2(u)dy^2, \text{ where } u = \tfrac{1}{\sqrt{2}}(t-z),\ v = \tfrac{1}{\sqrt{2}}(t+z). \tag{4.35}$$

a) Compute the Ricci tensor using the coordinates (u, v, x, y). What is the condition on the functions $a(u)$ and $b(u)$ for the Ricci tensor to vanish?
b) Determine the Killing vectors and show that the infinitesimal transformations corresponding to them indeed leave the metric invariant. What is the physical interpretation of these Killing vectors?

3. A four-dimensional spacetime is given by the metric

$$d\tau^2 = 2dudv - dx^2 - dy^2 - \omega(x^2+y^2)du^2, \tag{4.36}$$

where ω is a constant. Verify that the signature is $(-1, 1, 1, 1)$ and find the Killing vectors (some are obvious and don't require any calculations). Calculate the energy-momentun tensor for the matter or radiation that acts as the source for this solution.

4. This problem is about the power emitted in the form of gravitational radiation from a rotating massive body. The moment of inertia is given by $I_{ij} = \int d^x x_i x_j \rho(\mathbf{r})$. Assume the coordinate axes are directed so that the rotation is along a principle axis (and in the z-direction) and that the moment of inertia is diagonal with $I_{ij} = diag(I_{11}, I_{22}, I_{33})$ and all elements being different.
a) Compute the formula for the power. The answer should be given as a function of the angular frequency Ω, the moment of inertia $I := I_{11} + I_{22}$ and the ellipticity $e := \frac{I_{11}-I_{22}}{I_{11}+I_{22}}$.
b) Does a body with circular symmetry around the rotation axes radiate?
c) Specialise to the case of one massive body in a circular orbit. Compute the power due to the orbital motion of Jupiter. Is it measurable?

5. Consider two massive bodies oscillating back and forth along the x-axis. This is similar to an antenna emitting electromagnetic radiation but here we assume that the (unphysical) motion of the two bodies A and B both of mass M is given by

$$x_A = \frac{a}{2}(1 - \sin \Omega\, t) = -x_B. \tag{4.37}$$

Use some every-day sounding values of the parameters like $a = 1m$, $M = 1kg$ and $\Omega = 1s^{-1}$ and compute both the energy of the system and the power emitted in the form of gravitational radiation. Is the effect detectable?

4.7 Symmetric spaces and cosmology

1. A generic Friedman-Robertson-Walker cosmology is described by the metric

$$d\tau^2 = dt^2 - a^2(t)(dx^2 + dy^2 + dz^2), \tag{4.38}$$

for some function $a(t)$. Make a suitable coordinate transformation and show that the metric can be re-written as

$$d\tau^2 = f^2(t)(dt^2 - dx^2 - dy^2 - dz^2). \tag{4.39}$$

Find the function $f(t)$ and compute the geodesic equation in the new metric.

2. Einstein originally introduced the cosmological constant into his equations so that they would allow for time-independent solutions corresponding to a static universe. Consider the Friedmann equations in the form

$$\frac{\ddot{a}}{a} = \frac{\Lambda}{3} - \frac{4\pi G}{3}(\rho + 3p), \tag{4.40}$$

$$(\frac{\dot{a}}{a})^2 = \frac{\Lambda}{3} + \frac{8\pi G}{3}\rho - \frac{k}{a^2}, \tag{4.41}$$

$$\tag{4.42}$$

where $k = 0, \pm 1$ is the parameter describing the properties of the maximally symmetric universe. Assume that ρ and $p = \omega\rho$ describe ordinary "matter", i.e., $\rho > 0$ and $\omega \geq 0$.
a) Give the condition for these equations to have static solutions. What does it imply for the geometry of the universe?
b) For a dust-dominated universe with $p = 0$, in view of a), what is the relation between a and ρ?
c) Are the solutions in a) stable?