3.10 Lectures 17 - 18: Cosmology

The cosmological principle: All positions in the universe are essentially equivalent.

1) Homogeneity: This means not in detail but at the scale of $10^8 - 10^9$ light years (clusters or super-clusters of galaxies), i.e., if matter in each cell of this size is smeared out then all cells look identical (the Copernican Principle).
Q: How is the structure observed in the universe generated?

2) Isotropy: The universe looks the same in all directions as seen from Earth.
Q: How do we understand the fine details of the non-isotropy seen in the CMB[23]?
Q: How can we explain the causality, and other, problems due to the observed isotropy over large distances?

1) and 2) together imply that the universe with its matter content is a maximally symmetric space embedded in a non-maximally symmetric spacetime: the cosmological principle.

Current observations indicate that the universe is
i) isotropic with 10^{-5} deviations (in the CMB discovered by the COBE satellite in 1992),
ii) in a state of accelerated expansion.
iii) has de Sitter geometry with a very small cosmological constant Λ. The de Sitter interpretation is however still somewhat controversial, see Di Valentino et al, Nature Astronomy, Vol. 4, February 2020, page 196-203.

One plausible solution to the questions above is *inflation*.

The universe is analysed in two steps
a) Cosmographically: The cosmological principle alone implies the Roberson-Walker (RW) metric ansatz

$$d\tau^2 = dt^2 - a^2(t)(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2), \quad k = 1, 0, -1, \tag{3.329}$$

where we will refer to $k = 0$ as *flat*, $k = +1$ as *closed* and $k = -1$ as *open*[24].
b) Cosmologically: Solving Einstein's equations with the RW metric, some non-zero stress tensor and an equation of state for the matter/radiation/dark energy content of the universe.

Note that if the whole spacetime is maximally symmetric it is regarded as a *vacuum solution*, i.e., a solution to Einstein's equations without a stress tensor but with a cosmological constant: $\Lambda = 0$ is then Minkowski space ($\mathbf{R^4}$), $\Lambda > 0$ de Sitter (dS) space ($\mathbf{R} \times S^3$) and $\Lambda < 0$ anti-de Sitter (AdS) space ($\mathbf{R^4}$ when considering the covering space).

[23]The Cosmic Microwave Background, black body radiation at 2.74 K.
[24]Note that maximally symmetric euclidean hyperbolic spaces can be made closed (and unbounded) by dividing them by subgroups of the isometry group.

3.10.1 Cosmography

In the RW metric t is the "cosmic time" and the (r, θ, ϕ) are so called "co-moving coordinates" (see below). This means that galaxies etc are located at fixed (r, θ, ϕ) and the fact that the proper distance between them is observed to increase with time is handled by the "cosmic scale factor" $a(t)$. Thus the RW metric contains two unknowns $k = 0, \pm 1$ and $a(t)$ which must be determined by observations and then explained in b) by the dynamical equations together with the equation of state.

The stress tensor $T^{\mu\nu}$ has a space-space part that is maximally symmetric, i.e., $T_{ij} \propto g_{ij}$ as a result of the theorem about form invariant second rank (having two symmetric indices) tensors. The rest of $T_{\mu\nu}$ will be discussed later together with the implications of it being conserved (divergence free).

We will need the affine connections below when discussing the geodesic motion of galaxies. If we write the RW metric more generally as

$$d\tau^2 = -g_{\mu\nu} dx^\mu dx^\nu = dt^2 - a^2(t)\tilde{g}_{ij} dx^i dx^j, \tag{3.330}$$

we can fairly easily obtain the affine connection and the Riemann tensor as follows. The non-zero spacetime metric components are

$$g_{tt} = -1, \;\; g_{ij} = a^2(t)\tilde{g}_{ij}(r, \theta, \phi), \tag{3.331}$$

where the tilde metric $\tilde{g}_{ij}$, as well as all other tilde quantities below, refer to the maximally symmetric subspace (of spacetime) which is here the universe. This gives the non-zero affine connections

$$\Gamma^t_{ij} = a\dot{a}\tilde{g}_{ij}, \;\; \Gamma^i_{tj} = \delta^i_j \frac{\dot{a}}{a}, \;\; \Gamma^i_{jk} = \tilde{\Gamma}^i_{jk}. \tag{3.332}$$

The Riemann tensor then reads

$$R^t{}_{itj} = a\ddot{a}\,\tilde{g}_{ij}, \;\; R^i{}_{jkl} = \tilde{R}^i{}_{jkl} + \dot{a}^2\,(\delta^i_k \tilde{g}_{lj} - \delta^i_l\,\tilde{g}_{kj}), \tag{3.333}$$

giving the Ricci tensor

$$R_{tt} = -3\frac{\ddot{a}}{a}, \;\; R_{ij} = (a\ddot{a} + 2\dot{a}^2)\tilde{g}_{ij} + \tilde{R}_{ij}. \tag{3.334}$$

Finally, the Ricci scalar is

$$R = 6(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}) + \frac{\tilde{R}}{a^2}. \tag{3.335}$$

The 3-dimensional maximally symmetric space-space part of the metric given by $\tilde{g}_{ij}$ leads to

$$\tilde{R}_{ij}{}^{kl} = 2k\delta^{kl}_{ij}, \;\; \tilde{R}_{ij} = 2k\tilde{g}_{ij}, \;\; \tilde{R} = 6k. \tag{3.336}$$

Note that the coordinates are all dimensionless since it is the scale factor $a(t)$ that has dimension length. These results will be used later in the cosmology part when we deal with the Einstein equations.

At this point it is convenient to discuss the geodesic equation. Using the result above it reads

$$\frac{d^2x^t}{d\tau^2} + \Gamma^t_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} = 0, \;\; \frac{d^2x^i}{d\tau^2} + \Gamma^i_{jk}\frac{dx^j}{d\tau}\frac{dx^k}{d\tau} + 2\Gamma^i_{jt}\frac{dx^j}{d\tau}\frac{dx^t}{d\tau} = 0. \tag{3.337}$$

Thus due to the fact that the coordinates used here are co-moving, i.e., matter has $\frac{dx^i}{dt} = 0$, we have $d\tau = dt$ and

$$\frac{d^2x^\mu}{dt^2} = 0, \tag{3.338}$$

i.e., the galaxies feel no gravitational force and are thus in free fall in this coordinate system (the crucial fact here is really that $\Gamma^\mu_{tt} = 0$).

Turning to the stress tensor $T^{\mu\nu}$ and the "galaxy" number density current J^μ_G, we get from imposing the maximal symmetry of 3-space (spacetime submanifold at constant t) that
T^{00} is a 3-scalar,
$T^{0i} = 0$ since it is a 3-vector,
$T^{ij} \propto \tilde{g}_{ij}$,
and similarly
J^0_G is a 3-scalar,
$J^i_G = 0$.
The *perfect fluid* assumption for the stress tensor (see SW sect. 2.10) means that the non-zero components should be written

$$T^{00} = \rho(t), \;\; T^{0i} = 0, \;\; T^{ij} = p(t)g^{ij}, \;\; J^0_G = n_G(t), \;\; J^i_G = 0, \tag{3.339}$$

where $\rho(t)$ is the energy density, $p(t)$ the pressure and $n_G(t)$ the galaxy density. Note that the metric in T^{ij} is not the tilde metric but the space-space part of $g_{\mu\nu}$. This just gives the proper interpretation of the pressure, and a nice covariant expression for $T^{\mu\nu}$, as we will see below.

In fact this stress tensor and current can be written in a covariant fashion using the co-moving 4-velocity $u^\mu = (1, 0, 0, 0)$ as follows:

$$T^{\mu\nu} = pg^{\mu\nu} + (p + \rho)u^\mu u^\nu, \;\; J^\mu_G = n_G u^\mu. \tag{3.340}$$

This is called "the perfect fluid form" of the stress tensor and is a direct consequence of the cosmological principle. Still without imposing any dynamical equations (Einstein's equations) we can get some information from the conservation laws. This will also help us to get the correct interpretation of e.g. the number density n_G. Thus imposing

$$\nabla_\mu J^\mu_G = 0 \Rightarrow \frac{d}{dt}(\sqrt{g}n_G) = 0 \Rightarrow n_G(t)a^3(t) = const \tag{3.341}$$

where we have used $g = -detg_{\mu\nu} = \frac{a^6 r^4 \sin^2\theta}{1-kr^2}$. Thus n_G is the number density per unit proper volume and $n_G a^3$ per unit coordinate volume which is constant in the co-moving coordinates.

For the stress tensor we have that only the $\nu = 0$ equation is non-trivial

$$\nabla_\mu T^{\mu\nu} = 0 \Rightarrow a^3 \dot{p} = \frac{d}{dt}(a^3(\rho + p)), \tag{3.342}$$

which can be rewritten as the so called *fluid equation*

$$\frac{\dot{\rho}}{\rho} + 3\frac{\dot{a}}{a}(1 + \frac{p}{\rho}) = 0. \tag{3.343}$$

We may also note that if we introduce an *equation of state* $p = p(\rho)$, often in the form $p = \omega\rho$ for some constant $\omega \geq -1$, we can solve the previous equation as will be done later. The components relevant for the universe are *dust* with $\omega = 0$, *radiation* with $\omega = \frac{1}{3}$ and a cosmological constant Λ corresponding to $\omega = -1$. In fact, $\frac{d\rho}{da} = \frac{\dot{\rho}}{\dot{a}}$ implies that

$$\frac{d}{da}(\rho a^3) = 3a^2\rho + \frac{d\rho}{da}\,a^3 = 3a^2\rho + \frac{\dot{\rho}}{\dot{a}}\,a^3, \tag{3.344}$$

which means that either of the two previous equations gives

$$\nabla_\mu T^{\mu\nu} = 0 \;\Rightarrow\; \frac{d}{da}(\rho a^3) = -3pa^2. \tag{3.345}$$

We thus conclude that adding an equation of state $p = p(\rho)$ makes it possible to solve for $\rho(a)$!

Finally, the proper distance to some other galaxy far away from us with coordinate r_G is

$$s(t) = \int_0^{r_G} \sqrt{g_{rr}(r')}dr' = a(t)\int_0^{r_G} \frac{dr'}{\sqrt{1-kr^2}}, \tag{3.346}$$

where the integral just gives a fixed constant number. Thus taking the time derivative gives

$$\dot{s} = \dot{a}\int_0^{r_G} \frac{dr'}{\sqrt{1-kr^2}} = \frac{\dot{a}}{a}s(t), \tag{3.347}$$

which is the *Hubble law* (1929) saying that, at any given (cosmic) time t, the velocity of a galaxy far away from us is proportional to its distance. Of course, since the universe is a maximally symmetric space this statement applies to any point in the universe not just as seen from the Earth. The Hubble law is often written, at any time t, in terms of the *Hubble parameter* as

$$v = Hs, \text{ where } H := \frac{\dot{a}}{a}. \tag{3.348}$$

The present value H_0 is called the *Hubble constant* and is sometimes written in terms the dimensionless h whose value is around 1:

$$H_0 = 100\,h\,km\,s^{-1}\,Mpc^{-1} = 3.24 \times 10^{-18}\,hs^{-1}. \tag{3.349}$$

The current value $h \approx 0.72$ given in the book by Guidry (2019) gives

$$H_0 = 72\, km\, s^{-1}\, Mpc^{-1}. \tag{3.350}$$

There seems, however, to be an uncertainty about whether the values of H_0 deduced from information of the present universe and from older information like the CMB are the same or not, see, e.g., Verde et al, ArXiv astro-ph/1907.10625. If correct it might imply that something is missing in our current understanding of the standard model of cosmology (ΛCDM).

Other related quantities that are used to describe the universe are the *Hubble length* and the *Hubble time*:

$$d_H = \tfrac{c}{H_0}, \;\; t_H = \tfrac{1}{H_0}, \tag{3.351}$$

with values $d_H \approx h^{-1} 3\, Gpc$ and $t_H \approx h^{-1} 10\, Gyr$. Another useful parameter is the *deceleration parameter* q:

$$q := -\frac{a\ddot{a}}{\dot{a}^2}. \tag{3.352}$$

A final quantity that appears very often in various formulas is the critical density ρ_c. Its meaning can be seen from a Newtonian derivation (the same result is obtained in GR) as follows. Consider a (flat) universe with matter density ρ and the total mass M inside a big sphere with radius d. The escape velocity at the surface of the big sphere is then, for a test body with mass m given by

$$\frac{1}{2} m v^2 = \frac{GMm}{d} \;\; \Rightarrow \; \rho_c = \frac{3H_0^2}{8\pi G}, \tag{3.353}$$

where we have used $M = \frac{4\pi}{3}\rho_c d^3$ and $v = H_0 d$. The current value of the critical density is

$$\rho_c(t_0) = \frac{3H_0^2}{8\pi G} = 1.88 \times 10^{-29}\, h^2\, \frac{g}{cm^3} = 1.88 \times 10^{-32}\, h^2\, \frac{kg}{m^3}. \tag{3.354}$$

From this simple interpretation of ρ_c we have three different scenarios for the universe:
$\rho_{total} > \rho_c$: the expansion will stop in finite time,
$\rho_{total} = \rho_c$: the universe will eventually stop expanding as $t \to \infty$,
$\rho_{total} < \rho_c$: the universe will never stop expanding.
The density parameter is usually expressed in terms of $\Omega = \frac{\rho}{\rho_c}$ and then for each contribution to the energy density separately. As it turns out, observations tell us that with good accuracy

$$\Omega_m + \Omega_{rad} + \Omega_\Lambda = 1, \tag{3.355}$$

where we have included the cosmological term in the perfect fluid stress tensor which means that $\Omega_\Lambda = \frac{\rho_\Lambda}{\rho_c} = \frac{\Lambda}{3H_0^2}$.

We will, however, in the next part on cosmology have reason to add also $\Omega_k := -\frac{k}{a^2 H_0^2}$ which unlike the other Ωs is not strictly positive (when $\Lambda \geq 0$). With all the Ω terms present in this equation the standard cosmology theory discussed below gives exactly 1 from which we can then deduce the value $k = 1$ provided the error bars are small enough to conclude that it is non-zero (this issue is complicated but see e.g. Nature Astronomy, Vol 4, Februari 2020, page 196-203). The connection given above between the sign of $\rho - \rho_c$ and the question whether the expansion will stop or not is no longer valid with the cosmological constant present as we will see below. Even if flat ($k = 0$) the universe can either expand or contract, and be either open or closed. Note that in the simple Newtonian argument above only matter and radiation is involved.

The values we observe today are roughly the following:
Hubble constant: $H_0 = 67.8 \pm 0.9\, km\, s^{-1}\, Mpc^{-1}$
Age of the universe: $t_0 = 13.80 \pm 0.04\, Gyr$
Matter (Ω_m): 30 percent divided into
–visible (baryonic) matter: 4 percent
–dark matter: 26 percent
Radiation (Ω_{rad}): 0,01 percent
Curvature (Ω_k): <1 percent (?)
Dark energy (Ω_Λ): 70 percent.

Note that these different kinds of "matter" depend very differently on the scale factor as will be shown below. This fact means that the above numbers must have been very different in the far past and will become very different in the future. Why the numbers are of the same order of magnitude today is in fact rather strange. This is called "the coincidence problem".

The challenge for astronomers is to determine $a(t)$, k and Λ from observations.

One quantity that is obtained from observations and can be use to determine $a(t)$ is the *red shift* λ. See Weinberg p. 415!!

3.10.2 Cosmology

Einstein: "What is so incomprehensible about the Universe is that it is comprehensible".

We now add dynamics to the descriptive picture obtained above, that is Einstein's equations, together with an equation of state for the "matter content" (this can also be dark energy) of the universe. We thus want to solve

$$R_{\mu\nu} = 8\pi G S_{\mu\nu}, \tag{3.356}$$

where

$$S_{\mu\nu} = T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T_\rho{}^\rho = \frac{1}{2}(\rho - p) g_{\mu\nu} + (\rho + p) u_\mu u_\nu, \tag{3.357}$$

that is

$$S_{tt} = \frac{1}{2}(\rho + 3p), \;\; S_{ij} = \frac{1}{2}(\rho - p)a^2\tilde{g}_{ij}. \tag{3.358}$$

Einstein's equations now read

$$tt: \;\ddot{a} = -\frac{4\pi}{3}G(\rho + 3p)a, \tag{3.359}$$

$$ij: \;a\ddot{a} + 2\dot{a}^2 + 2k = 4\pi G(\rho - p)a^2. \tag{3.360}$$

Note that the first equation indicate that an ordinary positive pressure p will decelerate the universe, not accelerate it, and a negative p will actually accelerate it.

We will now do some massage on the second of these equations. By using the first of the equations to eliminate $\ddot{a}$ from it we find that p cancels and we get

$$\dot{a}^2 + k = \frac{8\pi G}{3}\rho a^2. \tag{3.361}$$

This equation together with the tt-equation above are the *Friedmann equations* and the solution using the RW metric is the FRW universe.
A third important equation was obtained from the conservation of the stress tensor:

$$\dot{p}\,a^3 = \frac{d}{dt}(a^3(\rho + p)). \tag{3.362}$$

Evaluating the time derivative this equation becomes

$$\frac{a^3\dot{\rho}}{\dot{a}} + 3\rho a^2 = -3pa^2. \tag{3.363}$$

Using then the fact that

$$\frac{d}{da}(\rho a^3) = 3a^2\rho + \frac{d\rho}{da}\,a^3 = 3a^2\rho + \frac{\dot{\rho}}{\dot{a}}\,a^3, \tag{3.364}$$

which is just the left hand side of the previous equation, we finally find that

$$\nabla_\mu T^{\mu\nu} = 0 \;\Rightarrow\; \frac{d}{da}(\rho a^3) = -3pa^2. \tag{3.365}$$

We thus conclude that adding to these equation an equation of state

$$p = p(\rho), \tag{3.366}$$

often in the form $p = \omega\rho$, makes it possible to solve for $\rho(a)$!

The task is now to apply these basic equations to the evolution of the expanding universe. The universe can be assumed to have gone through three major, and very different, stages, namely the current matter dominated, the one preceeding it, the radiation dominated, and an early inflation phase dominated by dark energy.

Matter dominated era (from about 400.000 years after the Big Bang until today):
Here it is a good approximation to set

$$p(t) = 0. \tag{3.367}$$

Neglecting the pressure in the last basic equation above gives

$$\frac{d}{da}(\rho a^3) = 0 \Rightarrow \rho \propto a^{-3}, \tag{3.368}$$

which is the correct behaviour of the matter density when the universe is expanding.

Radiation dominated era (from the end of inflation at 10^{-32} sec to about 400.000 years after the Big Bang):
In this case we use the radiation equation of state

$$p = \frac{1}{3}\rho. \tag{3.369}$$

This has the following nice implication

$$\frac{d}{da}(\rho a^3) = -3pa^2 \Rightarrow \frac{d}{da}(\rho a^4) = 0 \Rightarrow \rho \propto a^{-4}. \tag{3.370}$$

The extra factor of a^{-1} comes from the stretching of the wavelength as the universe expands.

Dark energy dominated era (Inflation era, $10^{-35} - 10^{-32}$ sec after the Big Bang.)
Also in this case the above equations can be used provided we treat the cosmological term as part of the stress tensor. This means that we must set, neglecting all other types of matter,

$$\rho_\Lambda = \frac{\Lambda}{8\pi G} = -p_\Lambda. \tag{3.371}$$

Note that in this case the universe is actually maximally symmetric also as a spacetime. Assuming that the energy density is positive, i.e., that $\Lambda > 0$, the pressure is *negative* and it therefore wants to increase the velocity of the expansion of the universe. We can now use directly one of Einstein's equations to get the time evolution

$$\dot{a}^2 + k = \frac{8\pi G}{3}\rho_\Lambda a^2 \Rightarrow \dot{a}^2 = H^2 a^2 \Rightarrow a(t) \propto e^{Ht}, \tag{3.372}$$

where we have neglected the a independent k term since we are here considering a very small universe (a very small) but a very large (and positive) Hubble constant H given by $H^2 := \frac{\Lambda}{3}$.

Returning to the matter and radiation dominated eras, we can also there obtain the time evolution by solving $\dot{a}^2 + k = \frac{8\pi G}{3}\rho a^2$ once we know $\rho(a)$. This defines the so called *Friedmann model* defined by the three equations to be solved:

$$\dot{a}^2 + k = \frac{8\pi G}{3}\rho a^2, \tag{3.373}$$

$$\frac{d}{da}(\rho a^3) = -3pa^2, \tag{3.374}$$

$$p = p(\rho) = \omega\rho. \tag{3.375}$$

Note that the remaining Einstein's equation involving $\ddot{a}$ is contained in the first two equations above. We have also introduced the constant ω since that provides a form that often occurs. In all these equations we should use the total energy density $\rho = \rho_m + \rho_{rad} + \rho_\Lambda$. Also the k term can be included in ρ as is often done when expressing ρ relative the critical density $\rho_c := \frac{3H_0^2}{8\pi G}$ as $\Omega_i = \frac{\rho_i}{\rho_c}$. Observations, as mentioned previously, tell us that to within 1 percent

$$\Omega_m + \Omega_\Lambda + \Omega_{rad} = 1. \tag{3.376}$$

However, recalling that today $H_0 = \frac{\dot{a}}{a}$ the equation $\dot{a}^2 + k = \frac{8\pi G}{3}\rho a^2$ can be rewritten as the statement that the current value of the sum of all contributions to Ω, i.e., including $\Omega_k = -\frac{k}{a^2 H_0^2}$, is exactly equal to one:

$$\dot{a}^2 + k = \frac{8\pi G}{3}\rho a^2 \Leftrightarrow \Sigma_i \Omega_i = 1 \text{ where } i = m, \Lambda, rad, k. \tag{3.377}$$

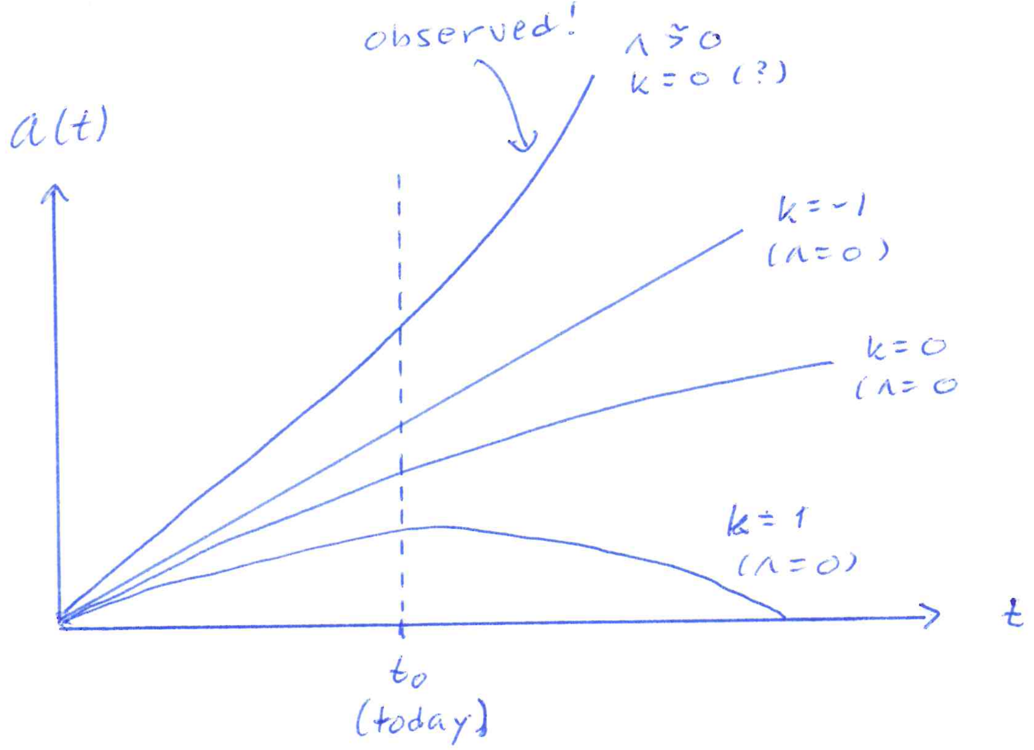
From these facts we find that with quite good accuracy that the universe is flat, i.e., that $k = 0$ which, however, is now being questioned by some groups, see Valentino et al, Nature Astronomy, Vol 4, Febr 2020, pages 196-203.

We can draw some basic conclusions about the history of the universe from these equations. Let us consider first the current or past situation ($t \leq t_0$) and then the future evolution. For $t \leq t_0$ the equation

$$\ddot{a} = -\frac{4\pi}{3}G(\rho + 3p)a \tag{3.378}$$

gives acceleration or deceleration depending on the sign of $\rho+3p$: Matter has $\rho+3p = \rho > 0$ and radiation $\rho + 3p = 2\rho > 0$ which means decelerated expansion while for dark energy $\rho + 3p = -2\frac{\Lambda}{8\pi G} < 0$ ($\Lambda > 0$ means accelerated expansion (like today and during inflation). Thus we see directly that the whether the expansion of the universe is accelerating or not is dictated by a delicate balance between the matter and dark energy content of the universe. This is independent of the value of k.

To get a feeling for how fast the universe will expand in the future we also need to discuss $\dot{a}$. Thus consider again $\dot{a}^2 + k = \frac{8\pi G}{3}\rho a^2$ in the current and future matter dominated era with $\rho \propto a^{-3}$, i.e., neglecting the dark energy. This gives a right hand side that behaves as $\rho a^2 \propto a^{-1}$ which goes to zero in the far future. In the very far future the cosmological constant will start ot dominate again if it is non-zero. But for now we assume it can be neglected which leads to three future scenarios depending on the value of k:
1) $k = -1:\ a(t) \propto t$ as soon as the ρ term is much smaller than 1,
2) $k = 0:\ \dot{a}^2 > 0$ but small since determined by $\frac{8\pi G}{3}\rho a^2 \propto a^{-1}$,
3) $k = +1:\ \dot{a}^2 = -1 + \frac{8\pi G}{3}\rho a^2$ which means that at some time in the future the right hand side will vanish. To these three cases, all corresponding to vanishing dark energy ($\Lambda = 0$), we can add the case that is the most likely interpretation of present observations namely $k = 0$ (now questioned) and $\Lambda > 0$ but very small. These four cases can be drawn in a diagram:


observed!
Λ ≠ 0
k = 0 (?)
a(t)
k = -1
(Λ = 0)
k = 0
(Λ = 0
k = 1
(Λ = 0)
t
t0
(today)