Fourieranalys MVE030 och Fourier Metoder MVE290 17.mars.2017 - Lösningar!

Betygsgränser: 3: 40 poäng, 4: 53 poäng, 5: 67 poäng.
Maximalt antal poäng: 80.
Hjälpmedel: BETA.
Examinator: Julie Rowlett.
Telefonvakt: Fanny Berglund, 5325.

1. Bevisa att Bessel funktionerna, J_n, uppfyller:

$$\sum_{n\in\mathbb{Z}} J_n(x)z^n = e^{\frac{x}{2}(z-1/z)}.$$

The proof here is already available in the big proofs document!

2. Bevisa att Hermite polynomen, $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$, är ortogonala i Hilbert-rummet $L^2_w(\mathbb{R})$ med $w(x) = e^{-x^2}$.

The proof here is already available in the big proofs document!

3. Beräkna:

$$\sum_{n=0}^{\infty} \frac{1}{1+n^2}.$$

(Hint: Utveckla e^x i Fourier-series i intervallet $(-\pi, \pi)$).

Okay, we follow the hint. We need to compute

$$\int_{-\pi}^{\pi} e^x e^{-inx} dx = \left. \frac{e^{x(1-in)}}{1-in} \right|_{x=-\pi}^{x=\pi} = \frac{e^{\pi}e^{-in\pi}}{1-in} - \frac{e^{-\pi}e^{in\pi}}{1-in} = (-1)^n \frac{2\sinh(\pi)}{1-in}.$$

Hence, the Fourier coefficients are

$$\frac{1}{2\pi}(-1)^n \frac{2\sinh(\pi)}{1-in},$$

and the Fourier series for e^x on this interval is

$$e^x = \sum_{-\infty}^{\infty} \frac{(-1)^n \sinh(\pi)}{\pi(1-in)} e^{inx}, \quad x \in (-\pi, \pi).$$

We can pull out some constant stuff,

$$e^x = \frac{\sinh(\pi)}{\pi} \sum_{-\infty}^{\infty} \frac{(-1)^n e^{inx}}{1-in}, \quad x \in (-\pi, \pi).$$

Now, we use the theorem which tells us that the series converges to the average of the left and right hand limits at points of discontinuity, like for example π. The left limit is e^{π}. Extending the function to be 2π periodic, means that the right limit approaching π is equal to $e^{-\pi}$. Hence

$$\frac{e^{\pi}+e^{-\pi}}{2} = \frac{\sinh(\pi)}{\pi}\sum_{-\infty}^{\infty}\frac{(-1)^n e^{in\pi}}{1-in}.$$

Now, we know that $e^{in\pi} = (-1)^n$, thus

$$\frac{e^{\pi}+e^{-\pi}}{2} = \frac{\sinh(\pi)}{\pi}\sum_{-\infty}^{\infty}\frac{1}{1-in}.$$

We now consider the sum, and we pair together $\pm n$ for $n \in \mathbb{N}$, writing

$$\sum_{-\infty}^{\infty}\frac{1}{1-in} = 1 + \sum_{n\in\mathbb{N}}\frac{1}{1-in} + \frac{1}{1+in} = 1 + \sum_{n\in\mathbb{N}}\frac{2}{1+n^2}.$$

Hence we have found that

$$\frac{e^{\pi}+e^{-\pi}}{2} = \frac{\sinh(\pi)}{\pi}\sum_{-\infty}^{\infty}\frac{(-1)^n e^{in\pi}}{1-in} = \frac{\sinh(\pi)}{\pi}\left(1+\sum_{n\geq 1}\frac{2}{1+n^2}\right).$$

The rest is mere algebra. On the left we have the definition of $\cosh(\pi)$. So, moving over the $\sinh(\pi)$ we have

$$\frac{\pi\cosh(\pi)}{\sinh(\pi)} = 1 + 2\sum_{n\geq 1}\frac{1}{1+n^2} \implies \left(\frac{\pi\cosh(\pi)}{\sinh(\pi)} - 1\right)\frac{1}{2} = \sum_{n\geq 1}\frac{1}{1+n^2},$$

and

$$\sum_{n\geq 0}\frac{1}{1+n^2} = \left(\frac{\pi\cosh(\pi)}{\sinh(\pi)} - 1\right)\frac{1}{2} + 1.$$

Wow.

Note: there is an alternate way to proceed once you've got the Fourier series. This goes through computing the L^2 norm,

$$\int_{-\pi}^{\pi}(e^x)^2 dx = \sinh(2\pi).$$

On the other hand, by orthonormality and completeness,

$$\sinh(2\pi) = \frac{\sinh(\pi)^2}{\pi^2}\sum_{-\infty}^{\infty}\left|\frac{(-1)^n}{1-in}\right|^2 ||e^{inx}||^2_{L^2(-\pi,\pi)}.$$

Using the fact that the square of the L^2 norm of e^{inx} is 2π here,

$$= \frac{2\sinh(\pi)^2}{\pi} \sum_{-\infty}^{\infty} \frac{1}{1+n^2} = \frac{2\sinh(\pi)^2}{\pi} \left(1 + \sum_{n\geq 1} \frac{2}{1+n^2} \right).$$

Re-arranging, we have

$$\frac{\pi \sinh(2\pi)}{2\sinh(\pi)^2} = 1 + 2\sum_{n\geq 1} \frac{1}{1+n^2}.$$

Re-arranging a little more,

$$\frac{1}{2}\left(\frac{\pi \sinh(2\pi)}{2\sinh(\pi)^2} - 1 \right) = \sum_{n\geq 1} \frac{1}{1+n^2}.$$

Are you concerned because this doesn't look the same as above? No need to worry, because the sinh has a double angle type formula analogous to the sine,

$$\sinh(2\pi) = 2\sinh(\pi)\cosh(\pi),$$

so we get

$$\frac{\pi \cosh(\pi)}{2\sinh(\pi)} - \frac{1}{2} = \sum_{n\geq 1} \frac{1}{1+n^2},$$

and

$$\frac{\pi \cosh(\pi)}{2\sinh(\pi)} + \frac{1}{2} = \sum_{n\geq 0} \frac{1}{1+n^2}.$$

4. Hitta siffrorna a_0, a_1, och $a_2 \in \mathbb{C}$ som minimerar

$$\int_0^\pi |x - a_0 - a_1\cos(x) - a_2\cos(2x)|^2 dx.$$

This is a straightforward best approximation problem, using the cosine expansion on $(0,\pi)$. The way it works is that we think of the function $f(x) = x$ on $(0,\pi)$, and we extend it to $(-\pi,\pi)$ to an even function. This extension is thus $f(x) = |x|$. Then, if we were to expand in a Fourier series, the sine coefficients would drop out. The cosine coefficients,

$$\alpha_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx = \frac{2}{\pi}\int_0^\pi f(x)\cos(nx)dx, \quad n \geq 0.$$

We just need to compute the first three of these guys, that is for $n = 0, 1, 2$. Thus,

$$\alpha_0 = \frac{2}{\pi} \int_0^\pi x dx = \frac{2}{\pi} \frac{\pi^2}{2} = \pi.$$

Next,

$$\alpha_1 = \frac{2}{\pi} \int_0^\pi x \cos(x) dx = \frac{-4}{\pi}.$$

Similarly, we compute

$$\alpha_2 = \frac{2}{\pi} \int_0^\pi x \cos(2x) dx = 0.$$

So the numbers we seek are

$$a_0 = \frac{\alpha_0}{2}, \quad a_1 = \alpha_1, \quad a_2 = \alpha_2.$$

5. Lös problemet:

$$u_t - u_{xx} = 0, \quad t > 0, \quad x \in \mathbb{R},$$

$$u(x, 0) = 20e^{-x^2}$$

Let's solve the homogeneous heat equation with initial data $20e^{-x^2}$. We know this to be

$$\frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-(x-y)^2/(4t)} 20e^{-y^2} dy.$$

6. Vi definerar

$$\widehat{LP_\alpha}(f) := \hat{f} \chi_{(-\alpha,\alpha)}.$$

Beräkna $LP_\alpha(f)$ med

$$f(x) = \frac{1}{1 + x^2}.$$

We go backwards. $\widehat{LP_\alpha}(f)$ is the product of the Fourier transform of f, together with the characteristic function. We look up in our handy BETA that the Fourier transform of $x^{-1} \sin(\alpha x) = \chi_{(-\alpha,\alpha)}$. So, we know that

$$\widehat{LP_\alpha}(f) = \hat{f} \widehat{\frac{\sin(\alpha x)}{x}}.$$

What does the Fourier transform do to convolutions? It turns them into products! Hence, we know that

$$LP_\alpha(f) = \int_{\mathbb{R}} f(x-y)y^{-1}\sin(\alpha y)dy.$$

It's fine to leave your answer this way. You could try to solve this doing a contour integral, but I couldn't find a contour to make it work. So, if you got this and you left it that way, fine. Alternatively, we can also look in our handy β to find that $\hat{f}(\xi) = \pi e^{-|\xi|}$. Then, we use the FIT to say that

$$\begin{aligned}
LP_\alpha(f)(y) &= \frac{1}{2\pi}\int_{\mathbb{R}} \chi_{(-\alpha,\alpha)}(x)\pi e^{-|x|}e^{ixy}dx \\
&= \frac{1}{2}\int_{-\alpha}^{\alpha} e^{-|x|}e^{ixy}dx
\end{aligned}$$

$$= \frac{1}{2}\left(\int_{-\alpha}^{0} e^{x+ixy}dx + \int_0^\alpha e^{-x+ixy}dx\right) = \frac{1}{2}\left(\frac{1}{1+iy} - \frac{e^{-\alpha(1+iy)}}{1+iy} + \frac{e^{\alpha(-1+iy)}}{-1+iy} - \frac{1}{-1+iy}\right)$$

$$\begin{aligned}
&= \frac{1}{2}\left(\frac{-1+iy}{-1-y^2} - \frac{e^{-\alpha}e^{-iy}(-1+iy)}{-1-y^2} + \frac{e^{-\alpha}e^{iy}(1+iy)}{-1-y^2} - \frac{1+iy}{-1-y^2}\right) \\
&= \frac{1}{2}\left(\frac{2}{1+y^2} + \frac{e^{-\alpha}2\cos(y)}{-1-y^2} + \frac{e^{-\alpha}(iy)(2i\sin(y))}{-1-y^2}\right) \\
&= \frac{1+e^{-\alpha}(y\sin(y)-\cos(y))}{1+y^2}.
\end{aligned}$$

7. Lös problemet:

$$u_t - u_{xx} = tx, \quad 0 < x < 4, \quad t > 0,$$

$$\begin{aligned}
u(x,0) &= 20, \\
u_x(4,t) &= 0, \\
u(0,t) &= 20.
\end{aligned}$$

The boundary conditions and initial condition are inhomogeneous. So, we first solve the homogeneous PDE with these inhomogeneous conditions. It's pretty simple, because the constant function 20 does the job.

Next, we solve the inhomogeneous PDE but with homogeneous BC and IC, specifically, we now solve

$$u_t - u_{xx} = tx, \quad 0 < x < 4, \quad t > 0,$$

$$\begin{aligned} u(x,0) &= 0, \\ u_x(4,t) &= 0, \\ u(0,t) &= 0. \end{aligned}$$

If we add the solution to the constant, 20, then the sum will do the job. First, we think about the homogeneous PDE, which would give us

$$\frac{T'}{T} - \frac{X''}{X} = 0 \implies \frac{X''}{X} = \frac{T'}{T} = \text{ constant.}$$

We have the nice boundary conditions for X,

$$X(0) = X'(4) = 0 \implies X_n(x) = \sin((n+1/2)\pi x/4),$$

$$X_n''(x) = -\lambda_n^2 X_n(x), \quad \lambda_n = \frac{(n+1/2)\pi}{4}.$$

up to constant factor. By the SLP theory, these guys form an orthogonal basis for $L^2(0,4)$, so we can expand the function tx in terms of this basis,

$$tx = t\sum_{n\geq 0} \widehat{x_n} X_n(x),$$

where

$$\widehat{x_n} = \frac{1}{2}\int_0^4 x\sin((n+1/2)\pi x/4)dx = \frac{8(-1)^n}{(n+1/2)^2}.$$

Now, we set up the PDE for

$$u(x,t) = \sum_{n\geq 0} c_n(t) X_n(x).$$

We apply the heat operator, and we want to solve

$$\sum_{n\geq 0} c_n'(t)X_n(x) - c_n(t)X_n''(x) = tx = \sum_{n\geq 1} t\widehat{x_n}X_n(x).$$

We use the equation satisfied by X_n to change this around to

$$\sum_{n\geq 0}\left(c_n'(t)+\lambda_n^2 c_n(t)\right)X_n(x)=\sum_{n\geq 1} t\widehat{x_n}X_n(x).$$

We equate coefficients,

$$c_n'(t)+\lambda_n^2 c_n(t)=t\widehat{x_n}.$$

This is an ODE. We also have the IC, that we want $c_n(0)=0$. A particular solution to the ODE is a linear function of t, that is

$$at+b.$$

Let's substitute a function of that type into the ODE above,

$$a+\lambda_n^2(at+b)=t\widehat{x_n}.$$

Then equating coefficients, we need that

$$a=a_n=\frac{\widehat{x_n}}{\lambda_n^2},\quad a+\lambda_n^2 b=0 \implies b=\frac{-a}{\lambda_n^2}=\frac{-\widehat{x_n}}{\lambda_n^4}.$$

The particular solution is then

$$\frac{\widehat{x_n}}{\lambda_n^2}\left(t-\lambda_n^{-2}\right).$$

We would like $c_n(0)=0$. However, this is not necessarily the case above. How to remedy this dilemma? We include a solution to the homogeneous ode,

$$f'+\lambda_n^2 f=0.$$

This is solved by constant multiples of $e^{-\lambda_n^2 t}$. So, the solution we seek is then

$$c_n(t)=\frac{\widehat{x_n}}{\lambda_n^2}\left(t-\lambda_n^{-2}\right)+b_n e^{-\lambda_n^2 t}.$$

Setting $c_n(0)=0$, we see that the constant we seek is

$$b_n=\frac{\widehat{x_n}}{\lambda_n^4}.$$

Thus

$$c_n(t)=\frac{\widehat{x_n}}{\lambda_n^2}\left(t-\frac{1}{\lambda_n^2}+\frac{1}{\lambda_n^2 e^{\lambda_n^2 t}}\right).$$

Our total solution is then

$$20+\sum_{n\geq 0} c_n(t)X_n(x).$$

8. Lös problemet:

$$u_{tt} - u_{xx} - u_{yy} = 0, \quad x^2 + y^2 < 1 \text{ och } y > 0, \quad t > 0,$$

I polara koordinaterna (r, θ),

$$u_{tt} - u_{rr} - r^{-1}u_r - r^{-2}u_{\theta\theta} = 0, \quad 0 < r < 1, \text{ och } 0 < \theta < \pi,$$

med

$$\begin{aligned} u(1,\theta,t) &= \sin(2\theta), & & t > 0, \\ u(r,\theta,0) &= 0, & 0 < r < 1, \quad & 0 < \theta < \pi, \\ u_t(r,\theta,0) &= 0, & 0 < r < 1, \quad & 0 < \theta < \pi. \end{aligned}$$

On inspection, we should separate θ from r and t. Doing this, we consider

$$v(r,t)\Theta(\theta).$$

In order to satisfy the boundary condition, we want

$$v(1,t) = 1, \quad \Theta(\theta) = \sin(2\theta).$$

Then, we know that $\Theta''(\theta) = -4\Theta(\theta)$. We insert this information into the PDE, so we consider

$$\Theta(v_{tt} - v_{rr} - r^{-1}v_r) - r^{-2}\Theta'' v = 0 \iff \frac{v_{tt} - v_{rr} - r^{-1}v_r}{v} - r^{-2}(-4) = 0.$$

The boundary condition $v(1,t) = 1$ shows that we should try to find a sort of steady state solution, that is look for a function $f(r)$ which satisfies

$$\frac{-f'' - r^{-1}f'}{f} + 4r^{-2} = 0.$$

We do this. The equation can be equivalently written as

$$-r^2 f''(r) - rf'(r) + 4f(r) = 0.$$

This is an Euler equation. We seek solutions of the form $f(r) = r^x$. This implies the equation for x,

$$-x(x-1) - x + 4 = 0 \implies x = \pm 2.$$

We don't want $f(0) = 0^{-2}$. So, we choose $x = 2$, and $f(r) = r^2$. Next, we want to solve for $v(r,t)$ to satisfy

$$\frac{v_{tt} - v_{rr} - r^{-1}v_r}{v} + 4r^{-2} = 0,$$

and

$$\begin{aligned}
v(1,\theta) &= 0, & && t > 0,\\
v(r,0) &= -r^2, & 0 < r < 1, && 0 < \theta < \pi,\\
v_t(r,0) &= 0, & 0 < r < 1, && 0 < \theta < \pi.
\end{aligned}$$

To do this, we separate variables, writing $v(r,t) = R(r)T(t)$, and we deal with R first, because it's got nice conditions, namely $R(1) = 0$. Separating variables, the PDE becomes

$$\frac{T''R - TR'' - r^{-1}TR'}{TR} + 4r^{-2} = 0 \implies \frac{T''}{T} = -4r^{-2} + \frac{R''}{R} + r^{-1}\frac{R'}{R}.$$

Hence both sides are equal to a constant, and we call it $-\lambda^2$. The equation for R is then

$$-\lambda^2 r^2 R = -4R + r^2 R'' + rR' \iff r^2R'' + rR' - 4R + \lambda^2 r^2 R = 0.$$

This should start to look familiar. To solve the ODE, we do the substitution $x = \lambda r$, $J(x) = R(\lambda r)$, then the equation for J becomes,

$$x^2 J''(x) + xJ'(x) + (x^2 - 4)J(x) = 0.$$

This is Bessel's equation of order 2. The solution is thus $J_2(x)$. That's why we chose the letter J for this function. Now, we want that $J_2(1) = 0$. Returning to our function

$$R(\lambda r) = J_2(x) = J_2(\lambda r), \quad J_2(\lambda) = 0 \implies \lambda_n = \text{ n-th positive root of } J_2.$$

Hence our solutions are

$$J_2(\lambda_n r),$$

and the equation for T is now

$$\frac{T''}{T} = -\lambda_n^2.$$

Thus,

$$T_n(t) = a_n \cos(\lambda_n t) + b_n \sin(\lambda_n t).$$

We have the initial condition that the derivative with respect to t of our solution should vanish at $t = 0$, so this shows that the sin part should drop out, leaving

$$T_n(t) = a_n \cos(\lambda_n t).$$

Next, we need to get our initial condition, so putting the T and R together, we write

$$\sum_{n\geq 1} J_2(\lambda_n r) a_n \cos(\lambda_n t).$$

Setting $t = 0$, we have

$$\sum_{n\geq 1} J_2(\lambda_n r) a_n = -r^2.$$

By some magical theorem, we know that $\{J_2(\lambda_n r)\}$ are an orthogonal basis for $L^2(0,1)$ with the measure rdr. So, we can expand the function $-r^2$ in terms of this basis. Doing this, we have

$$a_n = \int_0^1 -r^2 J_2(\lambda_n r) r dr \left(\int_0^1 J_2^2(\lambda_n r) r dr \right)^{-1}.$$

The total solution to our problem is then

$$\left(r^2 + \sum_{n\geq 1} a_n \cos(\lambda_n t) J_2(\lambda_n r) \right) \sin(2\theta).$$