MATEMATIK
Chalmers Tekniska Hogskola

Dugga

Fourieranalys/Fourier Metoder, lp1, 2018

Skriv ditt namn och personnummer - *tydligt!* Julie Rowlett and I'm not giving out my personal number!

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1. (1P) Define the Fourier coefficients c_n on $[-\pi, \pi]$ for a function $f(x)$.

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

2. (1P) Compute the Fourier series for $f(x) = e^x$ on $[-\pi, \pi]$. Use this to evaluate

$$\sum_{n \geq 1} \frac{1}{1+n^2}.$$

Okay, we follow the hint. We need to compute

$$\int_{-\pi}^{\pi} e^x e^{-inx} dx = \frac{e^{x(1-in)}}{1-in} \Big|_{x=-\pi}^{x=\pi} = \frac{e^{\pi} e^{-in\pi}}{1-in} - \frac{e^{-\pi} e^{in\pi}}{1-in} = (-1)^n \frac{2\sinh(\pi)}{1-in}.$$

Hence, the Fourier coefficients are

$$\frac{1}{2\pi} (-1)^n \frac{2\sinh(\pi)}{1-in},$$

and the Fourier series for e^x on this interval is

$$e^x = \sum_{-\infty}^{\infty} \frac{(-1)^n \sinh(\pi)}{\pi(1-in)} e^{inx}, \quad x \in (-\pi, \pi).$$

We can pull out some constant stuff,

$$e^x = \frac{\sinh(\pi)}{\pi} \sum_{-\infty}^{\infty} \frac{(-1)^n e^{inx}}{1-in}, \quad x \in (-\pi, \pi).$$

Now, we use the theorem which tells us that the series converges to the average of the left and right hand limits at points of discontinuity, like for example π. The left limit is e^{π}. Extending the function to be 2π periodic, means that the right limit approaching π is equal to $e^{-\pi}$. Hence

$$\frac{e^{\pi} + e^{-\pi}}{2} = \frac{\sinh(\pi)}{\pi} \sum_{-\infty}^{\infty} \frac{(-1)^n e^{in\pi}}{1-in}.$$

Now, we know that $e^{in\pi} = (-1)^n$, thus

$$\frac{e^{\pi} + e^{-\pi}}{2} = \frac{\sinh(\pi)}{\pi} \sum_{-\infty}^{\infty} \frac{1}{1-in}.$$

We now consider the sum, and we pair together $\pm n$ for $n \in \mathbb{N}$, writing

$$\sum_{-\infty}^{\infty} \frac{1}{1-in} = 1 + \sum_{n \in \mathbb{N}} \frac{1}{1-in} + \frac{1}{1+in} = 1 + \sum_{n \in \mathbb{N}} \frac{2}{1+n^2}.$$

Hence we have found that

$$\frac{e^{\pi} + e^{-\pi}}{2} = \frac{\sinh(\pi)}{\pi} \sum_{-\infty}^{\infty} \frac{(-1)^n e^{in\pi}}{1-in} = \frac{\sinh(\pi)}{\pi} \left(1 + \sum_{n \in \mathbb{N}} \frac{2}{1+n^2} \right).$$

The rest is mere algebra. On the left we have the definition of $\cosh(\pi)$. So, moving over the $\sinh(\pi)$ we have

$$\frac{\pi \cosh(\pi)}{\sinh(\pi)} = 1 + 2 \sum_{n \in \mathbb{N}} \frac{1}{1+n^2} \implies \left(\frac{\pi \cosh(\pi)}{\sinh(\pi)} - 1 \right) \frac{1}{2} = \sum_{n \in \mathbb{N}} \frac{1}{1+n^2}.$$

Wow.

3. (1P) Define the scalar product of two functions $f(x)$ and $g(x)$ on the interval $[-\pi, \pi]$.

$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)\overline{g(x)}dx.$$

4. (1P) State Bessel's inequality.
   For any $f \in \mathcal{L}^2$ on the interval $[-\pi, \pi]$ we have the inequality

$$\sum_{n\in\mathbb{Z}} |c_n|^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx.$$

   The general version for Hilbert spaces is fine too: if $\{\phi_n\}_{n\in\mathbb{N}}$ is an ONS in a Hilbert space then for any f in that Hilbert space

$$\sum_{n\in\mathbb{N}} |\langle f, \phi_n\rangle|^2 \leq ||f||^2.$$

5. (1P) Solve:

$$u(x,0) = \begin{cases} x+\pi, & -\pi \leq x \leq 0 \\ \pi - x, & 0 \leq x \leq \pi \end{cases}$$
$$u_t(x,0) = 0$$
$$u(-\pi,t) = u(\pi,t) = 0$$
$$u_t(x,t) - u_{xx}(x,t) = 0 \quad x \in [-\pi,\pi], \quad t > 0.$$

   This problem should have been u_{tt} above not u_t. If we ignore the point 0 at which the initial data is not differentiable, then just taking that initial data as a solution solves the PDE (except at 0). Technically speaking, however, it's not a legitimate solution because any solution to the heat equation with continuous initial data is then smooth for all $t > 0$. (This can be proven a variety of ways, but of course, you weren't expected to know this fact right now). So, ignoring this fact, the idea to use the initial data as the solution was pretty damn clever, and some people did this. However, technically, that solution is not really correct.

   So many of you got stuck somewhere because with the typo as above, it is an impossible problem. There simply is no solution which satisfies all these things. One way to prove that is to use the so-called heat kernel in this context, which is the function

$$H(x,y,t) = \sum_{n\geq 1} e^{-n^2t/4}\phi_n(x)\phi_n(y),$$

   with

$$\phi_n(x) = \begin{cases} \frac{\sin\left(\frac{nx}{2}\right)}{\sqrt{\pi}} & n \text{ even} \\ \frac{\cos\left(\frac{(2n+1)x}{2}\right)}{\sqrt{\pi}} & n \text{ odd} \end{cases}$$

   Then, the unique solution to the heat equation with initial data given by

$$v(x) := u(x,0)$$

   is

$$u(x,t) = \sum_{n\geq 1} e^{-n^2t/4}\phi_n(x)\hat{v}_n.$$

   This converges absolutely and uniformly for all $t > 0$. Hence, it is a smooth function of both x and t. Therefore, it cannot be equal to $v(x)$ for $t > 0$, because $v(x)$ is not differentiable at $x = 0$. Moreover, the derivative with respect to t at $t = 0$ is not identically zero. So there simply is no solution to this problem.

   This will NOT happen on the actual exam. However, since it was my bad making a typo, the consequence is that EVERYBODY gets one point for #5. Even if you didn't write anything at all about that problem. It wasn't fair, but hey, life isn't fair, and some PDEs simply CANNOT be solved. So, it's good for you to see this once at least, and in the end, you all receive an extra point in return for your mental frustration. Always remember that spending time trying to solve math problems is like working out your math brain. It does something, even if you don't solve the problem. The effort alone is beneficial.