FOURIER ANALYSIS & METHODS

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ABSTRACT. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

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The following proposition shows that any function that is bounded on a closed interval is an $\mathcal{L}^2$ function.

Proposition 1 (The Standard Estimate)**.** *Assume f is defined on some interval $[a, b]$. Assume that f satisfies a bound of the form $|f(x)| \le M$ for $x \in [a, b]$.*[1] *Then,*

$$\left| \int_a^b f(x) dx \right| \le (b - a)M.$$

Proof: Standard estimate!

$$\left| \int_a^b f(x) dx \right| \le \int_a^b |f(x)| dx \le \int_a^b M dx = M(b - a).$$

Exercise 1. *Use The Standard Estimate to prove that any function which is continuous on a closed, bounded interval $[a, b]$ is in $\mathcal{L}^2$ on that interval.*

Example 1. So, it seems that a lot of functions will be in $\mathcal{L}^2$. What are some functions which are *not* in $\mathcal{L}^2$? Let's consider the interval $[-\pi, \pi]$. The function $f(x) = \frac{1}{x}$ is not in $\mathcal{L}^2$ on that interval, because

$$\int_{-\pi}^{\pi} \frac{1}{x^2} dx$$

is infinite. We could still have unbounded functions on this interval which *are* in $\mathcal{L}^2$, as long as their integrals can be defined. For example, let's define

$$f(x) := \begin{cases} x^{-1/3} & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

[1] We actually only need this for "almost every" x, but to make that precise, we need some Lebesgue measure theory.

Then, we can integrate

$$\int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{|x|^{5/3}3}{5}\Big|_{-\pi}^{\pi} = \frac{6\pi^{5/3}}{5}.$$

So, the function doesn't have to be bounded for the integral to be finite, but it also can't blow up too badly.

2. Bessel's Inequality (L^2 convergence of Fourier series)

Today we're going to investigate the issue of convergence of Fourier series. To move towards this question of convergence, we prove an important estimate known as the Bessel Inequality. Bessel's Theorem will be a very important ingredient in the proof of our first big theorem which is one of the *theory items,* which can appear on the exam.

Theorem 2 (Bessel Inequality)**.** *Assume that f is square-integrable on $[-\pi, \pi]$. Then the Fourier coefficients $\{c_n\}_{n\in\mathbb{Z}}$ of f satisfy*

$$\sum_{n\in\mathbb{Z}} |c_n|^2 \le \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx.$$

Proof: It is sufficient to show that

$$2\pi \sum_{n=-N}^{N} |c_n|^2 \le ||f||^2 \quad \forall N \in \mathbb{N}.$$

Since on the right side we have the L^2 norm of a function, we would like to have the L^2 norm of a function. Recall the Pythagorean Theorem: when $a \perp b$ then the length of the vector $a + b = c$ is equal to $a^2 + b^2$. The same thing works in higher dimensions. In particular, since the functions e^{inx} are orthogonal for $n \neq m$, it is also true that $c_n e^{inx}$ are orthogonal for $n \neq m$, so we have

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$$||\sum_{n=-N}^{N} c_n e^{inx}||^2 = \sum_{n=-N}^{N} ||c_n e^{inx}||^2 = \sum_{n=-N}^{N} 2\pi|c_n|^2. \tag{2.1}$$

Now, let's write

$$S_N(x) := \sum_{n=-N}^{N} c_n e^{inx}.$$

This is the partial Fourier expansion of f. Let us compare it to f using the L^2 norm:

$$\begin{aligned} 0 \le ||S_N - f||^2 = \langle S_N - f, S_N - f\rangle &= \langle S_N, S_N - f\rangle - \langle f, S_N - f\rangle \\ &= \langle S_N, S_N\rangle - \langle S_N, f\rangle - \langle f, S_N\rangle + \langle f, f\rangle \\ &= ||S_N||^2 - \langle S_N, f\rangle - \langle f, S_N\rangle + ||f||^2. \end{aligned}$$

Let us compute the two terms in the middle:

$$\langle S_N, f\rangle = \int_{-\pi}^{\pi} \sum_{n=-N}^{N} c_n e^{inx}\overline{f(x)}dx = \sum_{n=-N}^{N} c_n \int_{-\pi}^{\pi} e^{inx}\overline{f(x)}dx = \sum_{n=-N}^{N} c_n \overline{\int_{-\pi}^{\pi} e^{-inx} f(x)dx}$$

$$= \sum_{n=-N}^{N} c_n 2\pi \overline{c_n}.$$

We compute:

$$\langle f, S_N \rangle = \int_{-\pi}^{\pi} f(x) \sum_{n=-N}^{N} \overline{c_n e^{inx}} dx = \sum_{n=-N}^{N} \overline{c_n} \int_{-\pi}^{\pi} f(x) e^{-inx} dx = \sum_{n=-N}^{N} \overline{c_n} 2\pi c_n.$$

Since

$$|c_n|^2 = c_n \overline{c_n}$$

we have

$$0 \le ||S_N - f||^2 = ||S_N||^2 - \langle S_N, f\rangle - \langle f, S_N \rangle + ||f||^2 = ||S_N||^2 - 2(2\pi) \sum_{n=-N}^{N} |c_n|^2 + ||f||^2.$$

By (2.1), we have

$$0 \le 2\pi \sum_{n=-N}^{N} |c_n|^2 - 2(2\pi) \sum_{n=-N}^{N} |c_n|^2 + ||f||^2 \implies 2\pi \sum_{n=-N}^{N} |c_n|^2 \le ||f||^2.$$

Corollary 3. *We have*

$$\sum_{n \in \mathbb{N}} |a_n|^2 + |b_n|^2 = 4|c_0|^2 + 2 \sum_{n \in \mathbb{Z} \setminus 0} |c_n|^2,$$

and

$$\lim_{|n| \to \infty} \star_n = 0, \quad \star = a, b, \text{ or } c.$$

Exercise 2. *The proof is an exercise. First, use the previous exercises where we expressed the a's and b's in terms of the c's. Next, what can you say about the terms of a non-negative, convergent series?*

2.1. **Pointwise convergence of Fourier Series.** By Bessel's inequality, we know that

$$\sum_{n \in \mathbb{Z}} |c_n|^2 \le \frac{1}{2\pi} \int_{-\pi}^{\pi} |f|^2.$$

Now, it's important to note that when the series of $|c_n|^2$ converges, then eventually $|c_n|^2 < 1$ so also $|c_n| < 1$. Then, $|c_n| > |c_n|^2$. So, just because the series of $|c_n|^2$ converges, the series with just c_n might not. For example,

$$\sum_{n \ge 1} \frac{1}{n^2} < \infty$$

whereas

$$\sum_{n \ge 1} \frac{1}{n} = \infty.$$

So Bessel's inequality doesn't tell us that the Fourier series

$$\sum_{n \in \mathbb{Z}} c_n e^{inx}$$

always converges. This is a bit of a concern, because we want to use our method to solve PDEs. In fact, we will see that Fourier series always converge 'in norm,' meaning with respect to the L^2 norm. However, to solve PDEs, we would like the series to converge at specific points. To state the theorem which tells us when and how a Fourier series converges, we need the following definition.

Definition 4. A function is piecewise $\mathcal{C}^k$ on a bounded interval, I, if there is a finite set of points in the interval (possibly empty set) such that f is $\mathcal{C}^k$ on $I \setminus S$. Moreover, we assume that the left and right limits of $f^{(j)}$ exist at all of the points in S, for all $j = 0, 1, \ldots, k$.

Now we are going to prove the great big theorem about pointwise convergence of Fourier series.

Theorem 5 (Convergence of Fourier series)**.** *Assume that f is piecewise $\mathcal{C}^1$ on $[-\pi, \pi]$. Define f on the rest of $\mathbb{R}$ to be a 2π periodic function. Denote the left limit at x by $f(x_-)$ and the right limit by $f(x_+)$, so that for each $x \in \mathbb{R}$,*

$$f(x_{)} := \lim_{t \to x, t < x} f(t), \quad f(x_+) := \lim_{t \to x, t > x} f(t).$$

Let

$$S_N(x) := \sum_{-N}^{N} c_n e^{inx},$$

where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx.$$

Then

$$\lim_{N \to \infty} S_N(x) = \frac{1}{2} \left(f(x_-) + f(x_+) \right), \quad \forall x \in \mathbb{R}.$$

Proof: This is a big theorem, because it requires several clever ideas in the proof. Smaller theorems can be proven by just "following your nose." So, to try to help with the proof, we're going to highlight the big ideas. To learn the proof, you can start by learning all the big ideas in the order in which they're used. Once you've got these down, then try to fill in the math steps starting at one idea, working to get to the next idea. The big ideas are like light posts guiding your way through the dark and spooky math.

Idea 1: Fix a point $x \in \mathbb{R}$. This first step is more getting into a frame of mind. Think of x as fixed. Then the numbers $f(x_-)$ and $f(x_+)$ are just the left and right limits of f at x, so these are also fixed. Our goal is to prove that:

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$$\lim_{N \to \infty} S_N(x) = \frac{1}{2} \left(f(x_-) + f(x_+) \right). \tag{2.2}$$

Idea 2: Expand the series $S_N(x)$ using its definition.

$$S_N(x) = \sum_{-N}^{N} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iny} dy e^{inx}.$$

Now, let's move that lonely e^{inx} inside the integral so it can get close to its friend, e^{-iny}. Then,

$$S_N(x) = \sum_{-N}^{N} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) e^{-iny+inx} dy.$$

We want to prove (2.2). Above we have $f(y)$ rather than $f(x)$. This leads us to...

Idea 3: Change the variable. Let $t = y - x$.

Then $y = t + x$. We have

$$S_N(x) = \sum_{-N}^{N} \frac{1}{2\pi} \int_{-\pi-x}^{\pi-x} f(t+x)e^{-int}dt.$$

Remember that very first fact we proved for periodic functions? It said that the integral of a periodic function of period P from any point a to $a+P$ is the same, no matter what a is. Here $P = 2\pi$. This leads to...

Idea 4: Shift the integral

$$\int_{-\pi-x}^{\pi-x} f(t+x)e^{-int}dt = \int_{-\pi}^{\pi} f(t+x)e^{-int}dt.$$

Thus

$$S_N(x) = \sum_{-N}^{N} \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t+x)e^{-int}dt = \int_{-\pi}^{\pi} f(t+x)\frac{1}{2\pi}\sum_{-N}^{N} e^{int}dt.$$

Idea 4: Define the N^{th} Dirichlet kernel, $D_N(t)$.

$$D_N(t) = \frac{1}{2\pi}\sum_{-N}^{N} e^{int}.$$

Idea 5: Collect the even and odd terms of D_N to compute its integral.

Recall that

$$n \in \mathbb{N} \implies e^{int} + e^{-int} = 2\cos(nt), n > 0.$$

Hence, we can pair up all the terms ± 1, ± 2, etc, and write

$$D_N(t) = \frac{1}{2\pi} + \sum_{n=1}^{N} \frac{1}{\pi}\cos(nt).$$

So, $D_N(t)$ is an even function. Moreover, since $\cos(nt)$ is 2π periodic and even,

$$\int_{-\pi}^{\pi} \cos(nt)dt = 0 \quad \forall n \geq 1,$$

so

$$\int_{-\pi}^{\pi} D_N(t)dt = \int_{-\pi}^{\pi} \frac{1}{2\pi}dt = 1.$$

Since $D_N(t)$ is even, we also have:

dnint (2.3)
$$\int_{-\pi}^{0} D_N(t)dt = \frac{1}{2} = \int_{0}^{\pi} D_N(t)dt. \tag{2.3}$$

Idea 6: Go back to the original definition of $D_N(t)$ and re-write it to look like a geometric series.

As it stands, $D_N(t)$ looks almost like a geometric series, but the problem is that it goes from minus exponents to positive ones. We can fix that by factoring out the largest negative exponent, so

$$D_N(t) = \frac{1}{2\pi}e^{-iNt}\sum_{n=0}^{2N} e^{int}.$$

We know how to sum a partial geometric series. This gives

dngeo (2.4)
$$D_N(t) = \frac{1}{2\pi}e^{-iNt}\frac{1-e^{i(2N+1)t}}{1-e^{it}} = \frac{e^{-iNt} - e^{i(N+1)t}}{2\pi(1-e^{it})}. \tag{2.4}$$

Since

$$S_N(x) = \int_{-\pi}^{\pi} f(t+x) D_N(t) dt,$$

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(2.2) is equivalent to

$$\lim_{N\to\infty} \left| \int_{-\pi}^{\pi} f(t+x) D_N(t) dt - \frac{1}{2} \left(f(x_-) + f(x_+) \right) \right| = 0.$$

The S_N business has an integral, but the $f(x_\pm)$ don't. They have got a convenient factor of one half, so...

Idea 7: Use our calculation of the integral of D_N to write

$$\frac{1}{2} f(x_-) = \int_{-\pi}^{0} D_N(t) dt f(x_-), \quad \frac{1}{2} f(x_+) = \int_{0}^{\pi} D_N(t) dt f(x_+).$$

Hence we are bound to prove that

$$\lim_{N\to\infty} \left| \int_{-\pi}^{\pi} f(t+x) D_N(t) dt - \int_{-\pi}^{0} D_N(t) f(x_-) dt - \int_{0}^{\pi} D_N(t) f(x_+) dt \right| = 0.$$

It is quite natural now to split the integral into the left and right sides, so that we must prove

$$\lim_{N\to\infty} \left| \int_{-\pi}^{0} D_N(t)(f(t+x) - f(x_-)) dt + \int_{0}^{\pi} D_N(t)(f(t+x) - f(x_+)) dt \right|.$$

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Idea 8: Use the second property (2.4) we proved about $D_N(t)$.

$$\left| \int_{-\pi}^{0} D_N(t)(f(t+x) - f(x_-)) dt + \int_{0}^{\pi} D_N(t)(f(t+x) - f(x_+)) dt \right| =$$

$$\left| \int_{-\pi}^{0} \frac{e^{-iNt} - e^{i(N+1)t}}{2\pi(1-e^{it})} (f(t+x) - f(x_-)) dt + \int_{0}^{\pi} \frac{e^{-iNt} - e^{i(N+1)t}}{2\pi(1-e^{it})} (f(t+x) - f(x_+)) dt \right|.$$

Since there are these factors of e^{-iNt} and $e^{i(N+1)t}$, this sort of looks like some twisted version of a Fourier coefficient. This observation leads us to...

Idea 9: Define a new function

$$g(t) = \begin{cases} \frac{f(t+x)-f(x_-)}{1-e^{it}} & t \in [-\pi, 0) \\ \frac{f(t+x)-f(x_+)}{1-e^{it}} & t \in (0, \pi] \end{cases}.$$

The function g is well-defined on the interval $[-\pi, \pi] \setminus \{0\}$ because the denominator does not vanish there. Moreover, it has the same properties as f has on this interval. We extend g to all of $\mathbb{R}$ to be 2π periodic. What happens to g when $t \to 0$?

$$\lim_{t\to 0_-} \frac{f(t+x) - f(x_-)}{1-e^{it}} = \lim_{t\to 0_-} \frac{t(f(t+x) - f(x_-))}{t(1-e^{it})} = \frac{f'(x_-)}{-ie^{i0}} = if'(x_-).$$

For the other side, a similar argument shows that

$$\lim_{t\to 0_+} \frac{f(t+x) - f(x_-)}{1-e^{it}} = if'(x_+).$$

Therefore, g has finite left and right limits at $t = 0$, because f does. Hence, g is also a piecewise differentiable and piecewise continuous 2π periodic function. Consequently, g is bounded on $[-\pi, \pi]$ so it is in $L^2([-\pi, \pi])$ and Bessel's inequality holds.

Idea 10: Recognize the Fourier coefficients of the new function

$$\int_{-\pi}^{0} \frac{e^{-iNt} - e^{i(N+1)t}}{2\pi(1-e^{it})}(f(t+x)-f(x_-))dt + \int_{0}^{\pi} \frac{e^{-iNt} - e^{i(N+1)t}}{2\pi(1-e^{it})}(f(t+x)-f(x_+))dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-iNt} g(t)dt - \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(N+1)t} g(t)dt.$$

The first term above is by definition G_N, the N^{th} Fourier coefficient of g, whereas the second term above is by definition G_{-N-1}, the $-N-1$ Fourier coefficient of g. By Bessel's inequality,

$$\lim_{N\to\infty} G_N = 0 = \lim_{N\to\infty} G_{-N-1}.$$

2.2. **Exercises for the week from [1].**

2.2.1. *Exercises to be demonstrated in the large group.*

(1) Compute the Fourier series of the function defined on $(-\pi, \pi)$

$$f(x) := \begin{cases} 0 & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}.$$

(2) Compute the Fourier series of the function defined on $(-\pi, \pi)$

$$f(x) := |\sin(x)|.$$

(3) Compute the Fourier series of the function defined on $(-\pi, \pi)$

$$f(x) := \begin{cases} 1 & -a < x < a \\ -1 & 2a < x < 4a \\ 0 & \text{elsewhere in } (-\pi, \pi). \end{cases}.$$

Here one ought to assume that $0 < a < \pi$ for this to make sense.

(4) Compute the Fourier series of the function defined on $(-\pi, \pi)$

$$f(x) = x^2.$$

2.2.2. *Exercises to be done by oneself (earlier in the week).*

(1) Compute the Fourier series of the function defined on $(-\pi, \pi)$

$$f(x) := x(\pi - |x|).$$

(2) Compute the Fourier series of the function defined on $(-\pi, \pi)$

$$f(x) = e^{bx}.$$

(3) Use the Fourier series for the function $f(x) = |\sin(x)|$ to compute the sum

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} = \frac{1}{2}, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{4n^2-1} = \frac{\pi-2}{4}.$$

(4) Use the Fourier series for the function $f(x) = x(\pi - |x|)$ to compute the sum

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3} = \frac{\pi^3}{32}.$$

(5) Let $f(x)$ be the periodic function such that $f(x) = e^x$ for $x \in (-\pi, \pi)$, and extended to be 2π periodic on the rest of $\mathbb{R}$. Let

$$\sum_{n \in \mathbb{Z}} c_n e^{inx}$$

be its Fourier series. Therefore, by Theorem 2.1

$$e^x = \sum_{n \in \mathbb{Z}} c_n e^{inx}, \quad \forall x \in (-\pi, \pi).$$

If we differentiate this series term-wise then we get $\sum inc_n e^{inx}$. On the other hand, we know that $(e^x)' = e^x$. So, then we should have

$$\sum inc_n e^{inx} = \sum c_n e^{inx} \implies c_n = inc_n \quad \forall n.$$

This is clearly wrong. Where is the mistake?

2.2.3. *Exercises to be demonstrated in the small groups.*

(1) Use the Fourier series of the function $f(x) = x(\pi - |x|)$, defined on $(-\pi, \pi)$ and extended to be 2π periodic on $\mathbb{R}$, to compute the sums:

$$\sum_{n \geq 1} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \sum_{n \geq 1} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

(2) Use the Fourier series of the function $f(x) = e^{bx}$, defined on $(-\pi, \pi)$ and extended to be 2π periodic on $\mathbb{R}$, to compute the sum:

$$\sum_{n \geq 1} \frac{1}{n^2 + b^2} = \frac{\pi}{2b} \coth(b\pi) - \frac{1}{2b^2}.$$

(3) Use the Fourier series of the function $f(x) = x^2$, defined on $(-\pi, \pi)$ and extended to be 2π periodic on $\mathbb{R}$, to compute the sums:

$$x^2 - \pi^2 x = 12 \sum_{n \geq 1} \frac{(-1)^n \sin(nx)}{n^3}, \quad x \in (-\pi, \pi)$$

$$x^4 - 2\pi^2 x^2 = 48 \sum_{n \geq 1} \frac{(-1)^{n+1} \cos(nx)}{n^4} - \frac{7\pi^4}{15}$$

$$\sum_{n \geq 1} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

2.2.4. *Exercises to be done by oneself (later in the week).*

(1) Determine the Fourier sine and cosine series of the function

$$f(x) = \begin{cases} x & 0 \leq x \leq \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} \leq x \leq \pi \end{cases}$$

(2) Expand the function

$$f(x) = \begin{cases} 1 & 0 < x < 2 \\ -1 & 2 < x < 4 \end{cases}$$

in a cosine series on $[0, 4]$.

(3) Expand the function e^x in a series of the form

$$\sum_{n\in\mathbb{Z}} c_n e^{2\pi inx}, \quad x \in (0,1).$$

(4) Define

$$f(t) = \begin{cases} t & 0 \le t \le 1 \\ 1 & 1 < t < 2 \\ 3-t & 2 \le t \le 3 \end{cases}$$

and extend f to be 3-periodic on $\mathbb{R}$. Expand f in a Fourier series. Determine, in the form of a Fourier series, a 3-periodic solution to the equation

$$y''(t) + 3y(t) = f(t).$$

References

[1] Gerald B. Folland, *Fourier Analysis and Its Applications,* Pure and Applied Undergraduate Texts Volume 4, (1992).