FOURIER ANALYSIS & METHODS

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ABSTRACT. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

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Proposition 1. *On the interval $[-\pi, \pi]$, the functions*

$$\phi_n(x) = \frac{e^{inx}}{\sqrt{2\pi}}$$

are an orthonormal set with respect to the scalar product,

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)\overline{g(x)} dx.$$

Proof: By definition, we consider

$$\int_{-\pi}^{\pi} \frac{e^{inx}}{\sqrt{2\pi}} \frac{\overline{e^{imx}}}{\sqrt{2\pi}} dx.$$

We bring the constant factor out in front of the integral the constant factor, and we recall that $\overline{e^{imx}} = e^{-imx}$, so we are computing

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx} e^{-imx} dx.$$

Exercise 1. *Why is*

$$\overline{e^{imx}} = e^{-imx}?$$

Explain in your own words or prove it algebraically.

So, we compute,

$$\int_{-\pi}^{\pi} e^{ix(n-m)} dx = \begin{cases} 2\pi & m = n \\ \frac{e^{ix(n-m)}}{n-m}\Big|_{x=-\pi}^{\pi} & n \neq m \end{cases}.$$

Now, we know that

$$e^{i\pi(n-m)} = \begin{cases} 1 & n-m \text{ is even} \\ -1 & n-m \text{ is odd.} \end{cases}.$$

To see this, I just imagine where we are on the Liseberghjul... Or you can write this out as

$$e^{i\pi(n-m)} = \cos(\pi(n-m)) + i\sin(\pi(n-m)).$$

The sine term is always zero since n and m are integers, and the cosine is either 1 or -1. Similarly,

$$e^{-i\pi(n-m)} = \begin{cases} 1 & n-m \text{ is even} \\ -1 & n-m \text{ is odd.} \end{cases}.$$

So in all cases, when $n \neq m$,

$$e^{i\pi(n-m)} - e^{-i\pi(n-m)} = 0.$$

Hence,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{inx}e^{-imx}dx = \begin{cases} \frac{2\pi}{2\pi} = 1 & n = m \\ 0 & n \neq m \end{cases}.$$

This is precisely what it means to be orthonormal!

So, now we know that $\{\phi_n(x)\}_{n\in\mathbb{Z}}$ are an orthonormal *set.* We want them to actually be an orthonormal *basis,* so that we can write for any $u_0(x)$,

$$u_0(x) = \sum_{n\in\mathbb{Z}} c_n\phi_n(x), \quad \phi_n(x) = \frac{e^{inx}}{\sqrt{2\pi}}.$$

In analogue to linear algebra, we should expect the coefficients to be the scalar product of our function $u_0(x)$ with the basis functions (vectors), $\phi_n(x)$. More generally, for a 2π periodic function $v(x)$, we hope to be able to write it as

$$v(x) = \sum_{n\in\mathbb{Z}} c_n\phi_n(x), \quad c_n = \int_{-\pi}^{\pi} v(x)\overline{\phi_n(x)}dx = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} v(x)e^{-inx}dx,$$

so that

$$v(x) = \sum_{n\in\mathbb{Z}}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\xi)e^{-in\xi}d\xi\right)e^{inx}.$$

This motivates:

Definition 2. Assume f is defined $[-\pi,\pi]$. The Fourier coefficients of f are

$$c_n := \frac{1}{2\pi}\langle f, e^{inx}\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx.$$

The Fourier series of f is

$$\sum_{n\in\mathbb{Z}} c_n e^{inx}.$$

1.1. **Computing Fourier series.** Let's start with the function $f(x) = |x|$. It satisfies $f(-\pi) = f(\pi)$. We will prove later that the Fourier series which is defined to be

$$\sum_{n\in\mathbb{Z}} c_n e^{inx}, \quad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$$

converges to $f(x)$ for all points $x \in (-\pi,\pi)$. What happens at the endpoints $\pm\pi$? We must postpone this question for now. Looking at the series, we make the following observation

$$\sum_{n\in\mathbb{Z}} c_n e^{in(x+2\pi)} = \sum_{n\in\mathbb{Z}} c_n e^{inx}.$$

Consequently, the series is 2π periodic. So, although the series will converge to $f(x) = |x|$ for $x \in (-\pi, \pi)$, because we are going to prove that it does, once we leave this interval, the series will no longer converge to $f(x) = |x|$. The series will converge to the function which is equal to $f(x) = |x|$ inside the interval $(-\pi, \pi)$, and which is 2π periodic on the whole real line. So, the function to which the series converges has a graph that looks like a zig-zag. It's really important to keep this in mind.

So, now let's compute the Fourier coefficients:

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} |x| e^{-inx} dx, \quad c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |x| dx = \frac{2\pi^2}{2(2\pi)} = \frac{\pi}{2}.$$

Since

$$|x| = \begin{cases} -x & x < 0 \\ x & x \geq 0 \end{cases}$$

we compute:

$$\int_{-\pi}^{0} -xe^{-inx} dx, \quad \int_{0}^{\pi} xe^{-inx} dx.$$

We do substitution in the first integral to change it:

$$\int_{-\pi}^{0} -xe^{-inx} dx = \int_{0}^{\pi} xe^{inx} dx = \left. \frac{xe^{inx}}{in} \right|_0^{\pi} - \int_0^{\pi} \frac{e^{inx}}{in} dx$$

$$= \frac{\pi e^{in\pi}}{in} - \frac{e^{in\pi}}{(in)^2} + \frac{1}{(in)^2}.$$

Similarly we also use integration by parts to compute

$$\int_0^{\pi} xe^{-inx} dx = \left. \frac{xe^{-inx}}{-in} \right|_0^{\pi} - \int_0^{\pi} \frac{e^{-inx}}{(-in)} dx$$

$$= \frac{\pi e^{-in\pi}}{-in} - \frac{e^{-in\pi}}{(-in)^2} + \frac{1}{(-in)^2}.$$

Adding them up and using the 2π periodicity, we get

$$\frac{2e^{in\pi}}{n^2} - \frac{2}{n^2} = \frac{2(-1)^n - 2}{n^2}.$$

OBS! We need to divide by 2π to get

$$c_n = \frac{(-1)^n - 1}{\pi n^2}, \quad n \in \mathbb{Z} \setminus \{0\}.$$

The Fourier series is therefore

$$\frac{\pi}{2} + \sum_{n \in \mathbb{Z}, \text{ odd}} e^{inx} \left(-\frac{2}{\pi n^2} \right).$$

Exercise 2. *Use these calculations to compute the series*

$$\sum_{n \geq 0} a_n \cos(nx) + b_n \sin(nx)$$

and to show that all of the b_n are equal to zero.

Now let's return to our example from Wednesday. We wish to solve the heat equation on a circular rod. Let

$$u(x,t) = \text{ the temperature at the point/angle } x \text{ and time } t.$$

Then the heat equation (physics!) dictates that

$$u_t - ku_{xx} = 0 \quad \forall x \in \mathbb{R}, \quad t > 0.$$

Above $k > 0$ is a constant which comes from - you guessed it - physics! There is some initial temperature along the rod as well,

$$u(x,0) = f(x).$$

Since the rod is circular,

$$u(x+2\pi, t) = u(x,t) \quad \forall x \in \mathbb{R},$$

so similarly,

$$f(x+2\pi) = f(x) \quad \forall x \in \mathbb{R}.$$

When we solved the heat equation using separation of variables we obtained a solution which could be written either using complex exponentials or using sines and cosines. For simplicity, I am taking the complex exponentials,

$$u(x,t) = \sum_{n\in\mathbb{Z}} e^{-n^2kt} c_n e^{inx}.$$

So, we would like

$$u(x,0) = \sum_{n\in\mathbb{Z}} c_n e^{inx} = f(x).$$

Now we know how to find the coefficients,

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

For the function, for example, $f(x) = |x|$ for $x \in (-\pi, \pi)$ which is defined on the rest of the real line to be 2π periodic, this is a function which makes sense as the initial temperature of the rod. We have computed these coefficients. The theory we will prove later will show that the Fourier series converges to $f(x)$ for all $x \in \mathbb{R}$. Moreover, the theory will show that our solution $u(x,t)$ is the unique solution to the heat equation with initial condition given by f. Nice!

We are not limited to computing Fourier series of periodic functions, it's just that Fourier series will always be periodic functions themselves. For example, consider the function $f(x) = x$ defined on $(-\pi, \pi)$. By the theory we shall prove later, the Fourier series will converge to this function inside the interval $(-\pi, \pi)$. Outside this interval, the series will converge to a function which is 2π periodic, and is equal to x for $x \in (-\pi, \pi)$. So this will have little jumps at the points $(2n+1)\pi$ for $n \in \mathbb{Z}$. It will be discontinuous there. We don't need to worry about that, it's no problem whatsoever. For the moment we just are content that the Fourier series will converge to $f(x) = x$ for $x \in (-\pi, \pi)$. This is because in our applications, we will use these series to solve PDEs in bounded intervals. For now we are working with the bounded interval $(-\pi, \pi)$ but later we'll see that we can use the same techniques to handle any bounded interval.

Exercise 3. *Compute in the same way the Fourier coefficients*

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} x e^{-inx} dx \quad n \in \mathbb{Z}.$$

Use that calculation to show that $a_n = 0$ for all n, and then compute the Fourier sine series,

$$\sum_{n \geq 1} b_n \sin(nx).$$

Exercise 4. *Look at these two Fourier series, that is the series for $|x|$ and x. Do the series converge? Do they converge absolutely? Compare and contrast them!*

1.2. **Introducing Hilbert spaces.** A Hilbert space is a complete normed vector space whose norm is induced by a scalar product.

Definition 3. A Hilbert space, H, is a vector space. This means that H is a set which contains elements. If f and g are elements of H, then for any $a, b \in \mathbb{C}$ we have

$$af + bg \in H.$$

This is what it means to be a vector space. Moreover, Hilbert spaces have two other nice features: a scalar product and a norm. Let us write the scalar product as

$$\langle f, g \rangle : H \times H \to \mathbb{C}.$$

To be a scalar product it must satisfy:

$$\langle af, g \rangle = a\langle f, g \rangle \quad \forall a \in \mathbb{C},$$

$$\langle h + f, g \rangle = \langle h, g \rangle + \langle f, g \rangle,$$

and

$$\langle f, g \rangle = \overline{\langle g, f \rangle}.$$

The norm is defined through the scalar product via:

$$||f|| := \sqrt{\langle f, f \rangle}.$$

The norm must satisfy

$$||f|| = 0 \iff f = 0, \quad ||f + g|| \leq ||f|| + ||g||.$$

Finally, what it means to be complete is that if a sequence $\{f_n\} \in H$ is Cauchy, which means that for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$||f_n - f_m|| < \varepsilon \quad \forall n, m \geq N,$$

then there exists $f \in H$ such that

$$\lim_{n \to \infty} f_n = f,$$

by which we mean that

$$\lim_{n \to \infty} ||f_n - f|| = 0.$$

Exercise 5. *As an example, we can take $H = \mathbb{C}^n$. For $z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n$ and $w = (w_1, \dots, w_n) \in \mathbb{C}^n$ the scalar product*

$$\langle z, w \rangle := \sum_{j=1}^{n} z_j \overline{w_j}.$$

Show that the scalar product defined in this way satisfies all the demands made upon it in the definition above. Why is $H = \mathbb{C}^n$ complete?

Now, let us fix a finite interval $[a, b]$. We shall be particularly interested in a Hilbert space known as $L^2([a, b])$ or once we have specified a and b, simply L^2. This is the actual grown-up mathematician definition of the Hilbert space, L^2. It can be gleefully ignored.

Definition 4 (The precise definition of L^2)**.** The Hilbert space $L^2([a, b])$ is the set of equivalence of classes of functions where f and g are equivalent if

$f(x) = g(x)$ for almost every $x \in [a, b]$ with respect to the one dimensional Lebesgue measure.

Moreover, for any f belonging to such an equivalence class, we require

$$\int_a^b |f(x)|^2 dx < \infty. \tag{1.1}$$

If f and g are each members of equivalence classes satisfying (1.1) the scalar product of f and g is then defined to be

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}dx. \tag{1.2}$$

One can prove that with this definition we obtain a Hilbert space.

Theorem 5. *The space $L^2([a, b])$ for any bounded interval $[a, b]$ defined as above, with the scalar product defined as above, is a Hilbert space.*

This theorem is beyond the scope of this course. Moreover, the precise mathematical definition of L^2 is overkill for what we would like to do (solve PDEs). This is why I offer you:

Definition 6 (Our working-definition of L^2)**.** $L^2([a, b])$ is the set of functions which satisfy (1.1), and is equipped with the scalar product defined in (1.2).

Although we don't necessarily need it right now, you may be happy to know that the L^2 scalar product satisfies a Cauchy-Schwarz inequality,

$$|\langle f, g\rangle| \leq ||f||||g||.$$

Exercise 6. *Use the Cauchy-Schwarz inequality to prove that for any $f \in L^2$ on the interval $[-\pi, \pi]$, the Fourier coefficients,*

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx,$$

satisfy

$$|c_n| \leq \frac{||f||}{\sqrt{2\pi}}.$$

2. Exercises to be done by oneself: Answers

1.3.7 Use separation of variables to find an infinite family of independent solutions to

$$u_t = ku_{xx}, \quad u(0, t) = 0, \quad u_x(\ell, t) = 0,$$

representing heat flow in a rod with one end held at temperature zero and the other end insulated.

Answer:

$$u_n(x,t) = e^{-(2n+1)^2\pi^2 kt/(4l^2)} \sin\left(\frac{(2n+1)\pi x}{2l}\right).$$