1 Dynamical systems

1.1 What are dynamical systems?

Dynamical system = Set of quantities (system) + Rule how these change with time (dynamical)
Linear dynamical systems
Most systems encountered in introductory courses.
Often exact solutions using methods based on linear superposition.
Two examples: Small-amplitude oscillations of simple pendulum ($\theta = A\cos\omega t$) and double pendulum.
Non-linear dynamical systems
Most real-world systems are (at least to some degree) non-linear
Allows for new types of solutions (compared to linear systems).
Examples: Large-amplitude oscillations of simple pendulum and double pendulum.
Angle of single pendulum no longer well approximated by $A\cos(\omega t)$.
Motion of double pendulum becomes chaotic:

- Unpredictable (appears to be random although system is deterministic).
- Sensitive dependence on initial conditions, Two arbitrarily closeby initial conditions will show different trajectories after some time.

Non-linear systems often show chaotic behaviour.

Examples where dynamical systems are encountered -

Example	Typical variables
Classical Mechanics	Positions and momenta
Electrical circuits	Currents
Population dynamics	Number of individuals of different species
Chemical reactions	Concentrations of chemicals

Plus everywhere else you encounter ODEs or recurrence equations (such as processes in living organisms, control theory, economics, etc.)

1.1.1 Mathematical description of dynamical system

Continuous dynamical systems can be written as systems of coupled ordinary differential equations:

$$
\begin{aligned}
\dot{x}_1 &= f_1(x_1, \ldots, x_n) \\
\dot{x}_2 &= f_2(x_1, \ldots, x_n) \\
&\vdots \\
\dot{x}_n &= f_n(x_1, \ldots, x_n)
\end{aligned}
$$

Time-dependent variables x_1, x_2, $\ldots x_n$ span the phase space of dimensionality n.
$\dot{x}$ denotes total time derivative: $\dot{x} \equiv \frac{\mathrm{d}}{\mathrm{d}t}x$.
Using vector notation $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{f} = (f_1, \ldots, f_n)$ we write more compactly

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$$

The vector field $\boldsymbol{f}$ is called flow and the solution $\boldsymbol{x}(t)$ is called trajectory.

Discrete dynamical systems can be written as coupled recurrence equations (on vector form):

$$\boldsymbol{x}_{i+1} = \boldsymbol{F}(\boldsymbol{x}_i)$$

$\boldsymbol{x}_i \equiv x_{1,i}, \ldots, x_{n,i}$ denotes n phase-space variables at discrete times $i = 0, 1, \ldots$.
The functions $\boldsymbol{F} = (F_1, \ldots, F_n)$ are called a map (from $\boldsymbol{x}_i$ to $\boldsymbol{x}_{i+1}$) and the solution $\boldsymbol{x}_i$ is called orbit.

Discrete dynamical systems appear upon discretisation of continuous dynamical systems, or by themselves, for example x_i could denote the population of some species a given year i.

In this course we focus on continuous dynamical systems. Discrete dynamical systems are treated in Computational Biology A (FFR110).

1.2 Example: overdamped pendulum

Consider a pendulum that is so heavily damped that oscillations are suppressed. The angle θ of such pendulum satisfies:

$$\dot{\theta} = -\sin\theta\,. \tag{1}$$

This equation is non-linear but solvable by separation of variables:

$$\frac{1}{\sin\theta}\mathrm{d}\theta = -\mathrm{d}t$$

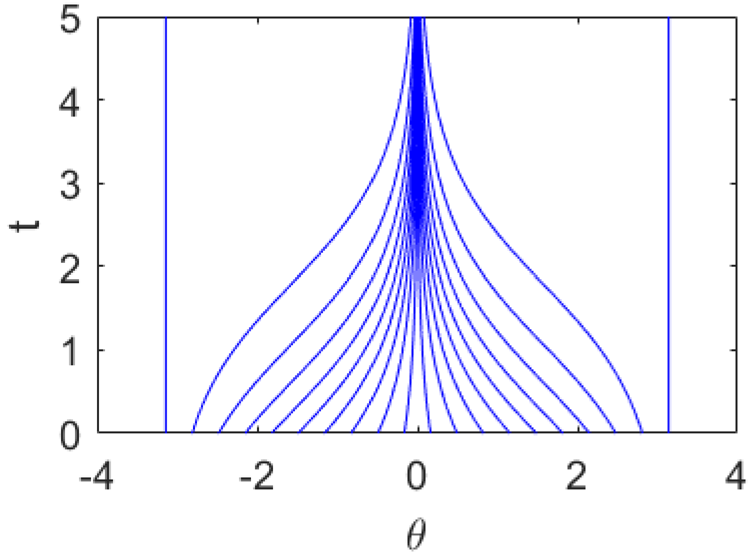
Integrate from $t = 0$ to T and from $\theta(0) \equiv \theta_0$ to $\theta(T) \equiv \theta_T$

$$I = \int_{\theta_0}^{\theta_T} \frac{1}{\sin\theta}\mathrm{d}\theta = \cdots = \left[\ln\left(\tan(\theta/2)\right)\right]_{\theta_0}^{\theta_T}$$
$$I = -\int_0^T \mathrm{d}t = -T$$

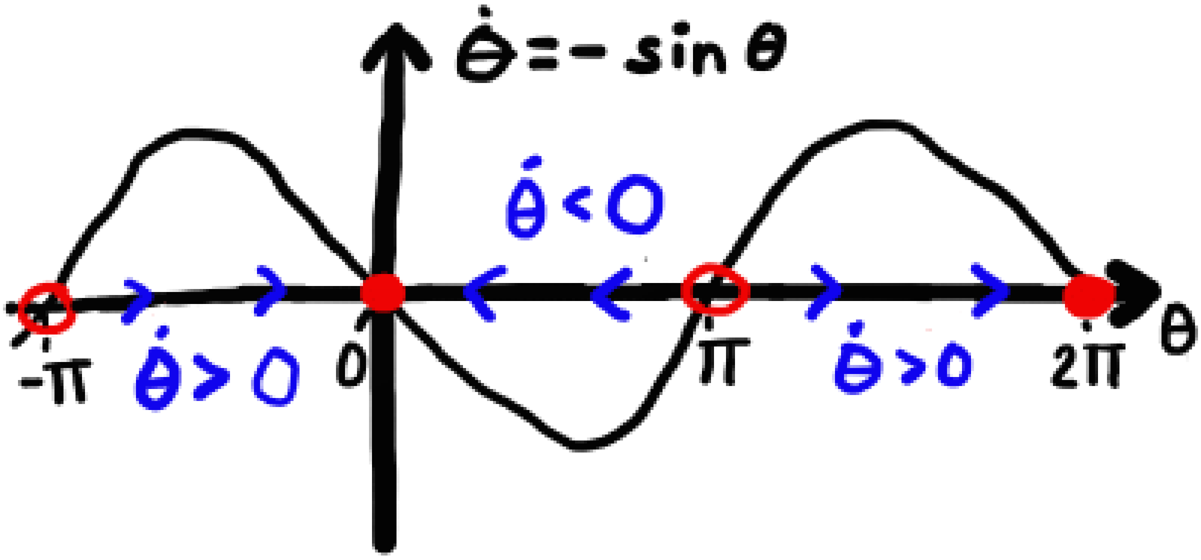
In conclusion

$$\ln\left(\frac{\tan(\theta_T/2)}{\tan(\theta_0/2)}\right) = -T \;\Rightarrow\; \boxed{\theta(t) = 2\mathrm{atan}(e^{-t}\tan(\theta_0/2))}$$

Trajectories starting with $-\pi < \theta_0 < \pi$ converge to $\theta = 0$ as $t \to \infty$ and trajectories starting at $\theta_0 = \pm\pi$ remain at $\pm\pi$:



Solution using a dynamical systems approach It is easier to solve the system geometrically. Plot $\dot{\theta} = f(\theta)$ against θ:



Arrows denote the directions of trajectories along the line (c.f. exact trajectories in previous figure).
Points with no flow ($\dot{\theta} = 0$): fixed points (also called: equilibrium points or steady states) correspond to constant solutions of the ODE.
● Stable fixed point (attractor/sink). Surrounding flow is directed towards the fixed point ⇒ dynamics is stable to small perturbations.
○ Unstable fixed point (repeller/source). Surrounding flow is directed away from the fixed point ⇒ small deviations from the fixed point grow with time, the fixed point is unstable to small perturbations.

The geometric solution gives the qualitative picture: all trajectories end up at $\theta = 0$ (or multiples of 2π), unless they start exactly at an unstable fixed point. Some details are missing but often it is enough to have qualitative information about the solution.

1.3 Flows on the line

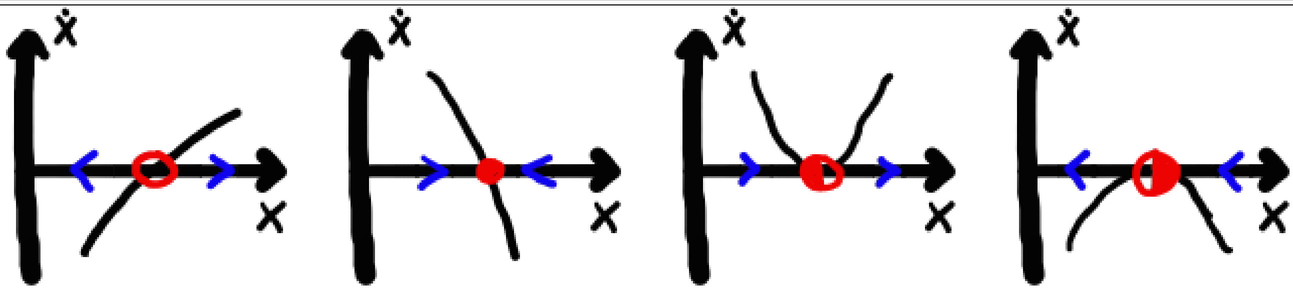
Dynamical systems of phase-space dimensionality $n = 1$

$$\dot{x} = f(x)$$

f is smooth and real-valued. x takes any real value. No explicit time dependence in f. One example is given in Eq. (1).

1.3.1 Types of fixed points

Assume x^* is an isolated fixed point on the line, $f(x^*) = 0$. The possible types are summarized as follows:



Type	Unstable ○	Stable ●	Half stable/semi stable ◐, ◑	
Slope	$f'(x^*) > 0$	$f'(x^*) < 0$	$f'(x^*) = 0$	$f'(x^*) = 0$
Example	$f(x) = x$	$f(x) = -x$	$f(x) = x^2$	$f(x) = -x^2$

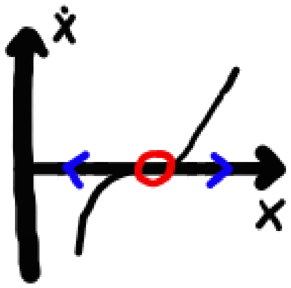
Half-stable fixed points:

◐ Dynamics attracted to the left of fixed point, repelled to the right.

◑ Repelled to the left, attracted to the right.

The case $f'(x^*) = 0$ is called marginal.

Note that $f'(x^*) = 0$ is not a sufficient condition for a fixed point to be half-stable, for example $f(x) = x^3$ is unstable:



2 Bifurcations and catastrophes

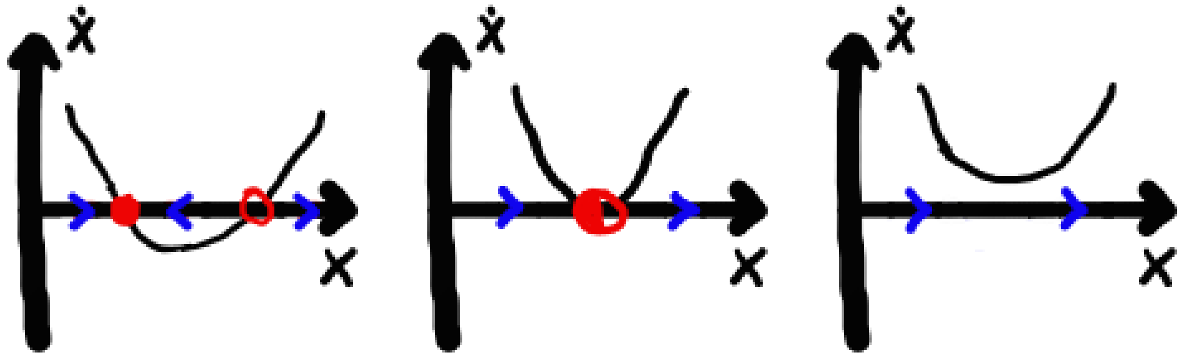
A bifurcation is a qualitative change in the dynamics (for example creation/annihilation or change in stability of fixed points) as a system parameter is varied. A bifurcation point is the value of the parameter where the bifurcation occurs.

2.1 Saddle-node bifurcation

Consider the system

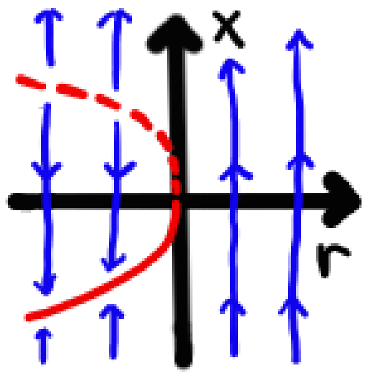
$$\dot{x} = r + x^2$$

for negative, zero, and positive values of r:



$r < 0$ $r = 0$ $r > 0$

As the bifurcation parameter r passes the bifurcation point r_c, two fixed points (one unstable and one stable) merge and disappear:



This is a bifurcation diagram, i.e. a plot of fixed points against the bifurcation parameter (often plotted without the blue flow). In bifurcation diagrams, solid lines denote stable fixed points and dashed lines denote unstable ones. The bifurcation at $r = 0$ is a saddle-node bifurcation. Saddle-node bifurcations is the typical mechanism for creation|annihilation of fixed points.

2.2 Analytical analysis

The geometrical approach considered so far gives the qualitative behaviour of the dynamics. To get more quantitative predictions, we consider analytical approaches.

2.2.1 Linear stability analysis

Consider general flow, $\dot{x} = f(x)$, with a fixed point $x = x^*$: $f(x^*) = 0$. A small deviation $\eta(t) = x(t) - x^*$ from the fixed point x^* evolves according to

$$\dot{\eta} = \dot{x} - \frac{\mathrm{d}}{\mathrm{d}t}x^* = \dot{x} = f(x)$$

Series expand the flow around the fixed point:

$$\begin{aligned}\dot{\eta} = f(x) &= \underbrace{f(x^*)}_{=0} + f'(x^*)\underbrace{(x - x^*)}_{=\eta} + \frac{1}{2}f''(x^*)\underbrace{(x - x^*)^2}_{=\eta^2} + \ldots \\ &\approx f'(x^*)\eta\end{aligned}$$

Solution:

$$\eta = \eta_0 e^{f'(x^*)t}$$

This is the general form of the solution close to an isolated fixed point. $\lambda = f'(x^*)$ is the stability exponent (a constant number); $1/|\lambda|$ is the characteristic time scale of the solution close to x^* (stability time). Note that when $\lambda < 0$ the deviation from the fixed point decreases exponentially fast, but the fixed point is not reached ($\eta = 0$) in a finite time.

For the saddle-node bifurcation above we have $f(x) = r + x^2$ and $f'(x) = 2x$:

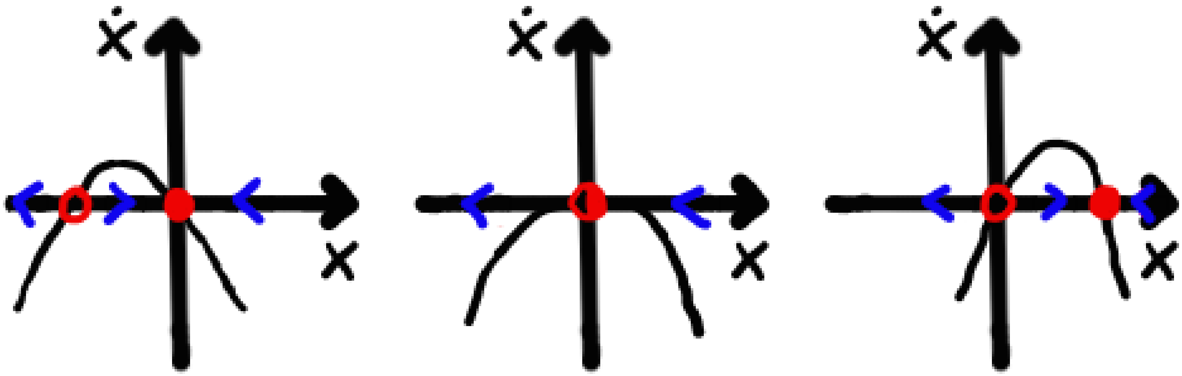
Parameter range	Fixed points	Stability exponents
$r < 0$	$x_1^* = -\sqrt{-r}$ $x_2^* = \sqrt{-r}$	$\lambda_1 = -2\sqrt{-r}$ (stable) $\lambda_2 = 2\sqrt{-r}$ (unstable)
$r = 0$	$x^* = 0$	$\lambda = 0$ (marginal)
$r > 0$	—	—

Note: The direction of a flow on the line is uniquely determined everywhere by its fixed points. Bifurcations only occur when fixed points are created, destroyed, or change stability. All these require $f'(x^*) = 0$, which is a necessary condition for bifurcations in flows on the line.

2.3 Transcritical bifurcation

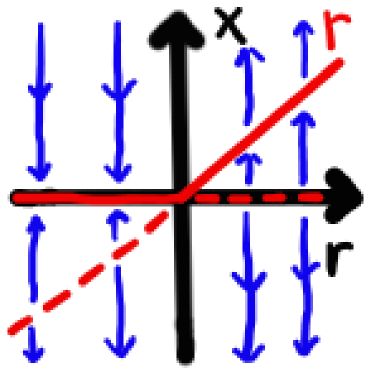
A transcritical bifurcation occurs when a fixed point exists for all values of a bifurcation parameter r surrounding r_c, but changes stability as r passes r_c. As for the saddle-node bifurcation, it is possible to derive a normal form valid close to any transcritical bifurcation:

$$\dot{x} = x(r - x) \tag{2}$$



$r < 0$ $r = 0$ $r > 0$

The normal form has a fixed point at $x^* = 0$ for all values of r, but stability changes as r passes the bifurcation point $r_c = 0$:



2.3.1 Example: Logistic growth

Let $N(t)$ be the population size of a species at time t. Assume that N changes due to births or deaths (no migration). Linear model (Malthus 1798):

$$\dot{N} = \underbrace{bN}_{b=\text{per capita birth rate } (b>0)} - \underbrace{dN}_{d=\text{per capita death rate } (d>0)}$$

Solution: $N(t) = N(0)e^{rt}$, with per capita growth rate $r \equiv b - d$. If $r > 0$ the population grows without bound. This is unrealistic, we expect population sizes to be limited due to a finite amount of resources and space. One way to model this limitation is to modify the per capita growth rate to decrease linearly with population size,

$$r \to r(1 - N/K)\,,$$

with a positive carrying capacity K. This gives a non-linear growth model

$$\dot{N} = Nr(1 - N/K)\,.$$

This is the Logistic equation (Verhulst 1836). The system has two fixed points $N_1^* = 0$ and $N_2^* = K$.
Introducing the rescaled variable $x = rN/K$ we obtain the normal form for transcritical bifurcations (2):

$$\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}N}\dot{N} = \frac{r}{K}Nr(1 - N/K) = x(r - x)\,.$$

Following the corresponding bifurcation diagram above, we have:

- For $r < r_{\rm c} = 0$ the birth rate is smaller than the death rate and the population goes extinct for any initial population size (the fixed point $x_1^* = 0$ is stable and $x_2^* = r$ is negative (unphysical)).
- For $r > r_{\rm c} = 0$ the population approaches the maximal sustainable limit for any initial population size (the fixed point $x_1^* = 0$ is unstable and $x_2^* = r$ is positive and stable).

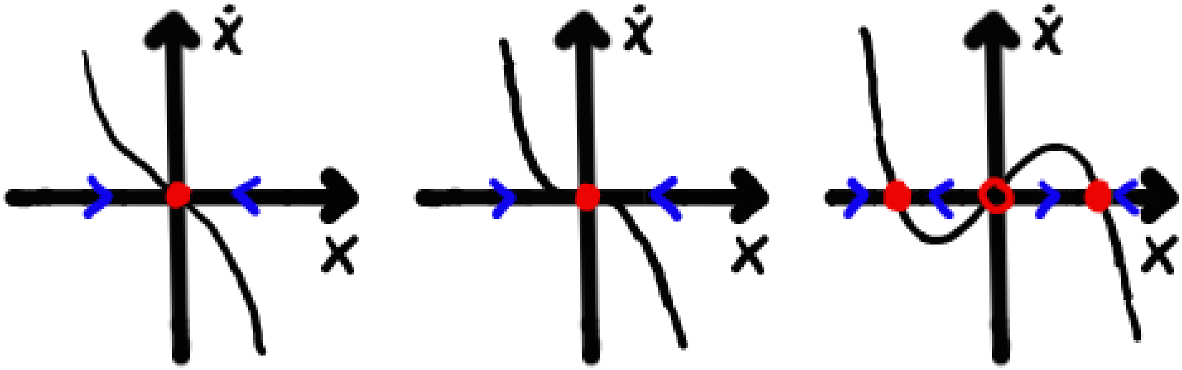
2.4 Pitchfork bifurcation

In a pitchfork bifurcation one fixed point splits into three.
The pitchfork bifurcation can be either supercritical or subcritical.
At bifurcation point, we must have triple root: $f(x^*) = f'(x^*) = f''(x^*) = 0$.

2.4.1 Supercritical pitchfork bifurcation

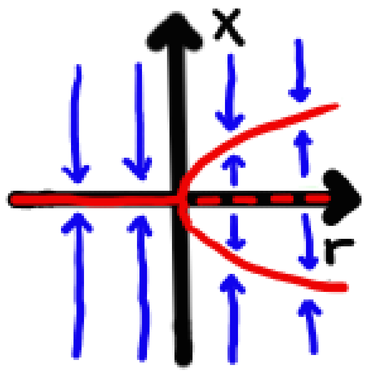
Normal form of supercritical pitchfork bifurcations:

$$\dot{x} = x(r - x^2)$$



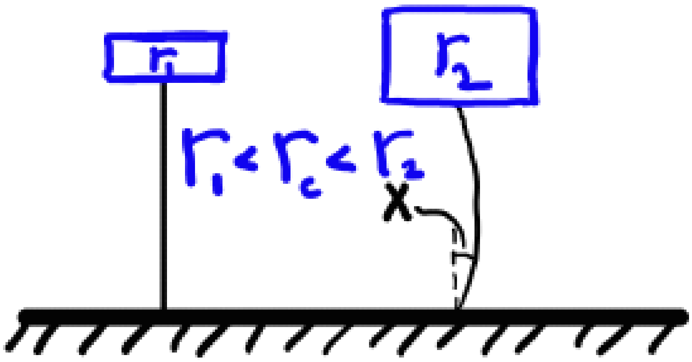
$r < 0$ $r = 0$ $r > 0$

Bifurcation diagram:



Example: Buckling of elastic ruler It may seem unlikely that three fixed points join at one point, but this often happens in systems with mirror symmetry (equations invariant under $x \to -x$).

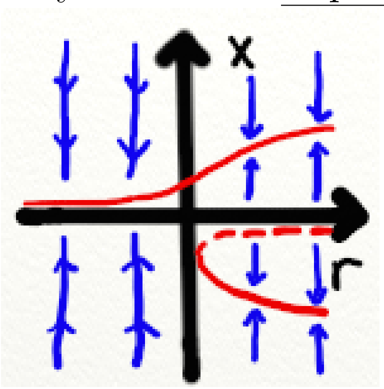
As an example, consider an up-standing perfectly mirror symmetric elastic ruler with a weight applied from above. Let r be the the mass of the weight and let x be the 'buckling angle':



The ruler can sustain a small weight r without deformation. If r is increased above a threshold (the bifurcation point r_c), the slightest asymmetry in the applied mass causes the ruler to buckle in the direction determined by the asymmetry. When the mass is lightened, the ruler moves back towards its original state ($x^* = 0$).

2.4.2 Imperfect bifurcation and catastrophes

If the symmetry of the ruler in the example above is not perfect, we may obtain an imperfect bifurcation.



Here small initial buckling angles in either direction makes the ruler buckle towards positive x. However, a large enough negative initial buckling angle makes the ruler buckle in the opposite direction (lower branch on the saddle-node bifurcation). Note that if the mass is slowly decreased from this state, the ruler makes a sudden switch to positive x as r becomes smaller than the saddle-node bifurcation point. This jump in the state of the system is a catastrophe (sudden change in state). If r is once again increased, the ruler does not flip back to negative x (hysteresis).

Cusp catastrophe Imperfect bifurcations are often described by addition of an imperfection parameter h to the normal form. For the supercritical pitchfork bifurcation we obtain:

$$\dot{x} = x(r - x^2) + h\,.$$

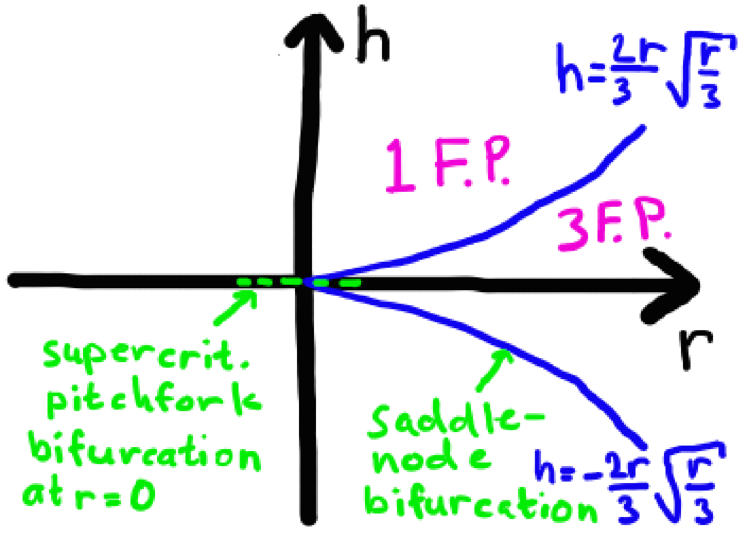
This is a two-parameter problem. When the perturbation h is zero, the normal form is reobtained. As discussed earlier, a necessary condition for bifurcations of fixed points is that both $f(x^*) = 0$ and $f'(x^*) = 0$. The condition $f'(x^*) = 0$ gives

$$0 = \frac{\partial}{\partial x}[x(r - x^2) + h]|_{x=x^*} = r - 3(x^*)^2$$

Inserting the solution $x^* = \pm\sqrt{r/3}$ into the condition $f(x^*) = 0$ gives

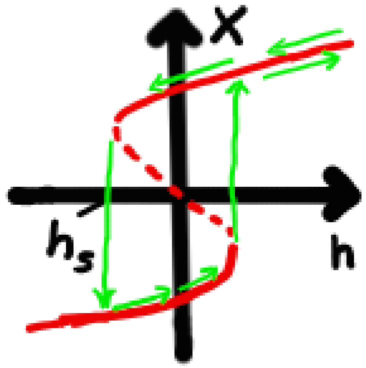
$$0 = \pm\sqrt{\frac{r}{3}}\left(r - \left[\pm\sqrt{\frac{r}{3}}\,\right]^2\right) + h \quad \Rightarrow \quad h = \mp\frac{2}{3}r\sqrt{\frac{r}{3}}$$

Thus, bifurcations involving at least two fixed points occur at curves $h = \mp\frac{2}{3}r\sqrt{\frac{r}{3}}$:



These curves separates regions with one fixed point from regions with three fixed points. For the bifurcation to involve three fixed points we must have a triple root, i.e. $0 = f''(x^*) = -6x^*$. This condition is only satisfied when $r = h = 0$. We can therefore conclude that the bifurcations occurring along $h = \mp\frac{2}{3}r\sqrt{\frac{r}{3}}$ with $h \neq 0$ involves two fixed points that are created out of the blue (saddle-node bifurcations), just as in the figure illustrating an imperfect bifurcation in example with the ruler above.

The bifurcation curve above is an example of a cusp catastrophe (named so because the two branches of saddle-node bifurcations meet tangentially in a cusp (peak) at the origin). The bifurcation diagram along constant $r > 0$ in the figure above is:



Assume that the system starts at the top fixed point with a large value of h. When h is decreased, the system eventually moves over the left saddle-node bifurcation point, h_s, and makes a big jump to a fixed point far away (a catastrophe). After the jump the system does not revert back to the original fixed point by a small increase in h (hysteresis). To move back to the original fixed point (remaining at constant r) we must increase h beyond the right saddle point, where a new jump (catastrophe) occurs (forming a hysteresis loop).

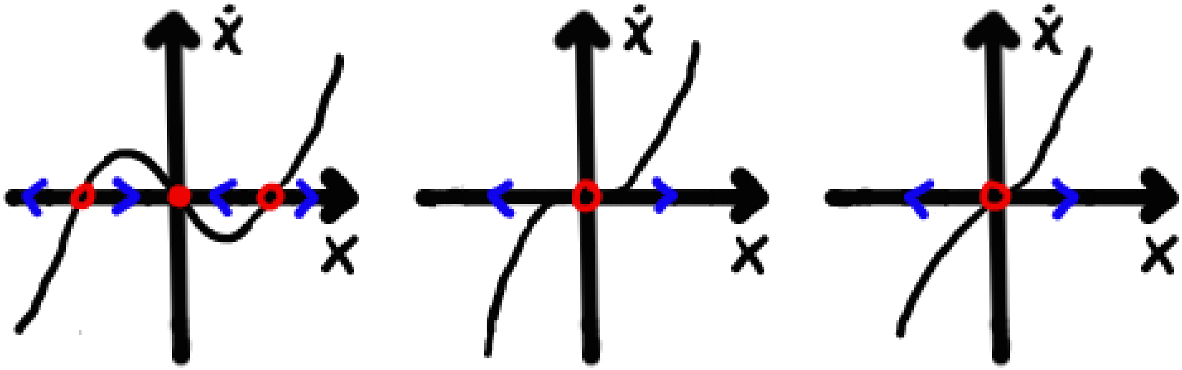
Some examples on catastrophes:

- A sudden change in equilibrium could be catastrophic for buildings and other constructions.
- The problem of hysteresis could be catastrophic for ecological systems: if the system makes a big jump to a new equilibrium (for example due to human influence), it may be very hard to restore the system to its original state due to hysteresis.
- Models in behavioural sciences [Scientific American article by Zeeman (1976)]

2.4.3 Subcritical pitchfork bifurcation

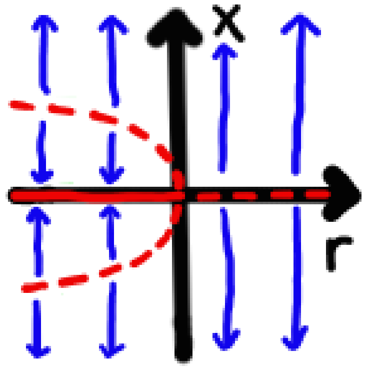
Normal form of subcritical pitchfork bifurcations:

$$\dot{x} = x(r + x^2)$$



$r < 0$ $r = 0$ $r > 0$

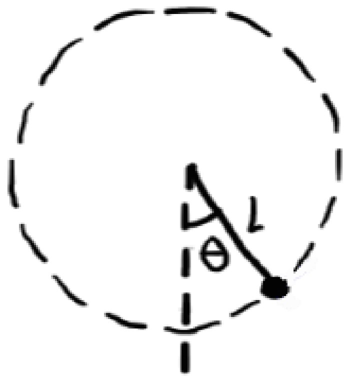
Bifurcation diagram:



As for the supercritical case, we have a stable fixed point at $x^* = 0$ for $r < r_c$. When r passes r_c there are no stable fixed points and a small deviation from $x = 0$ grows to infinity in a finite time (blow-up due to the cubic dynamics). Most physical systems have higher-order non-linear corrections that counteract the blow-up (the pitchfork bifurcation happens locally at small x and the system may have other fixed points at larger values of $|x|$). However, the system must make a jump to the new fixed points making subcritical pitchfork bifurcations potentially dangerous, similar to the catastrophes discussed in Section 2.4.2.

3 Linear 2D flows

3.1 Example: Rigid pendulum



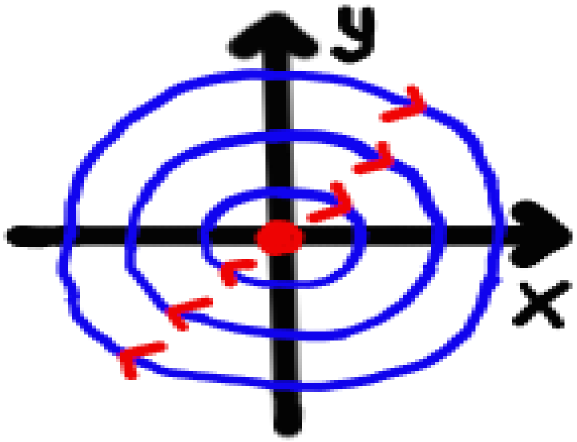
Angular dynamics of a damped pendulum of length l and mass m:

$$\ddot{\theta} = -\frac{g}{l}\sin\theta - \frac{\gamma}{m}\dot{\theta}\,. \tag{3}$$

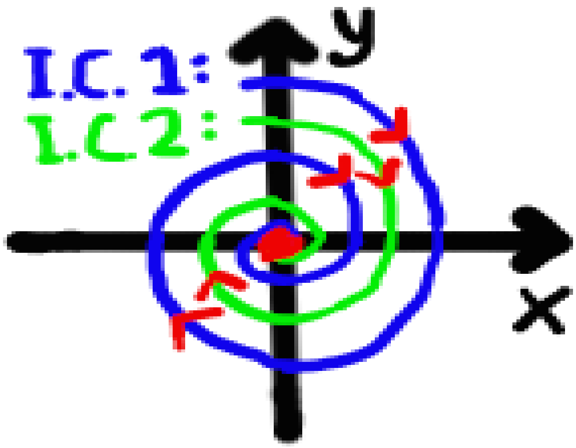
Here g is gravitation acceleration and γ is a damping rate. Consider small oscillations, $\sin\theta \approx \theta$ and write as a dynamical system with $x = \theta$, $y = \dot{\theta}$

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= -\frac{g}{l}x - \frac{\gamma}{m}y\end{aligned}$$

This is an example of a linear flow. It has a fixed point at $x^* = y^* = 0$. As for the one-dimensional systems we do a geometrical visualisation of a few representative trajectories (phase portrait) to understand the dynamics close to the fixed point. The trajectories are obtained by integration of the dynamical system starting from a suitable set of initial positions (x_0, y_0) (or by using StreamPlot[] in Mathematica):



$\gamma = 0$



$\gamma > 0$

Case $\gamma = 0$: the fixed point is surrounded by closed orbits in the form of ellipses of infinite density (which orbit is chosen depends on the initial condition). The fixed point is a center: nearby trajectories neither approach nor depart from it.
Physical interpretation: The fixed point $x^* = y^* = 0$ corresponds to the pendulum at rest, $\theta = \dot{\theta} = 0$. Non-zero initial conditions give closed orbits, corresponding to oscillations in the underlying dynamics [c.f. the ellipses formed by the explicit solution $(x, y) = (\theta, \dot{\theta}) = A_0(\cos(\omega_0 t + \phi_0), -\omega_0 \sin(\omega_0 t + \phi_0))$ with $\omega_0 = \sqrt{g/l}$].
Case $\gamma > 0$: the fixed point is a stable spiral: trajectories spiral inward towards the fixed point.
Physical interpretation: Due to the viscous damping ($\gamma > 0$) the magnitude of oscillations decreases with time.

3.2 Classification of linear flows

Two-dimensional flows have several additional types of fixed points compared to one-dimensional flows.
To find all possible types, consider a general linear flow (neglect constant terms, since they correspond to constant shifts in x and y):

$$\dot{x} = ax + by$$
$$\dot{y} = cx + dy$$

On matrix form:

$$\dot{\boldsymbol{x}} = \mathbb{A}\boldsymbol{x}\ ,\ \mathbb{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\ . \tag{4}$$

Assume $\mathbb{A}$ is diagonalizable, $\mathbb{A} = \mathbb{P}\mathbb{D}\mathbb{P}^{-1}$ with eigenvalue matrix

$$\mathbb{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

and P is a matrix spanned by the eigenvectors of $\mathbb{A}$. Then Eq. (4) can be written as

$$\begin{aligned} &\dot{\boldsymbol{x}} = \mathbb{P}\mathbb{D}\mathbb{P}^{-1}\boldsymbol{x} \\ \Rightarrow &\frac{\mathrm{d}}{\mathrm{d}t}[\mathbb{P}^{-1}\boldsymbol{x}] = \mathbb{D}\underbrace{\mathbb{P}^{-1}\boldsymbol{x}}_{\boldsymbol{\xi}} \\ \Rightarrow &\dot{\boldsymbol{\xi}} = \mathbb{D}\boldsymbol{\xi} \\ \Rightarrow &\boldsymbol{\xi}(t) = (e^{\lambda_1 t}\xi_1(0), e^{\lambda_2 t}\xi_2(0)) \end{aligned}$$

For the case of complex eigenvalues $\lambda = \mu \mp \mathrm{i}\omega$ with corresponding eigenvectors $\boldsymbol{v}$ and $\overline{\boldsymbol{v}}$, this solution becomes complex. Then choose

$$\mathbb{D} = \begin{pmatrix} \mu & -\omega \\ \omega & \mu \end{pmatrix}\ , \qquad \mathbb{P} = \begin{pmatrix} \mathrm{Re}[v_1] & \mathrm{Im}[v_1] \\ \mathrm{Re}[v_2] & \mathrm{Im}[v_2] \end{pmatrix}$$

such that $\boldsymbol{\xi}$ is real and $\dot{\boldsymbol{\xi}} = \mathbb{D}\boldsymbol{\xi}$, with solution (let $\xi_2(0) = 0$)

$$\boldsymbol{\xi}(t) = \xi_1(0)e^{\mu t}(\cos(\omega t), \sin(\omega t))$$

The solutions $\boldsymbol{\xi}(t)$ show the prototypic behaviour of trajectories in linear systems and is quantified by λ_1 and λ_2.
In the solution of the original problem, $\boldsymbol{x}(t) = \mathbb{P}\boldsymbol{\xi}(t)$, directions are rotated and rescaled compared to $\boldsymbol{\xi}$, but the topological properties of the system are the same (structure of trajectories is rotated and stretched but the relative order between trajectories remain intact).

The eigenvalues are determined by the characteristic equation:

$$0 = \det(\mathbb{A} - \lambda\mathbb{I}) = \lambda^2 - \tau\lambda + \Delta$$

with

$$\tau = \mathrm{Tr}\mathbb{A}$$
$$\Delta = \det\mathbb{A}\,.$$

The solutions of the characteristic equation are:

$$\lambda_1 = \frac{\tau + \sqrt{\tau^2 - 4\Delta}}{2}\,, \qquad \lambda_2 = \frac{\tau - \sqrt{\tau^2 - 4\Delta}}{2} \tag{5}$$

Example: Rigid pendulum in a viscous medium

$$\mathbb{A} = \begin{pmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{\gamma}{m} \end{pmatrix}$$

We have $\tau = -\frac{\gamma}{m}$, $\Delta = \frac{g}{l}$. Case $\gamma = 0$:

$$\lambda_1 = \mathrm{i}\sqrt{\frac{\mathrm{g}}{\mathrm{l}}}$$
$$\lambda_2 = -\mathrm{i}\sqrt{\frac{\mathrm{g}}{\mathrm{l}}}$$

As we saw in Section 3.1 this fixed point is a center. The eigenvalues are imaginary and the values correspond to the angular frequency $\omega_0 = \sqrt{g/l}$.

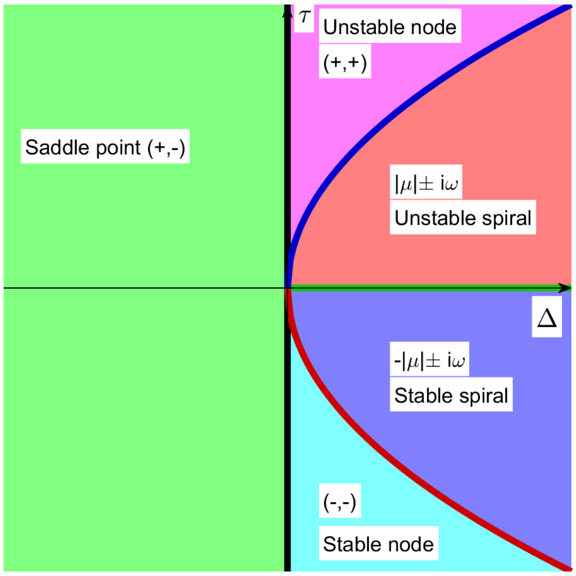
Case $\gamma > 0$ but small:

$$\lambda_1 = \frac{-\gamma/m + \mathrm{i}\sqrt{4\mathrm{g}/\mathrm{l} - (\gamma/\mathrm{m})^2}}{2} = -\frac{\gamma}{2m} + \mathrm{i}\sqrt{\frac{\mathrm{g}}{\mathrm{l}} - \frac{\gamma^2}{2\mathrm{m}^2}}$$
$$\lambda_2 = \frac{-\gamma/m - \mathrm{i}\sqrt{4\mathrm{g}/\mathrm{l} - (\gamma/\mathrm{m})^2}}{2} = -\frac{\gamma}{2m} - \mathrm{i}\sqrt{\frac{\mathrm{g}}{\mathrm{l}} - \frac{\gamma^2}{2\mathrm{m}^2}}$$

As we saw in Section 3.1 this fixed point is a stable spiral. It shows oscillating behaviour with angular frequency $\sqrt{g/l - \gamma^2/(2m^2)}$. The negative real part of the eigenvalues decreases the magnitude of the oscillations exponentially with time.

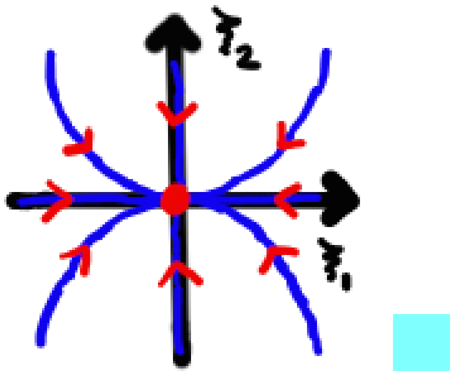
3.2.1 Different possibilities (the 'Zoo' of fixed points)

The type of fixed point depends on the relative sign of $\mathrm{Re}[\lambda_1]$ and $\mathrm{Re}[\lambda_2]$ and on whether $\mathrm{Im}[\lambda_{1,2}]$ vanishes or not. All fixed points can be classified in five major types plus a number of boundary cases. Parameterizing the eigenvalues by Δ and τ as in Eq. (5) we have:

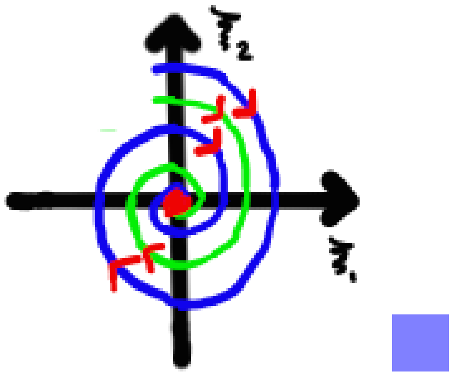


3.2.2 Major types

Stable fixed points If $\mathrm{Re}[\lambda_1] < 0$ and $\mathrm{Re}[\lambda_2] < 0$ the fixed point is stable: trajectories from all initial conditions move towards it. Moreover, if $\mathrm{Im}[\lambda] = 0$ we have a stable node, otherwise a stable spiral.

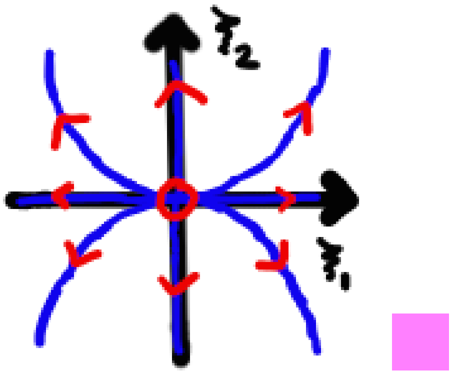


Stable node
$\tau < 0\,,\ 0 < \Delta < \tau^2/4$

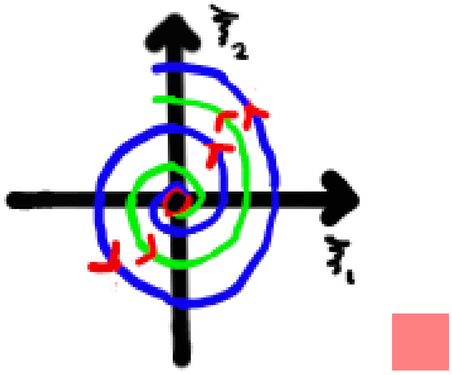


Stable spiral
$\tau < 0\,,\ \tau^2/4 < \Delta$

Unstable fixed points If $\mathrm{Re}[\lambda_1] > 0$ and $\mathrm{Re}[\lambda_2] > 0$ the fixed point is unstable: trajectories from all initial conditions move away from it.

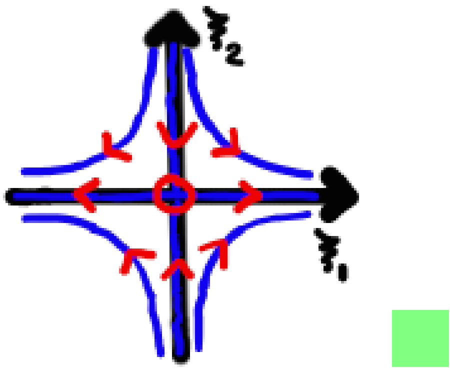


Unstable node
$\tau > 0\,,\; 0 < \Delta < \tau^2/4$



Unstable spiral
$\tau > 0\,,\; \tau^2/4 < \Delta$

Saddle points (unstable) If $\mathrm{Re}[\lambda_1] > 0$ and $\mathrm{Re}[\lambda_2] < 0$ the fixed point is a saddle point: it attracts in one direction and repels in another.



Saddle point
$\Delta < 0$

3.2.3 Boundary types

The boundaries between the different regions give rise to additional kinds of fixed points. We will not put focus on them in this course. One example is the centers we encountered for the undamped pendulum. These have $\mathrm{Re}[\lambda_1] = \mathrm{Re}[\lambda_2] = 0$ and $\mathrm{Im}[\lambda \neq 0]$ and lie on the green line line $\Delta > 0$ and $\tau = 0$ in the diagram above.

4 Phase plane

The previous section dealt with linear two-dimensional flows. This section considers non-linear two-dimensional flows living in a phase space of dimensionality two: the phase plane.

4.1 Geometrical approach: Phase portraits

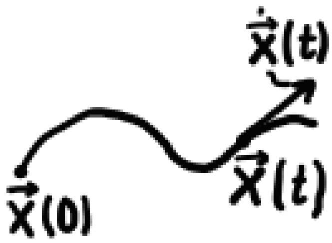
Consider a general dynamical systems of dimensionality two:

$$\dot{\boldsymbol{x}} = \boldsymbol{f}$$

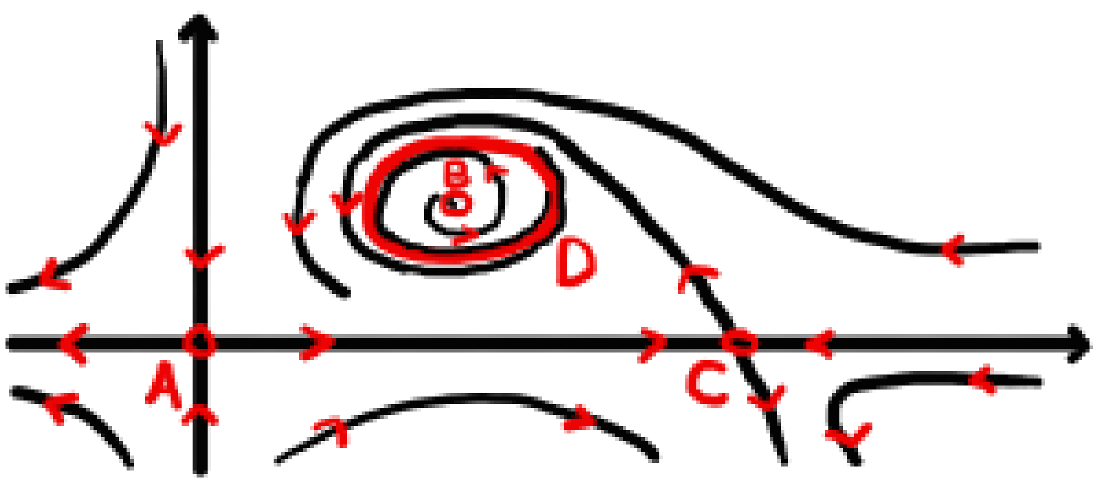
with $\boldsymbol{x} = (x_1, x_2)$ and $\boldsymbol{f}(\boldsymbol{x}) = (f_1(\boldsymbol{x}), f_2(\boldsymbol{x}))$. To have a cleaner notation without indices, we often use

$$x = x_1\,, \quad y = x_2\,, \quad f = f_1\,, \quad g = g_2\,, \quad \Rightarrow \quad \begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases}$$

The trajectory $\boldsymbol{x}(t)$ depends on the initial condition $\boldsymbol{x}(0)$:



In non-linear systems it is usually not possible to find $\boldsymbol{x}(t)$ analytically. Phase portraits are typically much more complicated compared to the linear flows considered so far. One example:



- Fixed points (A,B,C)

$$\boldsymbol{f}(\boldsymbol{x}^*) = 0$$

- Closed orbits (D) [periodic solution $\boldsymbol{x}(t) = \boldsymbol{x}(t+T)$].
- Arrangement of trajectories near different fixed points and different closed orbits may differ:
  - A, C saddle
  - B spiral
- Stability
  - A, B, C unstable
  - D stable

As for the one-dimensional case: if the flow is smooth, the initial-value problem has a unique solution. As a consequence different trajectories cannot intersect. If they did, there would be two solutions starting from the point of intersection, i.e. breaks the uniqueness condition.

4.1.1 Numerical computation of phase portraits

Using for example Matlab or Mathematica, one can use the built-in functions, e.g. StreamPlot[] in Mathematica to plot the flow, or NDSolve[] to find the trajectories.

4.1.2 Sketching the phase portrait by hand

To draw a phase portrait by pen and paper, it is often instructive to first determine the nullclines. These are the curves defined by

$$\dot{x} = 0 \qquad \text{or} \qquad \dot{y} = 0\,.$$

Along the nullclines the flow is either vertical ($\dot{x} = 0$) or horizontal ($\dot{y} = 0$). They divide the phase plane into regions where direction of flow is known or approximately known:

	$\dot{x} < 0$	$\dot{x} = 0$	$\dot{x} > 0$
$\dot{y} < 0$	$\swarrow$	$\downarrow$	$\searrow$
$\dot{y} = 0$	$\leftarrow$	$\cdot$	$\rightarrow$
$\dot{y} > 0$	$\nwarrow$	$\uparrow$	$\nearrow$

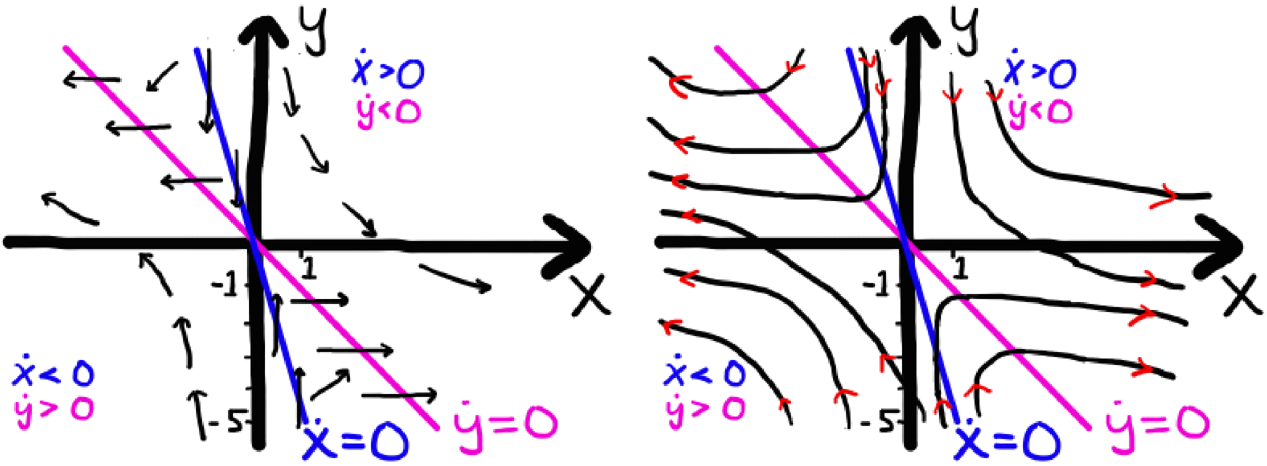
Intersection points between a nullcline with $\dot{x} = 0$ and one with $\dot{y} = 0$ give the fixed points of the flow. Since trajectories are not allowed to cross, the information given by the nullclines often allows to make a qualitative plot of the dynamics.

Linear example

$$\begin{aligned}\dot{x} &= 5x + y\\ \dot{y} &= -x - y\end{aligned}$$

Nullclines:

$$\begin{aligned}\dot{x} = 0: &\quad y = -5x\\ \dot{y} = 0: &\quad y = -x\end{aligned}$$



From the plotted trajectories we see that the fixed point at the intersection of the nullclines is a saddle point.

Consistency check:

$$\mathbb{A} = \begin{pmatrix} 5 & 1 \\ -1 & -1 \end{pmatrix} \quad \Rightarrow \quad \Delta = \det \mathbb{A} = -4 \quad \Rightarrow \quad \text{Saddle point}$$

4.2 Analytical approach: Linear stability analysis

A dynamical system of dimensionality two

$$\begin{aligned}\dot{x} &= f(x,y)\\ \dot{y} &= g(x,y)\end{aligned}$$

has fixed points (x^*, y^*) where $f(x^*, y^*) = g(x^*, y^*) = 0$. Linearize around the fixed point (c.f. Section 2):

$$\begin{aligned}&\eta = x - x^*\,, \qquad \mu = y - y^*\\ &\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\eta\\ \mu\end{pmatrix} = \mathbb{J}(x^*,y^*)\begin{pmatrix}\eta\\ \mu\end{pmatrix} + \dots\,, \qquad \text{with } \mathbb{J}(x^*,y^*) = \begin{pmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y}\\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}\end{pmatrix} \qquad (6)\end{aligned}$$

where $\mathbb{J}$ is called stability matrix, Jacobian matrix, or community matrix, and the derivatives are evaluated at the fixed point (x^*, y^*).

In linear stability analysis, we neglect the higher-order terms and the deviation (η, μ) satisfies a linear system that can be analyzed and classified as in Section 3.

4.2.1 Example on phase-plane analysis

Analyze the dynamical system:

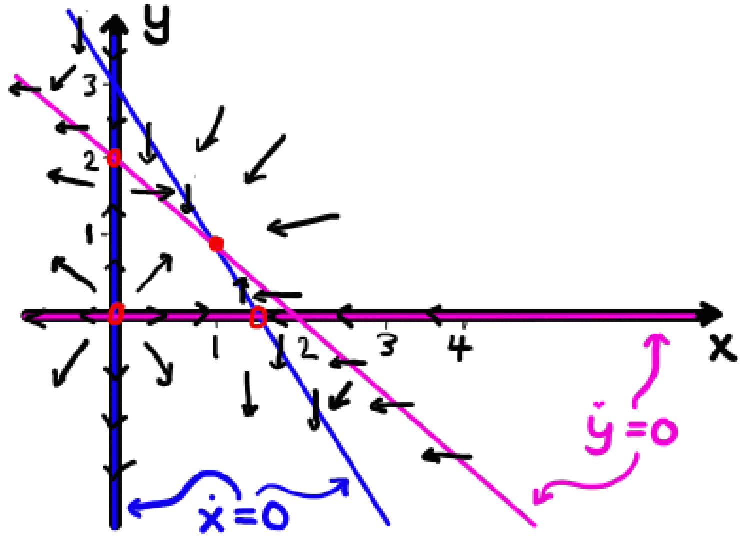
$$\begin{aligned}\dot{x} &= x(3 - 2x - y)\\ \dot{y} &= y(2 - x - y)\,.\end{aligned}$$

The nullclines are

$$\begin{aligned}\dot{x} = 0: &\quad x = 0 \quad \text{or} \quad x = (3-y)/2\\ \dot{y} = 0: &\quad y = 0 \quad \text{or} \quad y = 2 - x\,.\end{aligned}$$

On the nullclines the flow is one-dimensional and therefore straightforward to analyze:

Nullcline	$x = 0$	$x = (3-y)/2$	$y = 0$	$y = 2 - x$
Flow	$\dot{y} = y(2-y)$	$\dot{y} = y(1-y)/2$	$\dot{x} = x(3-2x)$	$\dot{x} = x(1-x)$



The system has 4 fixed points at the intersections of the two types of nullclines:

$$(x_1^*, y_1^*) = (0,0)\,, \quad (x_2^*, y_2^*) = (0,2)\,, \quad (x_3^*, y_3^*) = (3/2,0)\,, \quad (x_4^*, y_4^*) = (1,1)\,.$$

The nullclines give a rough picture of the flow, but it is complicated to figure out what happens close to the fixed points using nullclines only. Therefore, use linear stability analysis for the fixed points:

$$\begin{aligned}\mathbb{J} &= \begin{pmatrix} (3-2x-y)+x(-2) & x(-1) \\ y(-1) & (2-x-y)+y(-1) \end{pmatrix} \\ &= \begin{pmatrix} 3-4x-y & -x \\ -y & 2-x-2y \end{pmatrix}\end{aligned}$$

Fixed point (x^*, y^*)	$(0,0)$	$(0,2)$	$(3/2,0)$	$(1,1)$
$\tau \equiv \mathrm{Tr}\mathbb{J}(\mathrm{x}^*, \mathrm{y}^*)$	5	-1	-5/2	-3
$\Delta \equiv \det \mathbb{J}(x^*, y^*)$	6	-2	-3/2	1
$\lambda_{1,2} = (\tau \pm \sqrt{\tau^2 - 4\Delta})/2$	$(2,3)$	$(-2,1)$	$(-3,1/2)$	$(-3 \pm \sqrt{5})/2$
Type	Unstable node	Saddle	Saddle	Stable node
$\boldsymbol{v}_1$	$(0,1)$	$(0,1)$	$(1,0)$	$(1-\sqrt{5},2)$
$\boldsymbol{v}_2$	$(1,0)$	$(-3,2)$	$(-3/7,1)$	$(1+\sqrt{5},2)$

Stable|unstable directions The real part of the eigenvalues determine the stability of a fixed point. Small deviations from a fixed point along an eigenvector $\boldsymbol{v}_{\rm u}$ corresponding to an eigenvalue $\lambda_{\rm u}$ with $\mathrm{Re}\,\lambda_{\rm u} > 0$ remain in the direction of $\boldsymbol{v}_{\rm u}$ and grow exponentially fast. This follows from Eq. (6) using $(\eta, \mu) = \epsilon(t)\boldsymbol{v}_{\rm u}$ with $\epsilon \ll 1$:

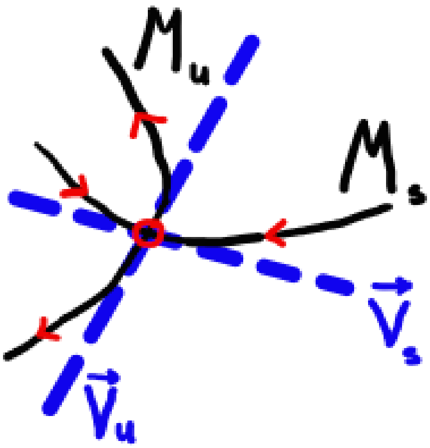
$$\frac{\mathrm{d}\epsilon}{\mathrm{d}t}\boldsymbol{v}_{\rm u} = \mathbb{J}(x^*, y^*)\epsilon\boldsymbol{v}_{\rm u} = \lambda_{\rm u}\epsilon\boldsymbol{v}_{\rm u} \quad \Rightarrow \quad \epsilon(t) = \epsilon(0)e^{\lambda_{\rm u}t}\,.$$

The (normed) eigenvector $\boldsymbol{v}_{\rm u}$ an unstable direction of the fixed point. Similarly, a normed eigenvector $\boldsymbol{v}_{\rm s}$ corresponding to $\lambda_{\rm s}$ with $\mathrm{Re}\,\lambda_{\rm s} < 0$ is a stable direction of the fixed point: small deviations in this direction shrink exponentially fast.

Stable|unstable manifolds The stable manifold $M_{\rm s}$ of a fixed point is either a point, curve, or surface in the phase-plane. It is defined as the set of points (including the fixed point) that approach the fixed point in the limit $t \to \infty$. Similarly, the unstable manifold $M_{\rm u}$ consists of the set of points that approach the fixed point in the limit $t \to -\infty$, i.e. if the flow is reversed, then $M_{\rm s}$ and $M_{\rm u}$ switch stability.

- In a **linear system** the stable|unstable manifold is given by the subspace spanned by the set of stable|unstable directions. For example, a saddle point has one negative and one positive eigenvalue, it attracts along the stable direction, but repels along the unstable direction. Its stable and unstable manifolds are lines in these directions. Attractors|repellers are stable|unstable in all directions and the stable|unstable manifold is a surface (the entire phase plane).

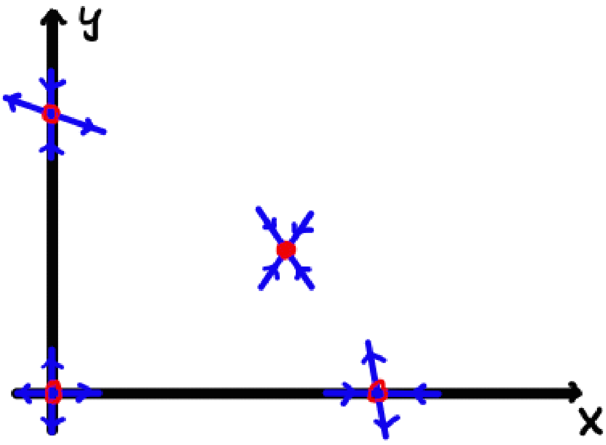
- For a **non-linear system**, $M_{\rm s}$ and $M_{\rm u}$ approach the manifolds of the linearized fixed point close to it, but may deviate further away due to non-linear effects. Example for a saddle point:



The stable|unstable manifold approaches the stable|unstable direction $v_s|v_u$ of the fixed point close to the fixed point.
The two-dimensional stable|unstable manifold of an attractor|repeller may become bounded.

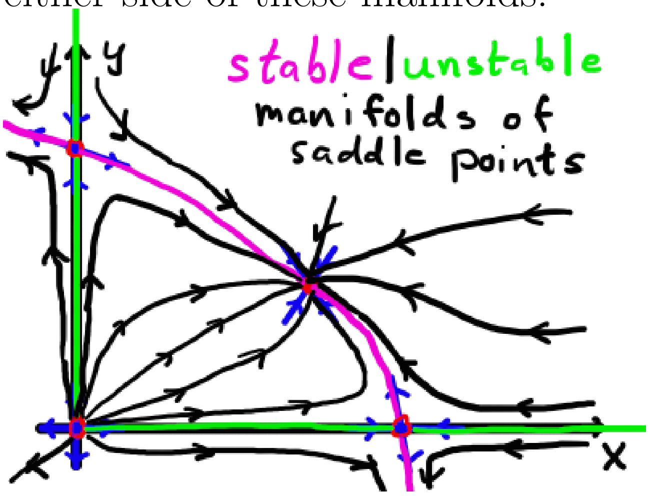
To numerically evaluate the stable|unstable manifold: start close to the fixed point in the stable|unstable direction and integrate the system backwards|forward in time.

Coming back to our example, from the table, the stable|unstable directions close to the fixed points are:



As shown by the nullclines, the flow aligns with the coordinate axes. Therefore, since trajectories cannot cross, the four quadrants are isolated from each other.

Consider first the upper-right quadrant. Since the flow is negative for large values of x or y, trajectories do not escape to infinity, and must therefore be attracted by the stable node at $(1, 1)$ (its stable manifold or <u>basin of attraction</u> is the upper right quadrant). In particular, the unstable manifolds of the saddle points must connect with the stable node. As a consequence, trajectories become trapped on either side of these manifolds:



This is a generic behaviour, the manifolds of saddle points often divide the phase space into regions of qualitatively different long-term dynamics.

One example is the stable manifolds of the saddle points along the coordinate axes: these separate dynamics that are attracted to the stable node, from the rest of the phase plane where trajectories run off to infinity. The stable manifolds of the saddle points are examples of separatrices (singular: separatrix): they divide the phase space into regions of different long-term behaviour.

Outside the upper-right quadrant, the unstable manifolds of the saddle points must run away to infinity (no attractor can attract them).

Note: The stable|unstable manifolds and the nullclines can sometimes coincide (the coordinate axes in the example above), but in general they are different curves, also close to the fixed point.

4.2.2 Effect of small non-linear terms

When is it safe to neglect quadratic terms in the stability analysis?

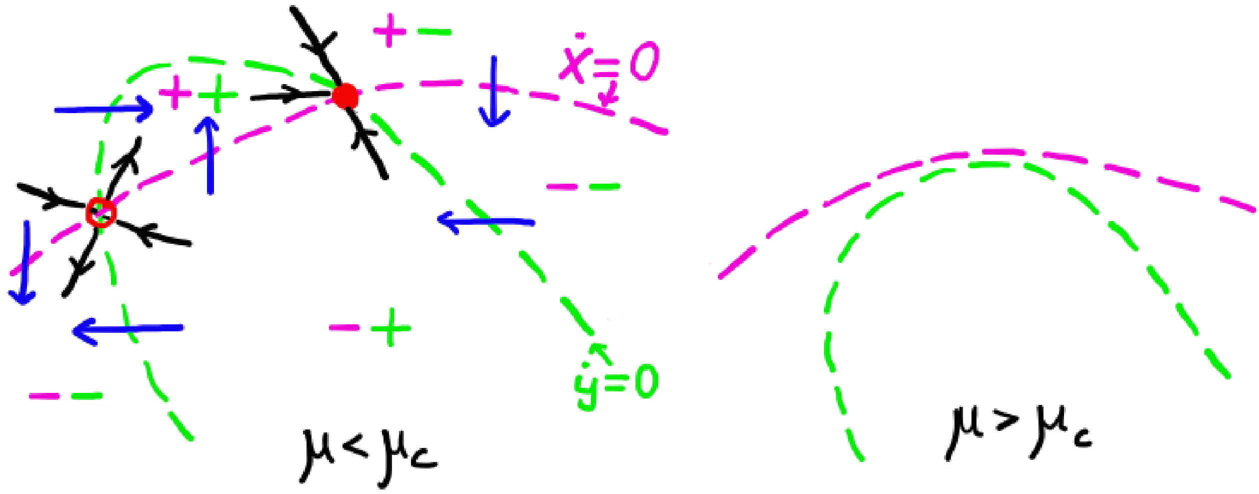
Linear stability analysis gives a qualitatively correct picture if the fixed point is a node, spiral, or saddle (as in the Example in Section 4.2.1). For the border-line cases, non-linear terms may (or may not) change the dynamics qualitatively from the border-line case.

5 Local two-dimensional bifurcations

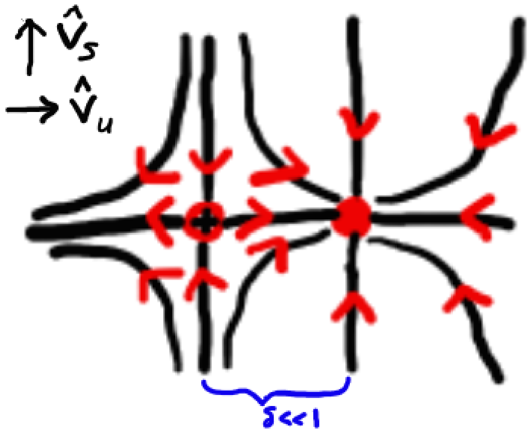
As in one-dimensional systems: fixed points may be created, destroyed, or change stability as parameters are varied (change of 'topological equivalence').

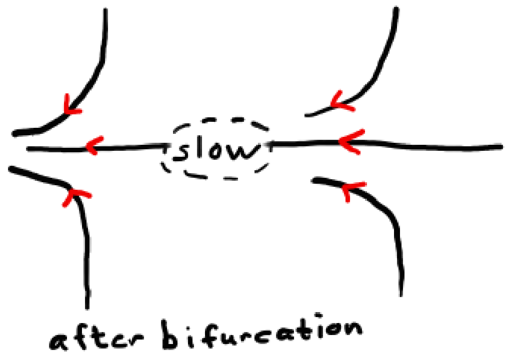
5.1 Saddle-node, transcritical, and pitchfork bifurcations

Assume that a saddle point and an attracting node collide as a parameter μ is varied. The mechanism of why the collision occurs at all (instead of the fixed points moving past each other): Fixed points are formed at intersections of nullclines. As μ is varied, the nullclines deform continuously. If they slip through each other the fixed points collide:



Change coordinates to the local eigenframe of the saddle point. Let the unstable direction of the saddle be $\hat{\boldsymbol{v}}_{\mathrm{u}} = (1, 0)$ and the stable direction $\hat{\boldsymbol{v}}_{\mathrm{s}} = (0, 1)$. When the node comes closeby, it must merge along the unstable manifold of the saddle [otherwise trajectories could not remain continuous and linear as the fixed points merge].





- The bifurcation is essentially one-dimensional (in any dimension). Normal form (in unstable|stable directions of saddle):

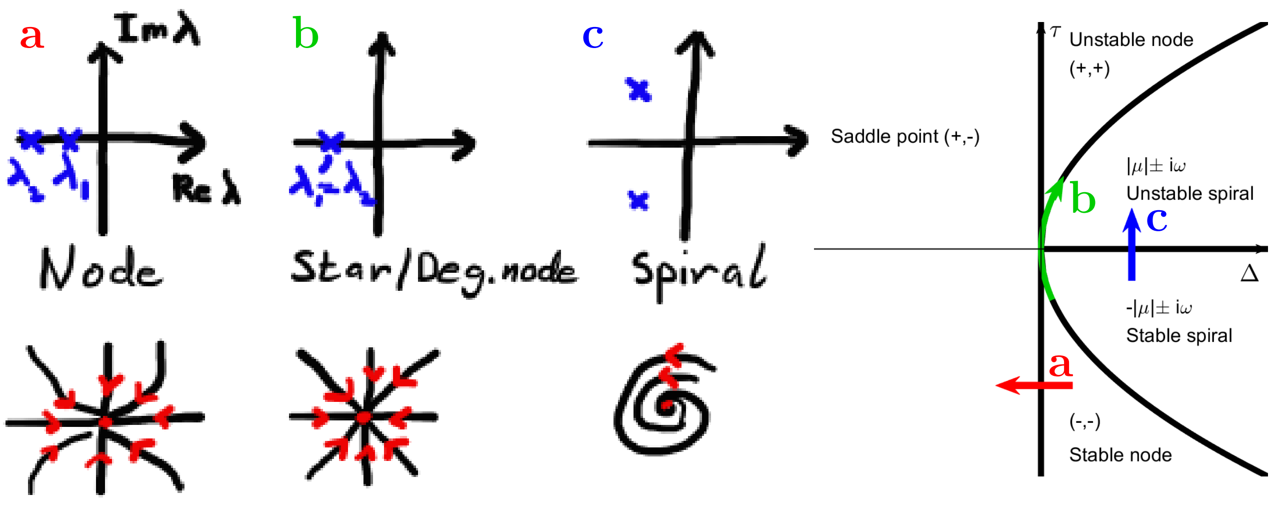
$$\dot{x} = -\mu - x^2 \qquad \text{(same as 1D)}$$
$$\dot{y} = -y$$

- Along the interconnecting manifold, the eigenvalues have opposite signs ⇒ at bifurcation (at least) one eigenvalue must vanish.
- Repelling node? ⇒ reverse the arrows!

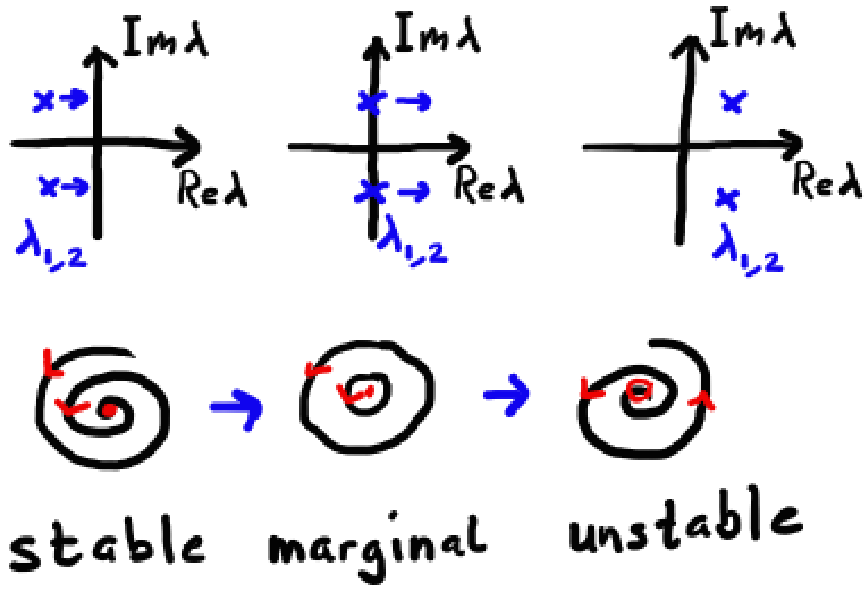
Similarly, the other bifurcations discussed in Section 2 (transcritical, subcritical pitchfork, supercritical pitchfork), occur in one-dimensional subspaces in higher-dimensional systems. Transversal directions are simply attracting or repelling. The bifurcations are summarized in the Table at the end of this section. The dynamics along the x-axis is that of 1D flows (x-component of flow plotted as black) and blue shows flow in 2D.

5.2 Hopf bifurcation

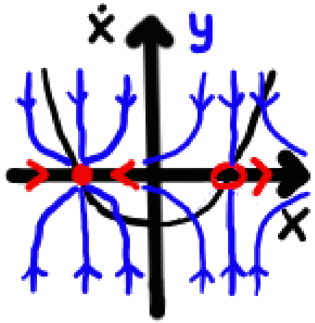
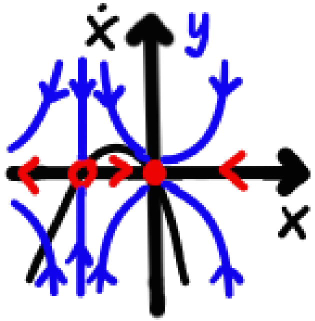
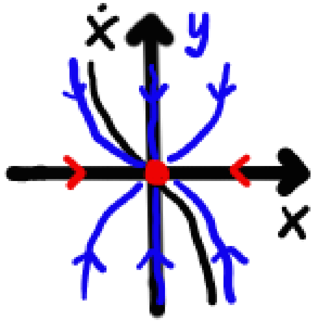
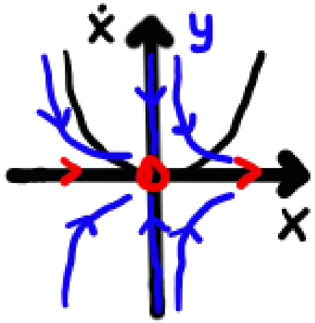
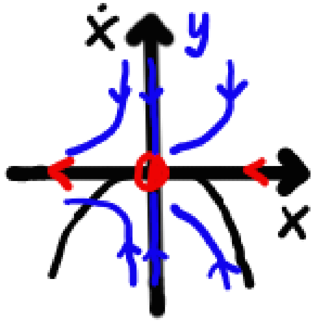
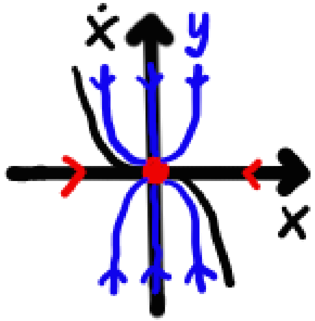
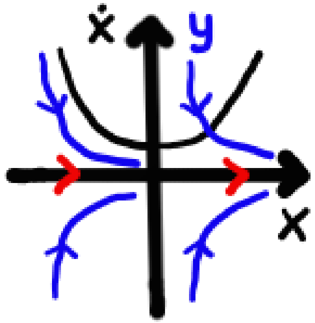
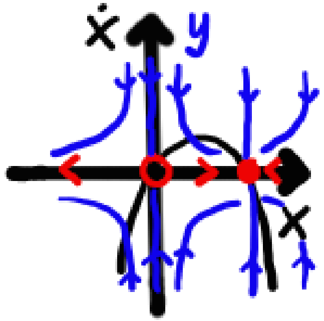
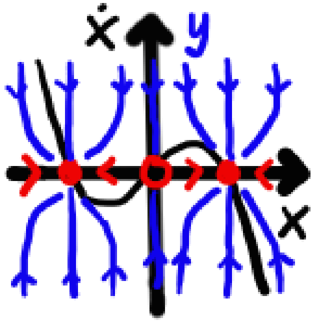
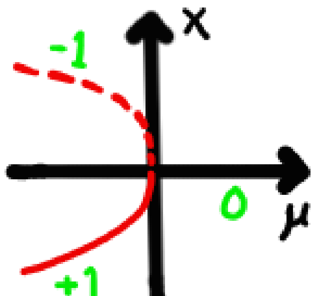
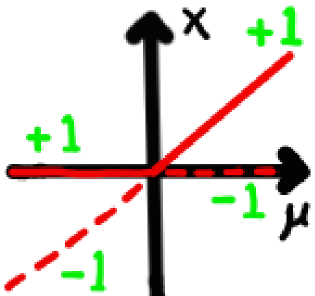
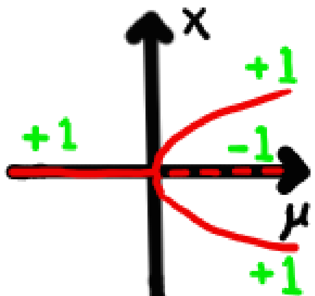
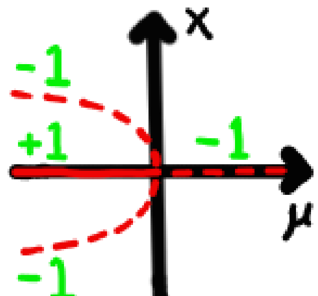
A stable fixed point has $\mathcal{R}e[\lambda_{1,2}] < 0$. A bifurcation to an unstable fixed point occurs if the maximal eigenvalue crosses zero. Consider the three possible bifurcations from stable to unstable in a linear system:



Cases **a** and **b** have $\mathcal{I}m[\lambda_{1,2}] = 0$, while case **c** has $\mathcal{I}m[\lambda_{1,2}] \neq 0$. Case **a** corresponds to saddle-node, transcritical, and pitchfork bifurcations above. Case **b** is marginal and therefore not so interesting. Case **c** is a Hopf bifurcation: a new type of bifurcation that does not exist in 1D systems. Consider the transition with $\mathcal{I}m[\lambda_{1,2}] \neq 0$:



Hopf bifurcations often lead to the formation of attracting closed orbits, limit cycles, discussed in Section 6.

Type Normal form	saddle-node $\dot{x} = \mu + x^2$ $\dot{y} = -y$	transcritical $\dot{x} = \mu x - x^2$ $\dot{y} = -y$	supercrit. pitchfork $\dot{x} = \mu x - x^3$ $\dot{y} = -y$	subcrit. pitchfork $\dot{x} = \mu x + x^3$ $\dot{y} = -y$
$\mu < 0$				Similar to super crit.
$\mu = 0$				
$\mu > 0$				
Diagram				

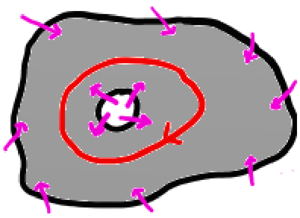
6 Closed orbits, limit cycles, and chaos

In Section 4 we connected the local dynamics close to fixed points in non-linear flows to the dynamics observed in linear flows. In this section we consider non-local dynamics due to non-lineaity.

6.1 Poincaré-Bendixson theorem

Assume a smooth flow in a bounded domain D of the plane. Assume further that D does not contain any fixed point and that there exists a trajectory that is confined in D for all times. Then at least one closed orbit exists in D. This is a consequence of the fact that trajectories for smooth flows cannot intersect in two dimensions.

To satisfy the condition that a confined trajectory exists, one can construct a trapping region, i.e. choose D such that the flow points inward everywhere. If it is possible to construct a trapping region, then the Poincaré-Bendixon theorem ensures that at least one closed orbit exists in D.



As a consequence, in two dimensions trajectories either end up close to fixed points or to closed orbits (or running away to infinity). In higher dimensions: infinite non-repeating trajectories (chaos) is possible.

6.2 Closed orbits

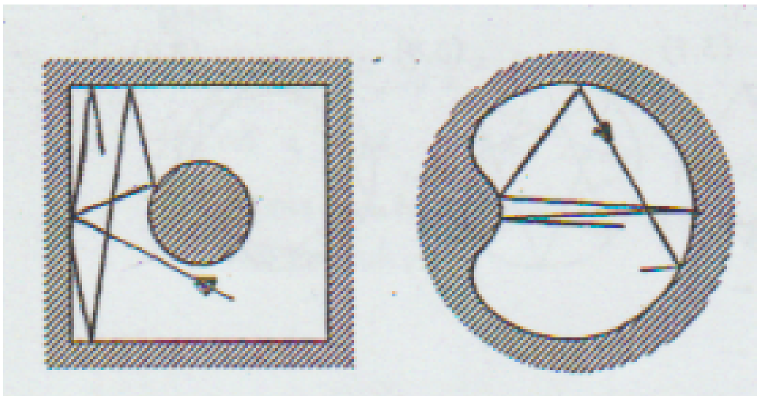
Closed orbits either occur as bands of periodic solutions (as around the center in Section 3.1) or as isolated attracting periodic solutions: limit cycles. Systems with limit cycles are useful in order to model self-sustained oscillations (oscillations without external periodic forcing), such as the firing of a pacemaker, cycles in the body , oscillating chemical reactions, unwanted or dangerous self-excitations in mechanical systems.

6.3 Chaotic systems

Chaotic dynamics exhibit the following properties

- Most trajectories show aperiodic long-term behaviour.
- System is deterministic, the irregular behavior is due to non-linearity of system and not due to stochastic forcing.
- Trajectories show sensitive dependence on initial condition (the 'butterfly effect').
- Must have dimensionality larger than two in continuous systems (otherwise chaos is ruled out by the Poincaré-Bendixon theorem)

6.3.1 Illustrative example: Convex billiards



6.3.2 More examples of chaotic systems

It is more a rule than an exception that systems exhibit chaos (often in the form of a mixture between chaotic and regular motion). Examples:

- **Biology** Population dynamics, arrythmia, epilepsy.
- **Physics** Double pendulum, helium atom, celestial mechanics, mixing of fluids, meteorological systems.
- **Computer science** Pseudo-random number generators.