FRACTALS

Contents

1. Algebras, sigma algebras, and measures

We begin by defining an algebra. This could also be called "an algebra of sets." Below we use the notation

$$P(X) = \text{the set of all subsets of } X.$$

Definition 1.1. Let X be a set. A subset $\mathcal{A} \subset P(X)$ is called an *algebra* if

(1) $X \in \mathcal{A}$
(2) $Y \in \mathcal{A} \implies X \setminus Y =: Y^c \in \mathcal{A}$
(3) $A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}$

$\mathcal{A}$ is a σ-algebra if in addition

$$\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{A} \implies \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}.$$

Remark 1. First, note that since $X \subset \mathcal{A}$, and algebras are closed under complementation, (yes it is a real word), one always has

$$\emptyset = X^c \in \mathcal{A}.$$

Moreover, we note that algebras are always closed under intersections, since for $A, B \in \mathcal{A}$,

$$A \cap B = (A^c \cup B^c)^c \in \mathcal{A},$$

since algebras are closed under complements and unions. Consequently, σ-algebras are closed under countable intersections.

We will often use the symbol σ in describing countably-infinite properties.

Exercise 1. *What is the smallest possible algebra? What is the next-smallest algebra? Continue building up algebras. Now, let X be a topological space. The Borel σ-algebra is defined to be the smallest σ-algebra which contains all open sets. What other kinds of sets are contained in the Borel σ-algebra?*

With the notion of σ-algebra, we can define a *measure.*

Definition 1.2. Let X be a set and $\mathcal{A} \subset P(X)$ a σ-algebra. A *measure* μ is a countably additive, set function which is defined on the σ-algebra, $\mathcal{A}$, such that $\mu(\emptyset) = 0$. The elements of $\mathcal{A}$ are known as *measurable sets.* We will only work with non-negative measures, but there is such a thing as a signed measure. Just so you know those beasties are out there. Countably additive means that for a countable disjoint collection of sets in the σ-algebra

$$\{A_n\} \subset \mathcal{A} \text{ such that } A_n \cap A_m = \emptyset \forall n \neq m \implies \mu\left(\bigcup A_n\right) = \sum \mu(A_n).$$

We shall refer to $(X, \mathcal{A}, \mu)$ as a *measure space.* What this means is that a measure space is comprised of a big set, X, and a certain collection of subsets of X, which is the σ-algebra, $\mathcal{A}$. Moreover, there is a *measure,* μ, which is a countably additive set function that is defined on all elements of $\mathcal{A}$.

Proposition 1.3 (Measures are monotone)**.** *Let $(X, \mathcal{A}, \mu)$ be a measure space. Then μ is finitely additive, that is if $A \cap B = \emptyset$ for two elements A, $B \in \mathcal{A}$, we have*

$$\mu(A \cup B) = \mu(A) + \mu(B).$$

Moreover, μ is monotone, that is for any $A \subset B$ which are both elements of $\mathcal{A}$ we have

$$\mu(A) \leq \mu(B).$$

Proof: First we make the rather trivial observation that if A and B are two elements of $\mathcal{A}$ with empty intersection, then

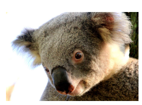
$$A \cup B = \cup A_j, \quad A_1 = A, \quad A_2 = B, \quad A_j = \emptyset \forall j \geq 3.$$

Then we have

$$\mu(A \cup B) = \mu(\cup A_j) = \sum_j \mu(A_j) = \mu(A) + \mu(B),$$

since $\mu(\emptyset) = 0$. For the monotonicity, if $A \subset B$ are two elements of $\mathcal{A}$, then

$$\mu(B) = \mu(B \setminus A \cup A) = \mu(B \setminus A) + \mu(A) \geq \mu(A),$$

since $\mu \geq 0$. 

So, in layman's terms, when we've got a measure space, we have a big set, X, together with a collection of subsets of X (note that X is a subset of itself, albeit not a *proper* subset), for which we have a notion of *size.* This size is the value of the function μ. So, if $Y \in \mathcal{A}$, then $\mu(Y)$ is the *measure* of Y. Roughly speaking, $\mu(Y)$ tells us how much space within X the set Y is occupying. For the case of the Lebesgue measure on $\mathbb{R}^n$, and the n-dimensional Hausdorff measure, we shall see that measure coincides with our usual notion of n-dimensional volume.

Proposition 1.4 (How to disjointify sets and countable sub-additivity)**.** *If $\{A_n\} \subset \mathcal{A}$ is a countable collection of sets, then we can find a disjoint collection $\{B_n\} \subset \mathcal{A}$ such that*

$$\cup A_n = \cup B_n.$$

Let μ be a measure defined on the σ-algebra, $\mathcal{A}$. Then countable sub-additivity holds for not-necessarily-disjoint countable collections of sets, which means that for all such $\{A_n\}$ as above,

$$\mu(\cup A_n) \leq \sum \mu(A_n).$$

Proof: We do this by setting

$$B_1 := A_1, \quad B_n := A_n \setminus \cup_{k=1}^{n-1} B_k, \qquad n \geq 2.$$

Then for $m > n$, note that

$$B_m \cap B_n = \left(A_m \setminus \cup_{k=1}^{m-1} B_k\right) \cap B_n = \emptyset$$

since

$$B_n \subset \cup_{k=1}^{m-1} B_k,$$

since $n \leq m-1$. Thus they are in fact disjoint. Moreover,

$$B_1 = A_1, \quad B_2 \cup B_1 = A_2 \setminus A_1 \cup A_1 = A_2 \cup A_1.$$

Similarly, by induction, assuming that

$$\cup_{k=1}^{n} B_k = \cup_{k=1}^{n} A_k,$$

we have

$$\cup_{k=1}^{n+1} B_k = B_{n+1} \cup \cup_{k=1}^{n} B_k = A_{n+1} \cup \cup_{k=1}^{n} B_k = A_{n+1} \cup \cup_{k=1}^{n} A_n$$

where in the last equality we used the induction hypothesis. Thus,

$$\cup_{n \geq 1} B_n = \cup_{n \geq 1} A_n.$$

Moreover, the way we have defined B_n together with the definition of the σ-algebra, $\mathcal{A}$, shows that $B_n \in \mathcal{A}$ for all n. By the monotonicity of μ,

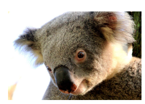
$$B_n \subset A_n \forall n \implies \mu(B_n) \leq \mu(A_n).$$

By the countable additivity for the disjoint sets, $\{B_n\}$, and since $\cup B_n = \cup A_n$

$$\mu(\cup A_n) = \mu(\cup B_n) = \sum \mu(B_n) \leq \sum \mu(A_n).$$

So, for not-necessarily disjoint sets, we have countable subadditivity, which means that

$$\mu(\cup A_n) \leq \sum \mu(A_n),$$

for all countable collections of sets $\{A_n\} \subset \mathcal{A}$. 

Definition 1.5. A measure space $(X, \mathcal{A}, \mu)$ is σ-finite if there exists a collection of sets $\{A_n\} \subset \mathcal{A}$ such that

$$X = \cup A_n, \quad \text{and} \quad \mu(A_n) < \infty \quad \forall n.$$

Exercise 2. *What are some examples of σ-finite measure spaces? What are some examples of measure spaces which are* not *σ-finite?*

One unfortunate fact about measures is that they're not defined on arbitrary sets, only on measurable sets (remember, those are the ones in the associated σ algebra). However, there is a way to define a set function which is almost like a measure and is defined for *every imaginable or unimaginable set.* This thing is called an *outer measure.*

Definition 1.6. Let X be a set. An outer measure μ^* on X is a map from $P(X) \to [0, \infty]$ such that

$$\mu^*(\emptyset) = 0, \quad A \subset B \implies \mu^*(A) \leq \mu^*(B),$$

and

$$\mu^*(\cup A_n) \leq \sum \mu^*(A_n).$$

Whenever things are indexed with n or some other letter and are not obviously indicated to be uncountable or finite, we implicitly are referring to a set indexed by the natural numbers.

1.1. **Carathéodory's outer measures.** We will require techniques from a great French mathematician, Carathéodory.

Proposition 1.7 (Outer Measures). *Let $E \subset P(X)$ such that $\emptyset \in E$. Let ρ be a map from elements of E to $[0,\infty]$ such that $\rho(\emptyset) = 0$. Then we can define for every element $A \in P(X)$*

$$\rho^*(A) := \inf\left\{\sum \rho(E_j) : E_j \in E, A \subset \cup E_j\right\},$$

where we assume that $\inf\{\emptyset\} = \infty$, so that if it is impossible to cover a set A by elements of E then $\rho^*(A) := \infty$. So defined, ρ^* is an outer measure.

Proof: Note that ρ^* is defined for every set. Now since $\emptyset \subset \emptyset = \cup E_j$, taking all $E_j = \emptyset \in E$ we have the cover for $\emptyset$ given by this particular choice of $\{E_j\} \subset E$. Therefore, since $\rho \geq 0$, we have that $\rho^* \geq 0$, and on the other hand since it is an infimum,

$$0 \leq \rho^*(\emptyset) \leq \sum_j \rho(\emptyset) = 0 \implies \rho^*(\emptyset) = 0.$$

This is the first condition an outer measure must satisfy.
Next, let's assume $A \subset B$. (By $\subset$ we always mean $\subseteq$). Then, since any covering of B by elements of E is also a covering of A by elements of E, it follows that the infimum over coverings of A is an infimum over a potentially *larger set of objects (namely coverings)* as compared with the infimum over coverings of B. Hence we have

$$\rho^*(A) = \inf\{\sum \rho(E_j) : E_j \in E, A \in \cup E_j\} \leq \inf\{\sum \rho(E_j) : E_j \in E, B \in \cup E_j\} = \rho^*(B).$$

This is the second condition.
Finally, we must show that ρ^* is countably subadditive. So, let $\{A_n\}$ be a collection of sets in $P(X)$. If for any n we have no cover of A_n by elements of E, then since

$$A_n \subset \cup_k A_k,$$

there is no cover of $\cup_k A_k$ by elements of E either. Hence we have

$$\rho^*(\cup A_n) = \infty, \quad \rho^*(A_n) = \infty \leq \sum \rho^*(A_k) \implies \rho^*(\cup A_n) = \infty = \sum \rho^*(A_n).$$

Thus countable subadditivity is verified in this case.
So, to complete the proof, we assume that each A_n admits at least one covering by elements of E. Let $\varepsilon > 0$. Since the definition of ρ^* is by means of an infimum, for each $j \in \mathbb{N}$ there exists a countable collection of sets $\{E_j^k\}_{k=1}^\infty$ where each $E_j^k \in E$, such that

$$\rho^*(A_j) \geq \sum_{k\geq 1} \rho(E_j^k) - \frac{\epsilon}{2^j} \implies \rho^*(A_j) + \frac{\epsilon}{2^j} \geq \sum_{k\geq 1} \rho(E_j^k).$$

Well then, the collection $\{E_j^k\}$ is a countable collections of elements of E which covers

$$\cup A_j.$$

Therefore by the definition of ρ^* as the infimum over such covers, we have

$$\rho^*(\cup A_j) \leq \sum_{j,k\geq 1} \rho(E_j^k).$$

Since for each E_j^k we have

$$\rho^*(A_j) + \frac{\epsilon}{2^j} \geq \sum_{k\geq 1} \rho(E_j^k),$$

summing over k we have

$$\sum_{j,k\geq 1} \rho(E_j^k) \leq \sum_{j\geq 1} \rho^*(A_j) + \frac{\epsilon}{2^j} = \epsilon + \sum_{j\geq 1} \rho^*(A_j).$$

Thus following all the inequalities we have

$$\rho^*(\cup A_j) \leq \epsilon + \sum_j \rho^*(A_j).$$

Since this inequality holds for arbitrary $\epsilon > 0$, we may let $\epsilon \to 0$, and the inequality also holds without that pesky ϵ. Hence we have verified countable subadditivity in this last case as well.

For each measure space, there is a canonically associated outer measure.

Corollary 1.8. *Let $(X, \mathcal{A}, \mu)$ be a measure space. Then, there is a canonically associated outer measure induced by μ defined by*

$$\mu^*(A) := \inf\{\sum \mu(E_j), \quad \{E_j\} \subset \mathcal{A}, \, A \subset \cup E_j\}.$$

Proof: By the definition of measure space, we have that $\emptyset \in \mathcal{A}$, and $\mu(\emptyset) = 0$. Moreover, $\mu : \mathcal{A} \to [0, \infty]$. Finally, we note that since for any $A \in P(X)$, $A \subset X \in \mathcal{A}$, we can always find a covering of such A by elements of $\mathcal{A}$. (In particular, one covering is to take $E_j = X$ for all j). Thus, μ^* is defined for all $A \in P(X)$. Moreover, μ and $\mathcal{A}$ satisfy the hypotheses of the preceding proposition. Therefore, since μ^* is defined in an analogous way to ρ^*, by the preceding proposition we also have that μ^* is an outer measure. 

Remark 2. For a measure space $(X, \mathcal{A}, \mu)$, we shall use μ^* to denote the *canonically associated outer measure,* which is defined according to the corollary. One of the reasons we require the notion of an outer measure is because it is used to define what it means for a measure space to be *complete.*

Exercise 3. *For those of you who have taken integration theory, what is the difference between the Lebesgue sigma algebra and the Borel sigma algebra? What is the definition of a measurable function $f : \mathbb{R} \to \mathbb{R}$?*

1.2. **Completeness.** If our notion of size (volume) defined in terms of the measure of sets belonging to a sigma algebra is a good notion, then if a certain set has size zero, anything contained within that set ought to also have size zero. *Eller hur?* It is precisely this observation that motivates the definition of a *complete* measure, which can be formulated in two different but equivalent ways.

Proposition 1.9 (Completeness Proposition)**.** *The following are equivalent for a measure space $(X, \mathcal{M}, \mu)$. If either of these hold, then μ is called* complete.

(1) If there exists $N \in \mathcal{M}$ with $\mu(N) = 0$, and $Y \subset N$, then $Y \in \mathcal{M}$.
(2) If $\mu^*(Y) = 0$ then $Y \in \mathcal{M}$.

Proof:
First let us assume (1) holds. Then if $Y \subset X$ with $\mu^*(Y) = 0$, by the definition of μ^* for each $k \in \mathbb{N}$ there exists

$$\{E_n^k\}_{n \geq 1} \subset \mathcal{M}, \quad Y \subset \cup_n E_n^k, \quad \sum_n \mu(E_n^k) < 2^{-k}.$$

Well, then

$$Y \subset N := \cap_k \cup_n E_n^k \in \mathcal{M},$$

where the containment holds because $\mathcal{M}$ is a σ-algebra. Since $N \subset \cup_n E_n^k$ for each $k \in \mathbb{N}$, by monotonicity of the measure

$$\mu(N) \leq \mu(\cup_n E_n^k) < 2^{-k} \forall k \in \mathbb{N} \implies \mu(N) = 0.$$

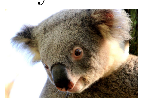
By the assumption of (1) since $Y \subset N \in \mathcal{M}$ and $\mu(N) = 0$, it follows that $Y \in \mathcal{M}$. So, every set with outer measure zero is measurable (that's what (2) says!)
Next, we assume (2) holds. Then if there exists $N \in \mathcal{M}$ with $\mu(N) = 0$ and $Y \subset N$, then

$$Y \subset \cup A_j, \quad A_1 := N, \quad A_j = \emptyset \forall j \geq 2,$$

and $\{A_j\} \subset \mathcal{M}$. So, by definition of outer measure,

$$0 \leq \mu^*(Y) = \inf \ldots \leq \sum \mu(A_j) = \mu(N) = 0.$$

Consequently $\mu^*(Y) = 0$, and by the assumption (2), $Y \in \mathcal{M}$. This shows that (2) $\implies$ (1).

Hence, they are equivalent. 

1.3. **Homework: Measure theory basics.**

(1) Let X be a finite set. How many elements does $P(X)$ contain? Prove your answer!
(2) Given a measure space $(X, \mathcal{A}, \mu)$ and $E \in \mathcal{A}$, define

$$\mu_E(A) = \mu(A \cap E)$$

for $A \in \mathcal{A}$. Prove that μ_E is a measure.
(3) Prove that the intersection of arbitrarily many σ-algebras is again a σ-algebra. Does the same hold for unions?
(4) Let A be an infinite σ-algebra. Prove that A contains uncountably many elements.
(5) Let $X = \mathbb{N}$, and define the algebra $\mathcal{A} = P(X)$. Prove that all elements of $\mathcal{A}$ are either countably infinite, finite, or empty. Define the measure to be 1 on a single element of $\mathbb{N}$ and 0 on the empty set. Prove that this satisfies the definition of a measure space. Will it also work to take $X = \mathbb{R}$, and let $\mathcal{A} = P(\mathbb{R})$, using the same definition of the measure? Do we get a measure space? Why or why not?

2. Completion of a measure, creating a measure from an outer measure, and pre-measures

Theorem 2.1 (Completion of a measure)**.** *Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\mathcal{N} := \{N \in \mathcal{M} \mid \mu(N) = 0\}$ and*

$$\bar{\mathcal{M}} = \{E \cup F \mid E \in \mathcal{M} \text{ and } F \subset N \text{ for some } N \in \mathcal{N}\}.$$

Then $\bar{\mathcal{M}}$ is a σ-algebra and $\exists!$ extension $\bar{\mu}$ of μ to a complete measure on $\bar{\mathcal{M}}$. Moreover, if $\mathcal{A}$ is a σ-algebra which contains $\mathcal{M}$, such that $(X, \mathcal{A}, \nu)$ is a complete measure space, and ν restricted to $\mathcal{M}$ is equal to μ, then $\mathcal{A} \supset \bar{\mathcal{M}}$. In this sense, $(X, \bar{\mathcal{M}}, \bar{\mu})$ is the minimal complete extension of $(X, \mathcal{M}, \mu)$ to a complete measure space.

Proof: First we show that $\overline{\mathcal{M}}$ is a σ-algebra. We observe that every element of $\mathcal{M}$ can be written as itself union with $\emptyset$, and $\emptyset \subset \emptyset \in \mathcal{N}$. So it follows that every element of $\mathcal{M}$ is an element of $\bar{\mathcal{M}}$. Next, assume that $\{A_n\} \subset \bar{\mathcal{M}}$ and $\{E_n, N_n\} \subset \mathcal{M}$ such that

$$A_n = E_n \cup F_n, \quad F_n \subset N_n \in \mathcal{N}.$$

Then

$$N := \cup N_n \in \mathcal{M}, \quad \text{and} \quad \mu(\cup N_n) \leq \sum \mu(N_n) = 0.$$

Since $\emptyset \subset N$, we have by the monotonicity of μ that

$$0 = \mu(\emptyset) \leq \mu(N) \leq \sum \mu(N_n) = 0.$$

We also have that

$$E := \cup E_n \in \mathcal{M}.$$

Then, let us define $F := \cup F_n \subset N$. It follows that

$$\cup A_n = E \cup F \in \bar{\mathcal{M}}.$$

Consequently $\bar{\mathcal{M}}$ is closed under countable unions. What about complements? If $A = E \cup F \in \bar{\mathcal{M}}$ with $F \subset N \in \mathcal{N}$ then note that

$$(E \cup F)^c = E^c \cap F^c = ((E^c \cap N) \cup (E^c \cap N^c)) \cap F^c,$$

and since $F \subset N \implies F^c \supset N^c$, the intersection of the last two terms is just $E^c \cap N^c$, so

$$(E \cup F)^c = (E^c \cap N \cap F^c) \cup (E^c \cap N^c).$$

Since $E, N \in \mathcal{M} \implies E^c \cap N^c \in \mathcal{M}$, and $E^c \cap N \cap F^c \subset N \in \mathcal{N}$ we see that $(E \cup F)^c \in \bar{\mathcal{M}}$. So, $\bar{\mathcal{M}}$ is closed under complements. Hence, we have shown that $\bar{\mathcal{M}}$ is a σ-algebra which contains $\mathcal{M}$.

Next, we must demonstrate that $\bar{\mu}$ is a well-defined, complete, and unique extension of μ. It is natural to ignore the subset of the zero-measure set, so we define

$$\bar{\mu}(E \cup F) := \mu(E).$$

If we have another representation of $E \cup F = G \cup H$ with $G \in \mathcal{M}$ and $F, H \subset N, M \in \mathcal{N}$, respectively, then

$$\bar{\mu}(E \cup F) = \mu(E).$$

Since $E \subset E \cup F = G \cup H \subset G \cup M$, with $G \cup M \in \mathcal{M}$, we have by the monotonicity of μ,

$$\mu(E) \leq \mu(G \cup M) \leq \mu(G) + \mu(M) = \mu(G).$$

Above, we have used countable subadditivity and the fact that $M \in \mathcal{N}$. Then, we note that

$$\bar{\mu}(G \cup H) = \mu(G),$$

as we have defined $\bar{\mu}$. So, following the equalities and inequalities, we have

$$\bar{\mu}(E \cup F) = \mu(E) \leq \mu(G) = \bar{\mu}(G \cup H).$$

To complete the argument, we use the Shakespeare technique: what is in a name? Would not a rose by any other name smell as sweet? Simply repeat the same argument above, replacing E by G and F by H, that is we do the same mathematical argument but we simply swap the names. Then we obtain

$$\bar{\mu}(G \cup H) \leq \bar{\mu}(E \cup F).$$

Hence we have shown that

$$\bar{\mu}(E \cup F) = \bar{\mu}(G \cup H).$$

We conclude that $\bar{\mu}$ is well-defined.

Now, let's show that $\bar{\mu}$ is a measure which extends μ. By definition, for $E \subset \mathcal{M}$

$$\bar{\mu}(E) = \bar{\mu}(E \cup \emptyset) = \mu(E).$$

So, this shows that

$$\bar{\mu}|_{\mathcal{M}} = \mu.$$

We also observe that since

$$\emptyset \in \mathcal{M} \implies \bar{\mu}(\emptyset) = \mu(\emptyset) = 0.$$

Next we wish to show monotonicity. If

$$A = E \cup F, \quad E \in \mathcal{M}, \quad F \subset N \in \mathcal{N},$$

and

$$A \subset B = G \cup H, \quad G \in \mathcal{M}, \quad H \subset M \in \mathcal{N},$$

then we have

$$E \subset A \subset B = G \cup H \subset G \cup M \implies$$
$$\bar{\mu}(A) = \mu(E) \leq \mu(G \cup M) \leq \mu(G) + \mu(M) = \mu(G) = \bar{\mu}(B).$$

We therefore have shown that $\bar{\mu}$ is monotone.

Next we wish to show that $\bar{\mu}$ is countably additive. Assume that $\{A_n\} = \{E_n \cup F_n\} \subset \bar{\mathcal{M}}$ are disjoint. Then

$$A_n \cap A_m = E_n \cup F_n \cap (E_m \cup F_m) \supset E_n \cap E_m,$$

which shows that

$$E_n \cap E_m = \emptyset, \quad \forall n \neq m.$$

Consequently,

$$\bar{\mu}(\cup A_n) = \mu(\cup E_n) = \sum \mu(E_n) = \sum \bar{\mu}(A_n).$$

So, $\bar{\mu}$ is countably additive. We have therefore proven that $\bar{\mu}$ is a measure on $\bar{\mathcal{M}}$.
Let's show that $\bar{\mu}$ is complete. Assume that $Y \in \bar{\mathcal{M}}$ with $\bar{\mu}(Y) = 0$. Then we can write

$$Y = E \cup F, \quad E \in \mathcal{N}, \quad F \subset N \in \mathcal{N}.$$

Hence, in particular,

$$Y \subset E \cup N \in \mathcal{N}.$$

Therefore $Z \subset Y \subset N$. We can therefore write Z as

$$Z = \emptyset \cup Z, \quad \emptyset \in \mathcal{M}, \quad Z \subset N \in \mathcal{N}.$$

It follows from the definition of $\mathcal{M}$ that $Z \in \bar{\mathcal{M}}$. Thus, any subset of a $\bar{\mathcal{M}}$ measurable set which has $\bar{\mu}$ measure zero is also an element of $\bar{\mathcal{M}}$, which is the first of the equivalent conditions required to be a complete measure.
Finally the uniqueness. Let's assume ν also extends μ to a complete measure on $\mathcal{M}$. This means that

$$\nu|_{\mathcal{M}} = \bar{\mu}|_{\mathcal{M}} = \mu.$$

For $Y = E \cup F \in \bar{\mathcal{M}}$, we also have $Y \subset E \cup N$, so by countable subadditivity,

$$\nu(Y) \leq \nu(E) + \nu(N) = \mu(E) + \mu(N) = \mu(E) = \bar{\mu}(Y).$$

Conversely

$$\bar{\mu}(Y) = \mu(E) = \nu(E) \leq \nu(E \cup F) = \nu(Y).$$

So, we've got equality all across, and in particular, $\nu(Y) = \bar{\mu}(Y)$.
Finally, let us assume that there is some other extension, φ, of μ to a complete measure on some σ-algebra $\mathcal{A}$ which contains $\mathcal{M}$. Thus, $(X, \mathcal{A}, \varphi)$ is a complete measure space, and

$$\varphi|_{\mathcal{M}} = \mu.$$

Then

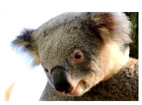
$$\varphi(N) = 0 \quad \forall N \in \mathcal{N}.$$

Now, let $E \cup F \in \mathcal{M}$. Then $E \in \mathcal{M}$, and thus $E \in \mathcal{A}$ is also true. Moreover, $F \subset N \in \mathcal{N}$, and so

$$N \in \mathcal{A}, \quad \varphi(N) = \mu(N) = 0.$$

Since $\mathcal{A}$ is complete, by the completeness proposition, we have that

$$F \in \mathcal{A} \implies E \cup F \in \mathcal{A}.$$

We have therefore proven that $\bar{\mathcal{M}} \subset \mathcal{A}$. 

Proposition 2.2 (Null Set Proposition)**.** *Let $(X, \mathcal{M}, \mu)$ be a non-trivial measure space, meaning there exist measurable subsets of positive measure. Then*

$$\mathcal{N} := \{Y \in \mathcal{M} : \mu(Y) = 0\}$$

is not *a σ-algebra, but it* is *closed under countable unions.*

Proof: If $\{N_n\} \subset \mathcal{N}$ is a countable collection, then since $\mathcal{M}$ is a σ-algebra,

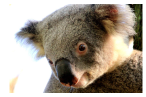
$$\cup N_n \in \mathcal{M}.$$

Moreover, we have

$$\mu(\cup N_n) \leq \sum \mu(N_n) = 0 \implies \mu(\cup N_n) = 0.$$

This shows that $\mathcal{N}$ is closed under countable unions. Why is it however, *not* a σ-algebra? It's not even an algebra! This is because it is not closed under complements. What is always an element of $\mathcal{N}$? The $\emptyset$ is always measurable and has measure zero. Hence $\emptyset \in \mathcal{N}$. What about its complement? This is where the non-triviality hypothesis plays a role. There is some $Y \in \mathcal{M}$ such that $\mu(Y) > 0$. Since $Y \subset X$, by monotonicity

$$\mu(X) \geq \mu(Y) > 0 \implies X = \emptyset^c \notin \mathcal{N}.$$



We shall now see that once we have an outer measure, we can build a sigma algebra and a measure, and obtain a complete measure space!

Theorem 2.3 (Carathéodory: creating a measure from an outer measure)**.** *Let μ^* be an outer measure on X. A set $A \subset X$ is called measurable with respect to $\mu^* \Leftrightarrow \forall\ E \subset X$ the following equation holds:*

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c). \qquad (*)$$

Then $\mathcal{M} := \{A \subset X | A \text{ is } \mu^* \text{ measurable}\}$ is a σ-algebra and $\mu^*\big|_{\mathcal{M}}$ is a complete measure.

Proof: Note that $A \in \mathcal{M} \Rightarrow A^c \in \mathcal{M}$ because (*) is symmetric in A and A^c. Since $\mu^*(\emptyset) = 0$, we have

$$\mu^*(E \cap \emptyset) + \mu^*(E \cap \emptyset^c) = \mu^*(\emptyset) + \mu^*(E \cap X) = 0 + \mu^*(E) = \mu^*(E).$$

Consequently, $\emptyset \in \mathcal{M}$.
Next we will show that $\mathcal{M}$ is closed under finite unions of sets. For $A, B \in \mathcal{M}$ and $E \subset X$ we get, by multiple use of (*):

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) = \mu^*((E \cap A) \cap B) + \mu^*((E \cap A) \cap B^c) + \mu^*((E \cap A^c) \cap B) + \mu^*((E \cap A^c) \cap B^c).$$

Furthermore, we can write $A \cup B = (A \cap B) \cup (A \cap B^c) \cup (A^c \cap B)$, so that

$$E \cap (A \cup B) = (E \cap (A \cap B)) \cup (E \cap (A \cap B^c)) \cup (E \cap (A^c \cap B)),$$

so by countable subadditivity of outer measures, we have

$$\mu^*(E \cap A \cap B) + \mu^*(E \cap A^c \cap B) + \mu^*(E \cap A \cap B^c) \geq \mu^*(E \cap (A \cup B))$$

Since $E \cap A^c \cap B^c = E \cap (A \cup B)^c$, using this inequality in the above equation gives us:

$$\mu^*(E) \geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c).$$

Moreover, by countable subadditivity of outer measures,

$$\mu^*(E) = \mu^*[(E \cap (A \cup B)) \cup (E \cap (A \cup B)^c)] \leq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)).$$

So the inequality is actually an equality, since we have shown that

$$\mu^*(E) \geq \mu^*(E \cap (A \cup B)) + \mu^*(E \cap (A \cup B)^c) \geq \mu^*(E).$$

Hence $A \cup B \in \mathcal{M}$.
Next we show that μ^* is finitely-additive:

$$\forall\, A, B \in \mathcal{M},\ A \cap B = \emptyset \Rightarrow \mu^*(A \cup B) = \mu^*((A \cup B) \cap A) + \mu^*((A \cup B) \cap A^c) = \mu^*(A) + \mu^*(B).$$

Now we will show that $\mathcal{M}$ is actually a σ-algebra: For $\{A_j\}_{j \in \mathbb{N}} \subset \mathcal{M}$ we can define a sequence of disjoint sets $\{B_j\}_{j \in \mathbb{N}} \subset \mathcal{M}$ fulfilling $\bigcup_{j \in \mathbb{N}} A_j = \bigcup_{j \in \mathbb{N}} B_j$ by:

$$B_1 := A_1, \quad B_n := A_n \setminus \cup_{k=1}^{n-1} B_k, \quad n \geq 2.$$

Let us also define

$$\tilde{B}_n := \bigcup_{j=1}^{n} B_j.$$

Then since $\mathcal{M}$ is closed under finite unions of sets and also closed under complementation, both

$$\tilde{B}_n \in \mathcal{M}, \quad B_n \in \mathcal{M}.$$

So, we need to show that

$$\bigcup_{j \in \mathbb{N}} A_n = \bigcup_{j \in \mathbb{N}} B_j \in \mathcal{M}.$$

For $E \subset X$, since $B_n \in \mathcal{M}$,

$$\mu^*(E \cap \tilde{B}_n) \overset{(*)}{=} \mu^*(E \cap \tilde{B}_n \cap B_n) + \mu^*(E \cap \tilde{B}_n \cap {B_n}^c) \; = \; \mu^*(E \cap B_n) + \mu^*(E \cap \tilde{B}_{n-1})$$

Thus $\mu^*(E \cap \tilde{B}_n) = \mu^*(E \cap B_n) + \mu^*(E \cap \tilde{B}_{n-1})$. Repeating this argument, we have $\mu^*(E \cap \tilde{B}_{n-1}) = \mu^*(E \cap B_{n-1}) + \mu^*(E \cap \tilde{B}_{n-2})$. Continuing inductively, we have:

$$\mu^*(E \cap \tilde{B}_n) = \mu^*(E \cap B_n) + \mu^*(E \cap B_{n-1}) + \mu^*(E \cap \tilde{B}_{n-2}) = ... = \sum_{k=1}^{n} \mu^*(E \cap B_k)$$

Using this result together with the fact that $\tilde{B}_n \in \mathcal{M}$, we get:

$$\mu^*(E) = \mu^*(E \cap \tilde{B}_n) + \mu^*(E \cap \tilde{B}_n^c) = \sum_{k=1}^{n} \mu^*(E \cap B_k) + \mu^*(E \cap \tilde{B}_n^c)$$

$$\geq \sum_{k=1}^{n} \mu^*(E \cap B_k) + \mu^*(E \setminus (\bigcup_{k=1}^{\infty} B_k))$$

Above, we have used that

$$E \cap \tilde{B}_n^c = E \cap (\cup_{k=1}^n B_n)^c = E \setminus \cup_{k=1}^n B_n \supset E \setminus \bigcup_{k=1}^{\infty} B_k,$$

together with the fact that outer measures are monotone. This inequality holds for any $n \in \mathbb{N}$, so we obtain

$$(**) \qquad \mu^*(E) \geq \sum_{k=1}^{\infty} \mu^*(E \cap B_k) + \mu^*(E \setminus (\bigcup_{k=1}^{\infty} B_k)).$$

Since

$$E \cap (\cup_{k=1}^{\infty} B_k) = \cup_{k=1}^{\infty} E \cap B_k,$$

by countable subadditivity of out measures,

$$\mu^*(E \cap (\bigcup_{k=1}^{\infty} B_k)) \leq \sum_{k=1}^{\infty} \mu^*(E \cap B_k).$$

We therefore obtain, combining this with the above inequality

$$\mu^*(E) \geq \mu^*(E \cap (\bigcup_{k=1}^{\infty} B_k)) + \mu^*(E \setminus (\bigcup_{k=1}^{\infty} B_k)) = \mu^*(E \cap (\bigcup_{k=1}^{\infty} B_k)) + \mu^*(E \cap (\bigcup_{k=1}^{\infty} B_k)^c).$$

Since $E \subset (E \cap Y) \cup (E \cap Y^c)$, by countable subadditivity of outer measures, for any Y we have

$$\mu^*(E) \leq \mu^*(E \cap Y) + \mu^*(E \cap Y^c).$$

We therefore also have the inequality

$$\mu^*(E) \leq \mu^*(E \cap (\bigcup_{k=1}^{\infty} B_k)) + \mu^*(E \cap (\bigcup_{k=1}^{\infty} B_k)^c).$$

Combining with the reverse inequality we demonstrated above, we obtain

$$\mu^*(E) = \mu^*(E \cap (\bigcup_{k=1}^{\infty} B_k)) + \mu^*(E \cap (\bigcup_{k=1}^{\infty} B_k)^c).$$

This shows that $\cup B_k$ satisfies the definition of $\mathcal{M}$, so we have

$$\bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} B_k \in \mathcal{M}.$$

Hence $\mathcal{M}$ is a σ-algebra.

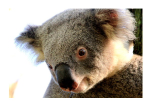
Now we want to show that $\mu^*\big|_{\mathcal{M}}$ is a measure. First we note that since μ^* is an outer measure, we have $\mu^*(\emptyset) = 0$. Moreover, outer measures are also monotone, so μ^* is monotone. Thus, we only need to show that μ^* restricted to $\mathcal{M}$ is countably additive. Let $\{B_k\}_{k\in\mathbb{N}} \subset \mathcal{M}$ be pairwise disjoint sets. Defining $E := \bigcup_{k=1}^{\infty} B_k$ and using (**), we get

$$\mu^*(\bigcup_{k=1}^{\infty} B_k) = \mu^*(E) \overset{(**)}{\geq} \sum_{k=1}^{\infty} \mu^*(E \cap B_k) + \mu^*(\emptyset) = \sum_{k=1}^{\infty} \mu^*(B_k) \geq \mu^*(\bigcup_{k=1}^{\infty} B_k)$$

$$\Longrightarrow \mu^*(\bigcup_{k=1}^{\infty} B_k) = \sum_{k=1}^{\infty} \mu^*(B_k)$$

So $\mu^*\big|_{\mathcal{M}}$ is a measure.
Finally, we show that it is a complete measure: For $Y \subset X$ such that $\mu^*(Y) = 0$, and for arbitrary $E \subset X$ we have by countable subadditivity of outer measures

$$\mu^*(E) \leq \mu^*(E \cap Y) + \mu^*(E \cap Y^c) \leq \mu^*(Y) + \mu^*(E) = \mu^*(E)$$

Therefore $Y \in \mathcal{M}$. 

Remark 3. We briefly discussed the proof of completion, and I shall add a remark here. Technically speaking, we should be considering

$$\mu^{**} : P(X) \to [0, \infty], \quad \mu^{**}(A) = \inf\{\sum_{j\geq 1} \mu^*(E_j) : A \subset \cup_{j\geq 1} E_j, \quad E_j \in \mathcal{M}\}.$$

If some set has $\mu^{**}(Y) = 0$, then for each $k \in \mathbb{N}$ there exists $\{E_j^k\} \in \mathcal{M}$ such that

$$Y \subset \cup_{j\geq 1} E_j^k, \quad \sum_{j\geq 1} \mu^*(E_j^k) < 2^{-k}.$$

Since $\mathcal{M}$ is a σ-algebra,

$$A_k := \cup_{j\geq 1} E_j^k \in \mathcal{M},$$

and

$$\mu^*(A_k) \leq \sum_{j\geq 1} \mu^*(E_j^k) < 2^{-k}.$$

Moreover, since $Y \subset A_k$ for all k, we have

$$Y \subset \cap_{k\geq 1} A_k,$$

and we also have that since $\mathcal{M}$ is a σ-algebra

$$\cap_{k\geq 1} A_k \in \mathcal{M}.$$

Since

$$\cap_{k\geq 1} A_k \subset A_n \quad \forall n \in \mathbb{N},$$

by monotonicity,

$$\mu^*(Y) \leq \mu^*(\cap_{k\geq 1} A_k) \leq 2^{-n} \quad \forall n \in \mathbb{N}.$$

This shows that $\mu^*(Y) = 0$. It is pretty straightforward to show that the converse holds as well, that is if $\mu^*(Z) = 0$ then $\mu^{**}(Z) = 0$. So, by the completeness proposition, our μ^* is complete!

Another important concept in measure theory is that of a *pre-measure.*

Definition 2.4. Let $\mathcal{A} \subset P(X)$ be an algebra. A function $\mu_0 : \mathcal{A} \to [0, \infty]$ is called a *pre-measure* if

(1) $\mu_0(\emptyset) = 0$

(2) If $\{A_j\}$ is a countable collection of disjoint elements of $\mathcal{A}$ such that

$$\cup A_j \in \mathcal{A},$$

then

$$\mu_0(\cup A_j) = \sum \mu_0(A_j).$$

Exercise 4. *We have shown how, given a measure space $(X, \mathcal{M}, \mu)$, we can obtain a minimal complete measure space, $(X, \bar{\mathcal{M}}, \bar{\mu})$. We have also shown how, given a measure, μ, we can canonically construct an outer measure, μ^*.*

(1) Using the canonically associated outer measure, μ^*, determine whether or not the set

$$\mathcal{A} := \{A \in P(X) : \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \text{ holds true for all } E \subset X\}$$

is equal to the set

$$\bar{\mathcal{M}} := \{E \cup F : E \in \mathcal{M}, \text{ and } F \subset N \in \mathcal{N}\},$$

where again $\mathcal{N}$ is the set of elements of $\mathcal{M}$ which have μ-measure zero.

(2) In this way, determine whether or not the spaces

$$(X, \mathcal{A}, \mu^*)$$

and

$$(X, \bar{\mathcal{M}}, \bar{\mu})$$

are the same? My sneaking suspicion is that they are the same, but I shall not spoil your fun in investigating this question.

2.1. **Homework: Constructing the Lebesgue measure.** The n-dimensional Lebesgue measure is the unique, complete measure which agrees with our intuitive notion of n-dimensional volume. To make this precise, first we define a generalized interval and our notion of intuitive volume.

Definition 2.5. A generalized interval in $\mathbb{R}^n$ is a set for which there exist real numbers $a_k \leq b_k$ for $k = 1, \ldots n$, such that this set has the form

$$I = \{x \in \mathbb{R}^n, x = \sum x_k e_k, \quad a_k < \text{ or } \leq x_k < \text{ or } \leq b_k, k = 1, \ldots, n\}.$$

Above we are using e_k to denote the standard unit vectors for $\mathbb{R}^n$. The intuitive volume function on $\mathbb{R}^n$ is defined on such a set to be

$$v_n(I) = \prod (b_k - a_k).$$

Next we can extend our intuitive notion of volume to *elementary sets.*

Definition 2.6. An *elementary subset* of $\mathbb{R}^n$ is a set which can be expressed as a finite disjoint union of generalized intervals. The collection of all of these is denoted by $\mathcal{E}_n$.

Exercise 1. *Prove that v_n is well-defined on $\mathcal{E}_n$.*

Exercise 2. *To make an algebra containing $\mathcal{E}_n$, in particular the smallest algebra containing $\mathcal{E}_n$, it is necessary to include compliments. Define*

$$\mathcal{A} := \{Y \subseteq \mathbb{R}^n \mid Y \in \varepsilon_n \text{ or } \exists Z \in \varepsilon_n \text{ s.t. } Y = Z^c\}$$

Prove that $\mathcal{A}$ is an algebra.

Exercise 3. *Show that ν_n is well-defined on $\mathcal{A}$ where*

$$\nu_n(\prod^n Ia_i, \alpha_i I) := \begin{cases} 0, & \text{if } a_i = \alpha_i \text{ for some } i \\ \prod(\alpha_i - a_i), & \text{else} \end{cases}$$

Exercise 4. *Show that ν_n is a pre-measure on $\mathcal{A}$.*

2.2. **Hints.**

(1) $\emptyset = \prod \mathrm{I}x, x\mathrm{I}$ for $x \in \mathbb{R}^n$. *Notation:* we use $\mathrm{I}a, b\mathrm{I}$ to denote either $]a, b[$, $[a, b]$, $]a, b]$ or $[a, b[$. Notation which is unnecessary shall be simplified when possible.

(2) Show that $\mathcal{A}$ is closed under compliments

(3) Let $A, B \in \mathcal{A}$. If $A, B \in \varepsilon_n$ then first consider the case where A, B are each single intervals i.e. $A = \prod \mathrm{I}a_i, \alpha_i\mathrm{I}, B = \prod \mathrm{I}b_i, \beta_i\mathrm{I}$ for $a_i \leq \alpha_i$, $b_i \leq \beta_i$. For each i, if $\mathrm{I}b_i, \beta_i\mathrm{I} \subset \mathrm{I}a_i, \alpha_i\mathrm{I}$ then note that

$$\mathrm{I}a_i, \alpha_i\mathrm{I} \backslash \mathrm{I}b_i, \beta_i\mathrm{I} = \mathrm{I}a_i, b_i\mathrm{I} \cup \mathrm{I}\beta_i, \alpha_i\mathrm{I}$$

If $\mathrm{I}b_i, \beta_i\mathrm{I} \not\subset \mathrm{I}a_i, \alpha_i\mathrm{I}$, then either $\mathrm{I}b_i, \beta_i\mathrm{I} \cap \mathrm{I}a_i, \alpha_i\mathrm{I} = \emptyset$ in which case $\mathrm{I}a_i, \alpha_i\mathrm{I} \backslash \mathrm{I}b_i, \beta_i\mathrm{I} = \mathrm{I}a_i, \alpha_i\mathrm{I}$, or $\mathrm{I}b_i, \beta_i\mathrm{I} \cap \mathrm{I}a_i, \alpha_i\mathrm{I} \neq \emptyset$ so that

$$\mathrm{I}a_i, \alpha_i\mathrm{I} \backslash \mathrm{I}b_i, \beta_i\mathrm{I} = \begin{cases} \mathrm{I}a_i, b_i\mathrm{I} \text{ if } b_i \leq \alpha_i (\Rightarrow \beta_i > \alpha_i) \\ \mathrm{I}\beta_i, \alpha_i\mathrm{I} \text{ if } a_i \leq beta_i (\Rightarrow b_i < a_i) \end{cases}$$

In both cases $\mathrm{I}a_i, \alpha_i\mathrm{I} \backslash \mathrm{I}b_i, \beta_i\mathrm{I}$ is the disjoint union of intervals. Repeating for each $i = 1, ..., n$ gives $A \backslash B \in \varepsilon_n$, and similarly $B \backslash A \in \varepsilon_n$. Note that $A \cap B = \prod \mathrm{I}x_i, y_i\mathrm{I}$ with $x_i = \max\{a_i, b_i\}$, $y_i = \min\{\alpha_i, \beta_i\}$ (and should $x_i \geq y_i$ then it is understood that $\mathrm{I}x_i, y_i\mathrm{I} = \emptyset$. Therefore,

$$A \cup B = (A \backslash B) \cup (B \backslash A) \cup (A \cap B) \in \varepsilon_n.$$

In fact, for $A = \prod \mathrm{I}a_i, \alpha_i\mathrm{I} \in \varepsilon_n$ note that

$$\begin{aligned} A^c =& \mathbb{R}^n \backslash A \\ =& \prod \mathrm{I} - \infty, a_i\mathrm{I} \cup \prod \mathrm{I}\alpha_i, \infty\mathrm{I} \end{aligned}$$

Allowing the endpoints x_i and/or y_i of $\mathrm{I}x_i, y_i\mathrm{I}$ to be $\pm\infty$, the same arguments for A, B as above show that $A^c \cup B$ and $A^c \cup B^c$ are elements of $\mathcal{A}$.

More generally, for $A = \bigcup_{j=1}^{k} I_j \in \varepsilon_n$ with $I_j \underset{k \neq j}{\cap} I_k = \emptyset$ and $B = \bigcup_{l=1}^{m} J_l \in \varepsilon_n$ with $J_l \underset{m \neq l}{\cap} J_m = \emptyset$ with end points possibly $\pm\infty$, repeated application of the above arguments shows that $I_1 \cup J_1 \in \varepsilon_n$, $(I_1 \cup J_1) \cup I_2 \in \varepsilon_n$, and so forth. Therefore, $A \cup B \in \varepsilon_n$. So $\mathcal{A}$ is closed under finite unions and hence $\mathcal{A}$ is an algebra.

(4) To show that ν_n is well-defined on $\mathcal{A}$ and that it is a pre-measure, first show that $\nu_n(\emptyset) = 0$.

(5) Next, let $\{A_m\}_{m \geq 1} \subset \mathcal{A}$ such that $\underset{m \geq 1}{\cup} A_m \in \mathcal{A}$, $A_m \underset{k \neq m}{\cap} A_k = \emptyset$ then $\exists \{I_j\}_{j=1}^{k}$ disjoint in $\mathcal{A}$ such that $\bigcup_{j=1}^{k} I_j = \bigcup_{m=1}^{\infty} A_m$.

By definition, $\nu_n(\bigcup_{m=1}^{M} A_m) = \sum_{m=1}^{M} v_n(A_m) \leq \nu_n(\bigcup_{j=1}^{k} I_j) = \sum_{j=1}^{k} v_n(I_j)$

$\forall M \in \mathbb{N}$, $\sum_{m=1}^{M} v_n(A_m) \leq \sum_{j=1}^{k} v_n(I_j) = \nu_n(\bigcup_{m=1}^{\infty} A_m) \leq \sum_{m=1}^{M} v_n(A_m)$

$\Rightarrow \nu_n(\bigcup_{m=1}^{\infty} A_m) = \sum_{m=1}^{M} v_n(A_m)$

3. Pre-measure extension theorem and metric outer measures

The name pre-measure is appropriate because it's almost a measure, it's just possibly not countably additive for *every* disjoint countable union, since these need not *always* be contained in a mere algebra (which is not necessarily a σ-algebra). However, Carathéodory can help us to extend pre-measures to measures. First, we require the following.

Proposition 3.1. *Let μ_0 be a pre-measure on the algebra $\mathcal{A} \subset P(X)$, and define*

$$\mu^*(Y) := \inf\{\sum_j \mu_0(A_j) : A_j \in \mathcal{A} \forall j, Y \subset \cup_j A_j\},$$

where the infimum is taken to be ∞ if there is no such cover of Y. Then we have:

1. *μ^* is an outer measure.*
2. *$\mu^*(A) = \mu_0(A) \forall A \in \mathcal{A}$.*
3. *Every set in $\mathcal{A}$ is μ^* measurable in the same sense as above, being that for arbitrary $E \subset X$, for $A \in \mathcal{A}$,*

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Proof: First, note that $\emptyset \in \mathcal{A}$ since $\mathcal{A}$ is an algebra. Moreover, the map μ_0 is defined on $\mathcal{A}$, with $\mu_0 : \mathcal{A} \to [0, \infty]$, and has $\mu_0(\emptyset) = 0$. Therefore by the Outer Measure Proposition, as we have defined μ^*, it is an outer measure.

Next, we wish to show that μ^* and μ_0 are the same when we restrict to the algebra, $\mathcal{A}$. To do this we will show that (1) pre-measures are finitely additive and (2) pre-measures are monotone.

Finite additivity of pre-measures: Next, we show that pre-measures are by definition finitely additive since for $A, B \in \mathcal{A}$ with $A \cap B = \emptyset$, then

$$A \cup B = \cup A_j, \quad A_1 = A, A_2 = B, A_j = \emptyset \forall j > 2,$$

gives

$$\mu_0(A \cup B) = \mu_0(\cup A_j) = \sum \mu_0(A_j) = \mu_0(A) + \mu_0(B).$$

Monotonicity of pre-measures: Assume that $A \subset B$ are both elements of $\mathcal{A}$. Then $B \setminus A = B \cap A^c \in \mathcal{A}$, so finite additivity gives

$$\mu_0(B) = \mu_0(B \setminus A) + \mu_0(A) \implies \mu_0(A) = \mu_0(B) - \mu_0(B \setminus A) \leq \mu_0(B).$$

Showing that $\mu^* = \mu_0$ on $\mathcal{A}$: Now, let $E \in \mathcal{A}$. If $E \subset \cup A_j$ with $A_j \in \mathcal{A}\ \forall j$, then let

$$B_n := E \cap (A_n \setminus \cup_1^{n-1} A_j).$$

Then

$$B_n \in \mathcal{A} \forall n, \quad B_n \cap B_m = \emptyset \forall n \neq m.$$

The union

$$\cup B_n = \cup E \cap (A_n \setminus \cup_1^{n-1} A_j) = E \cap \cup (A_n \setminus \cup_1^{n-1} A_j) = E \cap \cup A_n = E \in \mathcal{A}.$$

So by definition of pre-measure,

$$\mu_0(E) = \mu_0(\cup B_n) = \sum \mu_0(B_n) \leq \sum \mu_0(A_n),$$

since $B_n \subset A_n \forall n$. Taking the infimum over all such covers of E comprised of elements of $\mathcal{A}$, we have

$$\mu_0(E) \leq \mu^*(E).$$

On the other hand, $E \subset \cup A_j$ with $A_1 = E \in \mathcal{A}$, and $A_j = \emptyset \forall j > 1$. Then, this collection is considered in the infimum defining μ^*, so

$$\mu^*(E) \leq \sum \mu_0(A_j) = \mu_0(E).$$

We've shown the inequality is true in both directions, hence $\mu^*(E) = \mu_0(E)$ for any $E \in \mathcal{A}$.

Showing that $\mathcal{A}$ sets are μ^* measurable: Let $A \in \mathcal{A}$, $E \subset X$, and $\varepsilon > 0$. Since we always have by countable subadditivity

$$\mu^*(E) \leq \mu^*(E \cap A) + \mu^*(E \cap A^c),$$

if $\mu^*(E) = \infty$, then we also have

$$\infty \leq \mu^*(E \cap A) + \mu^*(E \cap A^c) \implies \mu^*(E \cap A) + \mu^*(E \cap A^c) = \infty,$$

so the equality holds. Now, let us assume that $\mu^*(E) < \infty$. Then, by its definition, there exists $\{B_j\} \subset \mathcal{A}$ with $E \subset \cup B_j$ and

$$\sum \mu_0(B_j) \leq \mu^*(E) + \varepsilon.$$

Since μ_0 is additive on $\mathcal{A}$,

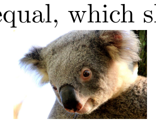
$$\mu^*(E)+\varepsilon \geq \sum \mu_0(B_j \cap A) + \mu_0(B_j \cap A^c) = \sum \mu_0(B_j \cap A) + \sum \mu_0(B_j \cap A^c) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Above we have used the definition of μ^* as an infimum, together with the fact that since $A \in \mathcal{A}$ and $B_j \in \mathcal{A}$ for all j, we have $B_j \cap A \in \mathcal{A}$ and $B_j \cap A^c \in \mathcal{A}$ for all j, and we also have

$$E \cap A \subset \cup B_j \cap A, \quad E \cap A^c \subset \cup B_j \cap A^c.$$

This is true for any $\varepsilon > 0$, so we have

$$\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c) \geq \mu^*(E).$$

So, these are all equal, which shows that A satisfies the definition of being μ^* measurable since E was arbitrary. 

Now we will prove that we can always extend a pre-measure to a measure. You will use this in the first exercise to complete the construction of the Lebesgue measure.

Theorem 3.2 (Pre-measure extension theorem)**.** *Let $\mathcal{A} \subset P(X)$ be an algebra, μ_0 a pre-measure on $\mathcal{A}$, and $\mathcal{M}$ the smallest σ-algebra generated by $\mathcal{A}$. Then there exists a measure μ on $\mathcal{M}$ which extends μ_0, namely*

$$\mu := \mu^* \text{ restricted to } \mathcal{M}.$$

If ν also extends μ_0 then $\nu(E) \leq \mu(E) \forall E \in \mathcal{M}$ with equality when $\mu(E) < \infty$. If μ_0 is σ-finite, then $\nu \equiv \mu$ on $\mathcal{M}$, so μ is the unique extension.

Proof: By its very definition, $\mathcal{M}$ is a σ-algebra, and all elements of $\mathcal{A}$ are contained in $\mathcal{M}$. Moreover, by the proposition,

$$\mu^*(A) = \mu_0(A), \quad \forall A \in \mathcal{A}.$$

Since $\emptyset \in \mathcal{A}$, we have

$$\mu^*(\emptyset) = 0.$$

Moreover, since μ^* is an outer measure, by the proposition, it is monotone. Consider the set

$$\{A \subset X | \mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \text{ holds true for all } E \subset X\}.$$

By the preceding proposition, this set contains all elements of $\mathcal{A}$. Moreover, since μ^* is an outer measure, in Caratheodory's Theorem, we proved that this set is a σ algebra, and μ^* restricted to this set is a measure. Hence, since it is a σ algebra which contains $\mathcal{A}$, it also contains $\mathcal{M}$. Therefore, μ^* restricted to $\mathcal{M}$ is countably additive, since μ^* on this larger (Carathéodory-Theorem-σ-algebra-set) is countably additive. Hence μ is a measure.

So, we only need to consider the statements about a possibly different extension ν which coincides with μ_0 on $\mathcal{A}$ and is a measure on $\mathcal{M}$. If $E \in \mathcal{M}$ and

$$E \subset \cup A_j, \quad A_j \in \mathcal{A} \forall j,$$

then

$$\nu(E) \leq \sum \nu(A_j) = \sum \mu_0(A_j).$$

This holds for any such covering of E by elements of $\mathcal{A}$, so taking the infimum we have

$$\nu(E) \leq \mu^*(E) = \mu(E) \text{ since } E \in \mathcal{M}.$$

If $\mu(E) < \infty$, let $\varepsilon > 0$. Then we may choose $\{A_j\} \subset \mathcal{A}$ which are WLOG (without loss of generality) disjoint (why/how can we do this?) such that

$$E \subset \cup A_j, \quad \mu(\cup A_j) = \sum \mu_0(A_j) < \mu^*(E) + \varepsilon = \mu(E) + \varepsilon,$$

since $E \in \mathcal{M}$. Note that $E \in \mathcal{M}$, $\{A_j\} \subset \mathcal{A}$, and $\mathcal{M}$ is a σ algebra containing $\mathcal{A}$. We therefore have

$$A := \cup A_j \in \mathcal{M}.$$

Then, we also have

$$\nu(A) = \lim_{n\to\infty} \nu(\cup_1^n A_j) = \lim_{n\to\infty} \sum_1^n \nu(A_j) = \lim_{n\to\infty} \sum_1^n \mu_0(A_j) = \mu(A).$$

Then we have since $E \in \mathcal{M}$, and $\{A_j\} \subset \mathcal{A}$, and $\mathcal{M}$ is a σ algebra containing $\mathcal{A}$ that $A \in \mathcal{M}$. By countable additivity of the measure μ, we have

$$\mu(\cup A_j) = \mu(A) = \mu(A \cap E) + \mu(A \setminus E) = \mu(E) + \mu(A \setminus E) < \mu(E) + \varepsilon$$

which shows that

$$\mu(A \setminus E) < \varepsilon.$$

Consequently, using monotonicity, the fact that $\mu(A) = \nu(A)$, the additivity of ν, and the fact that $\nu \leq \mu$, we obtain

$$\mu(E) \leq \mu(A) = \nu(A) = \nu(E \cap A) + \nu(A \setminus E) \leq \nu(E) + \mu(A \setminus E) < \nu(E) + \varepsilon.$$

This holds for all $\varepsilon > 0$, so

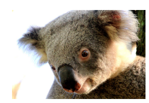
$$\mu(E) \leq \nu(E).$$

Consequently in this case $\mu(E) = \nu(E)$, whenever these are finite.
Finally, if $X = \cup A_j$ with $A_j \in \mathcal{A}$, $\mu_0(A_j) < \infty \forall j$, we may WLOG assume the A_j are disjoint. Then for $E \in \mathcal{M}$,

$$E = \cup(E \cap A_j),$$

which is a disjoint union of elements of $\mathcal{M}$. So by countable additivity

$$\mu(E) = \mu(\cup E \cap A_j) = \sum \mu(E \cap A_j) = \sum \nu(E \cap A_j) = \nu(\cup E \cap A_j) = \nu(E),$$

since $E \cap A_j \subset A_j$ shows that $\mu(E \cap A_j) \leq \mu(A_j) < \infty$, so $\mu(E \cap A_j) = \nu(E \cap A_j)$. 

3.1. **Introducing metric outer measures.** To define the Hausdorff measure, we will introduce *metric outer measures.* A metric outer measure requires an addition type of structure on the big set X: we need a notion of *distance between points.* Thus, metric outer measures are only defined when the set X also carries along a distance, d, also known as a *metric.* So, for a metric space (X, d) and for $A, B \subset X$ define

$$\operatorname{dist}(A, B) := \inf\{d(x, y) : x \in A, y \in B\}.$$

Define also the diameter of a set $A \subset X$

$$\operatorname{diam}(A) := \sup\{d(x, y) : x, y \in A\}, \operatorname{diam}(\emptyset) := 0.$$

Definition 3.3. Let μ^* be an outer measure defined on a metric space, (X, d). Then μ^* is called *metric outer measure* iff for each $A, B \subset X$ we have

$$\operatorname{dist}(A, B) > 0 \Rightarrow \mu^*(A \cup B) = \mu^*(A) + \mu^*(B).$$

Recall: $A \subset X$ is μ^*-measurable iff for each $E \subset X$

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^C).$$

Denote by $\mathcal{M}(\mu^*)$ the μ^*-measurable subsets. Recall that the Borel sets $\mathcal{B}(X)$ is the smallest σ-algebra generated by the topology of X (induced by the metric). In other words, it is the smallest σ-algebra which contains all open sets. We note that $\emptyset$ is both open and closed. A non-empty subset, U, of a metric space (X, d) is defined to be open precisely when

$$\forall x \in U \exists \delta > 0 \text{ such that } B_\delta(x) \subset U,$$

where

$$B_\delta(x) = \{y \in X | d(x, y) < \delta\}.$$

A subset of X is said to be closed precisely when its complement is open. We now prove a Theorem due to Carathéodory which states that the Borel sets in X are contained in $\mathcal{M}(\mu^*)$.

Theorem 3.4 (Carathéodory)**.** *Let μ^* be a metric outer measure on (X, d). Then we have $\mathcal{B}(X) \subset \mathcal{M}(\mu^*)$.*

Exercise 5. *Show that μ^* is a measure on $\mathcal{B}(X)$. Denote by μ the restriction of μ^* to $\mathcal{B}(X)$. Let $\mathcal{A}$ be defined as in the completion theorem, that is:*

$$\mathcal{A} = \{E \cup F : E \in \mathcal{B}(X), F \subset N \in \mathcal{B}(X), \quad \mu^*(N) = 0\}.$$

Define as in the completion $\bar{\mu}(E \cup F) = \mu(E)$. Is it true that $\mathcal{M}(\mu^*) = \mathcal{A}$? Prove your answer.

3.2. Homework: properties of the Lebesgue σ-algebra.

(1) In the previous homework, we proved that ν_n is a pre-measure on the algebra $\mathcal{A}$. Note by the definition of $\mathcal{A}$, it is the smallest algebra which contains ε_n. By the pre-measure extension theorem, since ν_n is σ-finite on $\mathcal{A}$, there exists a unique extension of ν_n to a measure $\bar{\mathcal{M}}$ on the smallest σ-algebra containing ε_n. It is unique, because $\mathbb{R}^n = \bigcup_{m\geq 1}[-M, M]^n = \bigcup_{m\geq 1} I_M$ and $\nu_m(I_M) = (2M)^n < \infty$ for each M. Canonically completing this measure to $\mathcal{M}$ by applying the completion theorem yields the Lebesgue measure and the Lebesgue σ-algebra, the smallest σ-algebra generated by ε_n such that the extension of ν_n to a measure with respect to this σ-algebra is complete. This is the construction of the Lebesgue measure. In this exercise, the task is to review the construction of the Lebesgue measure step-by-step, and make sure it makes sense to you.
(2) Prove that Borel sets are Lebesgue measurable.
(3) Prove $\mathcal{B} \subsetneq \mathcal{M}$
(4) It is difficult to construct sets $\not\subset \mathcal{M}$, but actually there are many natural examples... *Exercise*: Construct a subset of $\mathbb{R}^n$ which is not measurable. Recall that $f : \mathbb{R}^n \to \mathbb{R}^m$ is "measurable" usually is understood to mean that $\forall B \in \mathcal{B}^m$, $f^{-1}(B) \in \mathcal{M}^n$. More precisely, f is $(\mathbb{R}^n, \mathcal{B}^n), (\mathbb{R}^m, \mathcal{B}^m)$ measurable. In general, $f : X \to Y$ is $(X, \mathcal{A}), (Y, \mathcal{B})$ measurable if $\forall B \in \mathcal{B}$, $f^{-1}(B) \in \mathcal{A}$, where $\mathcal{A}$ and $\mathcal{B}$ are σ-algebras.
(5) Prove that all $n-1$ dimensional sets have $\mathcal{L}^n$ measure 0.

3.3. Hints.

(1) To prove that Borel sets are Lebesgue measurable, it suffices to show that open sets are Lebesgue measurable. So, let $\mathcal{O} \subset \mathbb{R}^n$ be open. Then we will show that $\mathcal{O} \in \mathcal{M}$.

First consider $\mathcal{O} = \prod]a_i, \alpha_i[\in \varepsilon_n \subset \mathcal{M}$. For an arbitrary open set $\mathcal{O}$, for each $x \in \mathcal{O}$ there exists $\varepsilon \in \mathbb{Q}, \varepsilon > 0$ such that $x \in \prod\,]q_m - \varepsilon, q_m + \varepsilon[\subset \mathcal{O}$, $q_m \in \mathbb{Q}$, $m = 1, ..., n$.

Taking the union of all such intervals, namely those contained in $\mathcal{O}$ such that endpoints are rational is a countable union. Countability of course follows since $\mathbb{Q}^n \subset \mathbb{R}^n$ is countable and $\mathbb{Q}$ is countable so a union of intervals with endpoints in $\mathbb{Q}^n$ is countable. Therefore, $\mathcal{O} \in \mathcal{M}$.

4. Metric outer measures

Theorem 4.1 (Carathéodory)**.** *Let μ^* be a metric outer measure on (X, d). Then we have $\mathcal{B}(X) \subset \mathcal{M}(\mu^*)$.*

Proof:
Note that since $\mathcal{M}(\mu^*)$ is a σ-algebra (by Thm. 2.3) it is enough to prove that every closed set is μ^*-measurable. (why does this suffice?) So let $F \subset X$ be a closed subset. Since the reverse inequality always holds, it will be enough to prove that for any set A we have

$$\mu^*(A) \geq \mu^*(A \cap F) + \mu^*(A \setminus F).$$

Define the sets

$$A_k := \left\{ x \in A : \operatorname{dist}(x, F) \geq \frac{1}{k} \right\}.$$

Then $\operatorname{dist}(A_k, A \cap F) \geq \frac{1}{k}$, so since μ^* is a metric outer measure we have

$$\mu^*(A \cap F) + \mu^*(A_k) = \mu^*(\underbrace{(A \cap F) \cup A_k}_{\subset A}) \leq \mu^*(A). \tag{+}$$

Let $x \in A \setminus F = A \cap F^c$. Since F^c is open, there exists $\delta > 0$ such that $B_\delta(x) \subset F^c$. Hence $d(x, F) \geq \delta$. So, in general, for all $x \in A \setminus F$, we have

$$\operatorname{dist}(x, F) > 0.$$

Consequently, we have

$$A \setminus F = \bigcup A_k.$$

The main and last step in the proof is to calculate the limit in (+) as $k \to \infty$. If the limit is infinity there is nothing to do, because it shows that

$$\mu^*(A) = \infty \geq \text{ anything we want, in particular } \geq \mu^*(A \cap F) + \mu^*(A \cap F^c).$$

So, let us assume that the limit in (+) is finite.
Note that $A_1 \subset A_2 \subset A_3 \subset \ldots$.
To get a bit of room between our sets, let us define

$$B_1 := A_1, \quad B_n := A_n \setminus A_{n-1}, \quad n \geq 2.$$

By definition, $A_k \subset A$, so we also have $B_k \subset A$ for all A. By definition of A_k and B_k, for all $x \in B_k$ we have

$$\frac{1}{k} \leq \operatorname{dist}(x, F) < \frac{1}{k-1},$$

where the second inequality follows since $B_k = A_k \setminus A_{k-1}$. Therefore if $j \geq k+2$, for all $y \in B_j$ we have

$$\frac{1}{j} \leq \operatorname{dist}(y, F) < \frac{1}{j-1} \leq \frac{1}{k+1} < \frac{1}{k} \leq \operatorname{dist}(x, F).$$

Let $\varepsilon > 0$ such that

$$\frac{1}{j-1} + \varepsilon < \frac{1}{k}.$$

By definition, there exists $z \in F$ such that

$$d(y, z) \leq \operatorname{dist}(y, F) + \varepsilon,$$

so

$$d(y, z) < \frac{1}{j-1} + \varepsilon < \frac{1}{k} \leq \operatorname{dist}(x, F) \leq d(x, z).$$

We therefore have by the triangle inequality,

$$d(x, y) \geq d(x, z) - d(z, y) \geq \frac{1}{k} - \left(\frac{1}{j-1} + \varepsilon \right) > 0.$$

Since $x \in B_k$ and $y \in B_j$ are arbitrary, and $\varepsilon > 0$ is fixed, we therefore have proven that $\operatorname{dist}(B_j, B_k) > 0$.
This means we can apply the metric outer measure property (for even and odd indices) and by induction we conclude that

$$\mu^* \left(\bigcup_{k=1}^{n} B_{2k-1} \right) = \sum_{k=1}^{n} \mu^*(B_{2k-1}),$$
$$\mu^* \left(\bigcup_{k=1}^{n} B_{2k} \right) = \sum_{k=1}^{n} \mu^*(B_{2k}).$$

These unions are each contained in A_{2n}, so we have the inequalities

$$\mu^* \left(\bigcup_{k=1}^{n} B_{2k-1} \right) = \sum_{k=1}^{n} \mu^*(B_{2k-1}) \leq \mu^*(A_{2n}),$$
$$\mu^* \left(\bigcup_{k=1}^{n} B_{2k} \right) = \sum_{k=1}^{n} \mu^*(B_{2k}) \leq \mu^*(A_{2n}).$$

Since $A_1 \subset A_2 \subset \ldots$, the values $\mu^*(A_{2n})$ are non-decreasing and by assumption bounded. Hence both sums above, since they are comprised of non-negative terms, are convergent as $n \to \infty$.
Therefore we conclude for any j

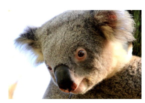
$$\begin{aligned}\mu^*(A \setminus F) &= \mu^*\left(\bigcup_i A_i\right)\\ &= \mu^*\left(A_j \cup \bigcup_{k \geq j+1} B_k\right)\\ &\leq \mu^*(A_j) + \sum_{k=j+1}^{\infty} \mu^*(B_k)\\ &\leq \lim_{n\to\infty} \mu^*(A_n) + \underbrace{\sum_{k=j+1}^{\infty} \mu^*(B_j)}_{\to 0, j\to\infty}.\end{aligned}$$

The last term tends to zero because it is comprised of the tails of two convergent series.
Since the latter sum goes to 0 by convergence we obtain

$$\mu^*(A \setminus F) \leq \lim_{n\to\infty} \mu^*(A_n).$$

Together with (+) this yields

$$\mu^*(A) \geq \lim_{k\to\infty} \mu^*(A_k) + \mu^*(A \cap F) \geq \mu^*(A \setminus F) + \mu^*(A \cap F)$$

which is the desired inequality. 

Corollary 4.2. *Let (X, d) be a metric space, and let μ^* be a metric outer measure on X. Then μ^* restricted to the Borel sigma algebra is a measure, that is $(X, \mathcal{B}(X), \mu^*)$ is a measure space.*

Proof: By the theorem, $\mathcal{M}(\mu^*) \supset \mathcal{B}(X)$. In a previous theorem, we proved that μ^* restricted to $\mathcal{M}(\mu^*)$ is a measure. Note that $\emptyset \in \mathcal{B}(X)$ and $\mu^*(\emptyset) = 0$. If $\{A_j\} \subset \mathcal{B}(X)$ are pairwise disjoint, then since they are also contained in $\mathcal{M}(\mu^*)$ we have

$$\mu^*(\cup A_j) = \sum \mu^*(A_j).$$

Hence μ^* vanishes on the empty set and is countably additive on $\mathcal{B}(X)$. Since μ^* is defined on $\mathcal{B}(X)$ which is a σ-algebra, we have that μ^* restricted to $\mathcal{B}(X)$ is a measure. 

4.1. **General results which shall be used to obtain the Hausdorff measure.** We shall obtain the Hausdorff measure using results which can be applied much more generally to obtain metric outer measures.

Definition 4.3 (Countable covers)**.** Let $\mathcal{C}$ denote a collection of sets in X. Assume $\emptyset \in \mathcal{C}$. Then for each $A \subset X$ we denote by $\mathcal{CC}(A)$ the collection of sets in $\mathcal{C}$ such that there is an at most countable sequence of sets $\{E_n\}_{n\in\mathbb{N}} \in \mathcal{CC}(A)$ such that

$$A \subset \bigcup_{n=1}^{\infty} E_n.$$

These are the *countable covers* of A by sets belonging to $\mathcal{C}$.

Definition 4.4. With $\mathcal{C}$ a collection of sets in X, let $\nu\colon \mathcal{C} \to [0,\infty]$ with $\nu(\emptyset) = 0$. We define the following set function depending on $\mathcal{C}$ and ν

$$\mu^*_{\nu,\mathcal{C}}(A) := \inf_{\mathbb{D}\in\mathcal{CC}(A)} \sum_{D\in\mathbb{D}} \nu(D). \tag{4.1}$$

If the infimum is empty, then we define $\mu^*_{\nu,\mathcal{C}}(A) = \infty$.

Theorem 4.5. *The set function given by (4.1), which for simplicity we denote here by μ^*, is an outer measure μ^* on X with*

$$\mu^*(A) \le \nu(A), \ \ A \in \mathcal{C}$$

For any other outer measure $\tilde{\mu}^*$ on X with the above condition we have

$$\tilde{\mu}^*(A) \le \mu^*(A), \ \ A \subset X.$$

So in this sense, μ^* is the unique maximal outer measure on X which satisfies $\mu^*(A) \le \nu(A)$ for all $A \in \mathcal{C}$.

Proof: Let $A \in \mathcal{C}$. Then, A covers itself, so we have by definition

$$\mu^*(A) \le \nu(A).$$

Next, we need to show that this μ^* is an outer measure. We have basically already done this in the Proposition on Outer Measures! Since $\nu \ge 0$, it follows that $\mu^* \ge 0$. Moreover, since $\emptyset$ is a countable cover of itself, we have

$$0 \le \mu^*(\emptyset) \le \nu(\emptyset) = 0.$$

Hence $\mu^(\emptyset) = 0$.
Monotonicity: Assume that $A \subset B$. Then, any countable cover of B is also a countable cover of A. However, there could be covers of A which do not cover B. Hence, the set of countable covers of A contains the set of countable covers of B, so the infimum over covers of A is smaller than the infimum over covers of B, and therefore

$$\mu^*(A) \le \mu^*(B).$$

Countable sub-additivity: Let $\{A_j\}$ be pairwise disjoint. We wish to show that

$$\mu^*(\cup A_j) \le \sum \mu^*(A_j).$$

Note that if for any j we have $\mu^*(A_j) = \infty$, we are immediately done. So, assume this is not the case for any j. Let $\varepsilon > 0$. Then for each j there exists a countable cover $\{D_j^k\}$ such that

$$A_j \subset \cup_k D_j^k, \quad \mu^*(A_j) + \frac{\varepsilon}{2^j} \ge \sum_k \nu(D_j^k).$$

Hence, we also have

$$\cup_j A_j \subset \cup_{j,k} D_j^k,$$

and so

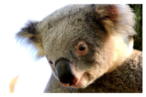
$$\mu^*(\cup A_j) = \inf ... \le \sum_{j,k} \nu(D_j^k) \le \sum_j \mu^*(A_j) + \frac{\varepsilon}{2^j} = \varepsilon + \sum_j \mu^*(A_j).$$

Since this holds for any $\varepsilon > 0$, we obtain the desired inequality.
Another outer measure: Assume that $\tilde{\mu}^*$ is another outer measure defined on X which has $\tilde{\mu}^*(A) \le \nu(A)$ for all $A \in \mathcal{C}$. If $\mu^*(A) = \infty$ there is nothing to prove. So assume that this is not the case. Let $\varepsilon > 0$. Then there exists a countable cover $\{D_j\}$ which contains A such that

$$\mu^*(A) + \varepsilon \ge \sum_k \nu(D_k) = \sum_k \tilde{\mu}^*(D_k) \ge \tilde{\mu}^*(\cup D_k) \ge \tilde{\mu}^*(A).$$

Above we have used that $\nu = \tilde{\mu}^*$ on the D_k, followed by countable sub-additivity of the outer measure $\tilde{\mu}^*$, followed by monotonicity of the outer measure $\tilde{\mu}^*$. Since this inequality holds for any $\varepsilon > 0$, we get that

$$\mu^*(A) \ge \tilde{\mu}^*(A).$$



Now, we shall specify to the case in which (X, d) is a metric space. For this, we recall that for a non-empty set $A \subset X$ we define its *diameter,*

$$\operatorname{diam}(A) := \sup\{d(x,y) : x \in A, \quad y \in A\}.$$

With this in mind, we can define the countable covers of diameter less than ϵ.

Definition 4.6. Let $\mathcal{C}$ be as above. For $\epsilon > 0$, define

$$\mathcal{C}_\epsilon := \{A \in \mathcal{C} : \operatorname{diam}(A) < \epsilon\}.$$

Now define the outer measure depending on this cover as a special case of (4.1), in particular we set

$$\mu^*_\epsilon(A) := \mu_{\nu, \mathcal{C}_\epsilon}(A).$$

If $\epsilon' < \epsilon$, then all covers which have diameter less than ϵ' also have diameter less than ϵ, so $\mathcal{C}_{\epsilon'} \subset \mathcal{C}_\epsilon$. Consequently, when we take the infimum to obtain μ^*_ϵ and $\mu^*_{\epsilon'}$, there are more elements considered in the infimum for $\mathcal{C}_\epsilon$ (i.e. more covers), so the infimum is smaller, and

$$\mu^*_\epsilon(A) \le \mu^*_{\epsilon'}(A).$$

The following theorem shows how, starting from any arbitrary set function ν which has $\nu(\emptyset) = 0$, we can construct a "canonical metric outer measure." We shall later see that for a particular choice of ν, we obtain the Hausdorff measure.

Theorem 4.7 (A canonical metric outer measure)**.** *The limit* $\mu^*_0(A) := \lim_{\epsilon \to 0} \mu^*_\epsilon(A), A \subset X$ *defines a metric outer measure.*

4.2. **Homework.**

(1) Prove that for any interval $I \subset \mathbb{R}^n$, there exists a series $\{B_j\}_{j \ge 1}$ such that
  (a) Each B_j is a ball in I.
  (b) It is $B_j \cap B_k = \emptyset$ for all $j \neq k$.
  (c) We have $\mathcal{L}_n(I \setminus \bigcup B_j) = 0$ (and therefore $\mathcal{L}_n(I) = \mathcal{L}_n(\bigcup B_j)$).

(2) Now for a bit of combinatorial fun... Let X be a non-empty set. Let $\{A_j\}_{j=1}^n$ be distinct, non-empty, proper subsets of X. How many elements does $\mathcal{A}$, the smallest algebra which contains $\{A_j\}_{j=1}^n$, have?

4.3. **Hints.** First note that $\mathcal{L}_n(I \setminus \mathring{I}) = 0$. So without loss of generality we can assume that I is open. For $x \in I$, there is $\delta \in \mathbb{Q}, \delta > 0$ such that $B_\delta(x) \subset I$. Also there exists $q \in \mathbb{Q}^n$ such that $|x - q| < \delta \cdot 10^{-6}$. This implies for every y with $|y - q| < (1 - 10^{-6})\delta$,

$$|y - x| \le |y - q| + |x - q| < \delta \implies y \in B_\delta(x) \subset I.$$

So we have

$$B_1 := B_{(1-10^{-6})\delta}(q) \subset I.$$

For $N \ge 1$ and $x \in I$, it is either $x \in \bigcup_{k=1}^N \overline{B_k}$ or not. We are assuming $\{B_k\}^N \subset I$ are disjoint balls with rational radii and rational centers (centers are elements of $\mathbb{Q}^n$). If $x \in \bigcup_{k=1}^N \overline{B_k}$ we consider $x \in I \setminus \bigcup_{k=1}^N \overline{B_k}$. Note that this set is *open.* So, if there exists $x \in I \setminus \bigcup_{k=1}^N \overline{B_k}$, then the same argument shows that there is a new ball,

$$x \in \ B_{N+1} \subset I \setminus \bigcup_{k=1}^N \overline{B_k}$$

with the center and radius of B_{N+1} rational (same argument as above). Then we note further that the set of balls

$$\{B_\delta(q) : \quad \delta \in \mathbb{Q}, \quad \text{ and } q \in \mathbb{Q}^n\}$$

is countable. Consequently, we require at most countably many of these balls to ensure that

$$I \subset \bigcup_{k=1}^{\infty} \overline{B_k} \text{ and } \mathcal{L}_n(\overline{B_k} \setminus B_k) = 0 \text{ for all } k \Rightarrow \mathcal{L}_n(\bigcup(\overline{B_k} \setminus B_k)) = 0.$$

So we get

$$\mathcal{L}_n(I) = \mathcal{L}_n(I \cap \bigcup B_k) + \mathcal{L}_n(I \setminus \bigcup B_k) = \mathcal{L}_n(\bigcup B_k) + \mathcal{L}_n(\bigcup \overline{B_k} \setminus B_k) = \mathcal{L}_n(\bigcup B_k).$$

5. Canonical metric outer measures and Hausdorff measure

Theorem 5.1 (A canonical metric outer measure)**.** *The limit $\mu_0^*(A) := \lim_{\epsilon \to 0} \mu_\epsilon^*(A), A \subset X$ defines a metric outer measure.*

Proof: Since μ_ε^* is non-decreasing as $\epsilon \downarrow 0$, the limit exists (since we allow ∞ as a limit value). We have already proven that each μ_ϵ^* is an outer measure.

Exercise 6. *Prove that the outer measure property is preserved under the limit as $\epsilon \to 0$, to show that μ_0^* is indeed an outer measure.*

Metric outer measure: Let $A, B \subset X$ be such that $\operatorname{dist}(A,B) > 0$. Since μ^* is an outer measure, by countable subadditivity,

$$\mu_0^*(A \cup B) \le \mu_0^*(A) + \mu_0^*(B).$$

We would like to prove the reverse inequality. The idea is that since A and B are at a positive distance away from each other, we can take ϵ small enough so that our μ_ϵ^* cover of the union splits into two disjoint covers. (Draw a picture!)
Let us make this precise. Since the distance between A and B is positive, there exists $n_0 \in \mathbb{N}$ such that

$$\operatorname{dist}(A,B) > \frac{1}{n}, \quad \text{for} \quad n > n_0.$$

Let $\delta > 0$ be some arbitrary positive number (this is our fudge factor which we shall later banish to zero). Then, cover the union $A \cup B$ with sets E_k^n such that

$$\mu^*_{\frac{1}{n}}(A \cup B) + \delta \ge \sum_{k=1}^{\infty} \nu(E_k^n)$$

and such that for each k we have $\operatorname{diam}(E_k^n) \le \frac{1}{n}$. Let us delete any E_k^n which has empty intersection with $A \cup B$, that is we delete any unneeded, extraneous, superfluous covers. Still denote this set by E_k^n for notational simplicity. We then still have

$$\mu^*_{\frac{1}{n}}(A \cup B) + \delta \ge \sum_{k} \nu(E_k^n).$$

Since the diameter of E_k^n is less than or equal to $\frac{1}{n}$ which is smaller than the distance between A and B, we have that the E_k^n intersect either A or B and not both in the sense that

$$E_k^n \cap A \neq \emptyset \Rightarrow E_k^n \cap B = \emptyset, \; E_k^n \cap B \neq \emptyset \Rightarrow E_k^n \cap A = \emptyset.$$

To see this, draw a picture. If some E_k^n intersected both A and B, then it would have to contain at least one point in A and at least one point in B. The distance between those points is strictly greater than $\frac{1}{n}$. Hence the diameter of such a set would need to exceed $\frac{1}{n}$, which is a contradiction.
So, with this consideration, let

$$E^n(A) := \{E_k^n : E_k^n \cap A \neq \emptyset\}, \quad E^n(B) := \{E_k^n : E_k^n \cap B \neq \emptyset\}.$$

Then, $E^n(A)$ and $E^n(B)$ have no sets in common and together they yield the sequence $(E_k^n)_{k=1}^{\infty}$. Since

$$A \cup B \subset \cup E_k^n \implies A \subset \cup_{E^n(A)}, \quad B \subset \cup_{E^n(B)}.$$

We therefore have

$$\mu^*_{\frac{1}{n}}(A) \le \sum_{E^n_k \in E^n(A)} \nu(E^n_k), \quad \mu^*_{\frac{1}{n}}(B) \le \sum_{E^n_k \in E^n(B)} \nu(E^n_k),$$

so the sum

$$\mu^*_{\frac{1}{n}}(A) + \mu^*_{\frac{1}{n}}(B) \le \sum_{E^n_k \in E^n(A)} \nu(E^n_k) + \sum_{E^n_k \in E^n(B)} \nu(E^n_k).$$

Now, the sum on the right side is just

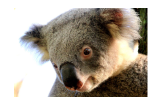
$$\sum_k \nu(E^n_k) \le \mu^*_{\frac{1}{n}}(A \cup B) + \delta.$$

So, we have proven that

$$\mu^*_{\frac{1}{n}}(A) + \mu^*_{\frac{1}{n}}(B) \le \mu^*_{\frac{1}{n}}(A \cup B) + \delta.$$

This holds for all $n \ge n_0$. So, letting $n \to \infty$, we obtain

$$\mu^*_0(A) + \mu^*_0(B) \le \mu^*_0(A \cup B) + \delta.$$

Finally, we let $\delta \downarrow 0$, which completes the proof that μ^*_0 is a metric outer measure. 

Remark 4. Making a special choice of the function ν, we shall obtain the Hausdorff measure, below. However, our preceding results are super general. If you are so inclined, it could be pretty interesting to play around with different functions, ν, satisfying the hypotheses, and thereby obtain different metric outer measures according to the theorem above.... Once you've got a metric outer measure, then you can use our results to obtain its sigma algebra of measurable sets. Moreover, our results prove that this sigma algebra contains the Borel sigma algebra. Our results also prove that this metric outer measure together with its sigma algebra of measurable sets yields a complete measure. So, now you have quite a collection of tools to build all kinds of different measures!

5.1. **The Hausdorff measure.** We shall use the general results from the preceding lecture to obtain the Hausdorff measure.

Definition 5.2. Let (X, d) be a metric space, $\delta > 0$ and $t \in (0, \infty)$. Then for $S \subset X$, define the set function

$$\mathcal{H}^t_\delta(S) := \inf \left\{ \sum_{i=1}^\infty (\text{diam} U_i)^t | S \subset \bigcup_{i=1}^\infty U_i, \text{diam}(U_i) < \delta \right\}$$

where the infimum is taken over all countable covers of S by sets $U_i \subset X$ with $\text{diam}(U_i) < \delta$.

Remark 5. In the definition if one requires the U_i's to be closed in this definition, the result is the same because

$$\text{diam}(U_i) = \text{diam}(\overline{U_i}).$$

If one requires the U_i's to be open, call the corresponding thing $\tilde{\mathcal{H}}^t_\delta$. Note that the infimum is now taken over fewer covers, since the U_i need to be open. So á priori one has $\tilde{\mathcal{H}}^t_\delta \ge \mathcal{H}^t_\delta$. For S such that

$$\mathcal{H}^t_\delta(S) = \infty,$$

then one also has

$$\tilde{\mathcal{H}}^t_\delta(S) = \infty,$$

so there is nothing to do. Let us assume this is not the case. Fix $\eta > 0$. Let $\{U_j\}$ be a cover which has

$$\mathcal{H}^t_\delta(S) + \eta \ge \sum_j \text{diam}(U_j)^t.$$

Then let

$$B_j = \{x \in X : d(x, U_j) < \epsilon_j 2^{-j-1}\},$$

Choose $\epsilon_j > 0$ so that the diameter of B_j, which is at most $\text{diam}(U_j) + \epsilon 2^{-j}$ is still less than δ. Since the diameter of U_j is strictly less than δ this is always possible. Without loss of generality assume that $\epsilon_j \le 1$ for all j. Then the B_j are an open cover of S, with diameter less than δ, so we have

$$\tilde{\mathcal{H}}^t_\delta(S) \le \sum_j (\text{diam}(B_j))^t \le \sum_j (\text{diam}(U_j) + \epsilon_j 2^{-j})^t.$$

As the $\epsilon_j \to 0$, the right side converges to $\sum_j \text{diam}(U_j)^t$. So, let this happen, to obtain

$$\tilde{\mathcal{H}}^t_\delta(S) \le \sum_j (\text{diam}(U_j))^t \le \mathcal{H}^t_\delta(S) + \eta.$$

Since $\eta > 0$ was arbitrary, letting now $\eta \to 0$ we obtain that $\tilde{\mathcal{H}}^t_\delta \le \mathcal{H}^t_\delta$. So it's still the same. Thus, if it's more convenient to consider (1) closed covers in definition of Hausdorff measure or (2) open covers in definition of Hausdorff measure, DO IT! There is no loss of generality.

Corollary 5.3 (Hausdorff measure)**.** *The set function $\mathcal{H}^t_\delta$ is an outer measure. Moreover,*

$$\mathcal{H}^t := \lim_{\delta \to 0} \mathcal{H}^t_\delta$$

is a metric outer measure. All Borel sets are $\mathcal{H}^t$ measurable, and these sets form a σ-algebra.

Proof: First, set

$$\nu(U) := \text{diam}(U)^t.$$

Then note that

$$\mathcal{H}^t_\delta(S) = \mu^*_{\nu, \mathcal{C}_\delta}(S)$$

is just a special case of the "canonical outer measure" theorem. By that theorem, we therefore obtain that

$$\mathcal{H}^t(S) := \lim_{\delta \to 0} \mathcal{H}^t_\delta(S)$$

is a metric outer measure. By an earlier theorem (2.3, all the Borel sets are $\mathcal{H}^t$-measurable. These Borel sets are contained in the σ-algebra of "$\mathcal{H}^t$-measurable sets from Theorem 5.1.

Moreover, by this same theorem, $\mathcal{H}^t$ on this σ-algebra is a complete measure.

We shall call $\mathcal{H}^t$ the t-dimensional Hausdorff measure. The reason for this is that if $t \in \mathbb{N}$ and A is t-dimensional, then the amount of A contained in a region of diam $= r$" should be proportional to r^t. This is because a ball in t-dimensional space has volume proportional to r^t. What exactly is the volume of a ball in $\mathbb{R}^n$ anyways?

5.2. **The volume of the unit ball in $\mathbb{R}^n$.**

Proposition 5.4. *The volume of the unit ball in $\mathbb{R}^n$ is*

$$w_n = Vol\ (B_1(0)) = \frac{2\pi^{\frac{n}{2}}}{n \cdot \Gamma(\frac{n}{2})}$$

Proof: Our goal is to compute

$$\int_{S_1(0)} \int_0^1 r^{n-1}\, \mathrm{d}r \mathrm{d}\sigma.$$

For starters, we would like to compute

$$\sigma_n := \int_{S_1(0)} \mathrm{d}\sigma,$$

that is the surface area of the unit ball. Let us start by computing a famous integral. Define

$$I_n := \int_{\mathbb{R}^n} e^{-\pi |x|^2} dx.$$

Note that $I_n = (I_1)^n$ by the Fubini-Tonelli theorem, since everything converges beautifully. So, in particular,

$$I_n = (I_2)^{2/n}.$$

I_2 is particularly lovely to compute:

$$I_2 = \int_0^{2\pi} \int_0^\infty e^{-\pi r^2} r \,\mathrm{d}r\mathrm{d}\theta = 2\pi \int_0^\infty e^{-s^2} \frac{sds}{\pi} = \int_0^\infty e^{-s^2} 2sds$$

$$= -e^{-s^2}\Big|_0^\infty = 1.$$

We have used the substitution $s = \sqrt{\pi} r$. So we see that $I_k = 1$ for all $k \in \mathbb{N}$. Then, we can apply this to compute σ_n.

$$1 = \int\limits_{\mathbb{R}^n} e^{-\pi|x|^2}\, dx = \int\limits_{S_1(0)} \int\limits_0^\infty e^{-\pi r^2} r^{n-1}\, \mathrm{d}r\mathrm{d}\sigma = \sigma_n \int\limits_0^\infty e^{-\pi r^2} r^{n-1}\, \mathrm{d}r.$$

Well, the latter integral we may be able to compute, because it is one-dimensional. Let $s = r^2\pi$. Then $\mathrm{d}s = 2r\pi dr$, so

$$r^{n-1} dr = \left(\frac{s}{\pi}\right)^{(n-1)/2} \frac{ds}{2\pi\sqrt{s/\pi}} = \frac{s^{n/2-1}}{2\pi^{n/2}}.$$

So,

$$1 = \frac{\sigma_n}{2\pi^{n/2}} \int_0^\infty e^{-s} s^{n/2-1} ds.$$

This looks familiar... Recall:

$$\Gamma(z) = \int\limits_0^\infty s^{z-1} e^{-s} \mathrm{d}s, \quad z \in \mathbb{C}, \quad \Re(z) > 1.$$

So,

$$\sigma_n = \frac{2\pi^{n/2}}{\Gamma(n/2)}.$$

We compute using integration by parts:

$$\Gamma(s+1) = \int\limits_0^\infty t^s e^{-t} \mathrm{d}t = \left[-t^s e^{-t}\right] - \int\limits_0^\infty -e^{-t} s t^{s-1} dt = s\Gamma(s).$$

Exercise 7. *Prove that the Γ function admits a meromorphic continuation to $\mathbb{C}$ which is holomorphic with the exception of simple poles at $0 \cup -\mathbb{N}$.*

Finally, we compute the volume of the ball:

$$\int\limits_{B_1(0)} dx = \mathrm{Vol}\ (B_1(0)) = \int\limits_{S_1(0)} \int\limits_0^1 r^{n-1}\, \mathrm{d}r\mathrm{d}\sigma = \sigma_n \int\limits_0^1 r^{n-1}\, \mathrm{d}r = \left[\sigma_n \frac{r^n}{n}\right]_0^1 = \frac{\sigma_n}{n} = w_n$$

Therefore, we have $w_n = \frac{2\pi^{n/2}}{n \cdot \Gamma(\frac{n}{2})}$, which finishes our proof.

Corollary 5.5. *$\forall x \in \mathbb{R}^n$ and $r > 0$, the area of $S_r(x)$ is $r^{n-1}\sigma_n$ and Vol $(B_r(x)) = w_n r^n$.*

Proof:

$$\int_{S_r(x)} \mathrm{d}\sigma = \int_{S_r(0)} \mathrm{d}\sigma = \int_{S_1(0)} r^{n-1} \mathrm{d}\sigma = r^{n-1}\sigma_n$$

Analogously for $B_r(x)$. 

Exercise 8. *Compute* $\Gamma(n/2)$ *for* $n \in \mathbb{N}$.

5.3. **Homework: Relationship between Lebesgue and Hausdorff measures.** To understand the relationship between Lebesgue and Hausdorff measures, we require the notion of *absolute continuity* of measures.

Definition 5.6. Let ν and μ be measures on $(X, \mathcal{M})$. Then ν is *absolutely continuous* with respect to μ and we write $\nu << \mu$ if $\nu(Y) = 0$ $\forall Y \in \mathcal{M}$ with $\mu(Y) = 0$. We say that μ and ν are *mutually singular* and write $\mu\nu$ if there exists $E, F \in \mathcal{M}$ with $E \cap F = \emptyset$, $E \cup F = X$, $\mu(E) = 0$, $\nu(F) = 0$.

Exercise 5. *Prove that* $\mathcal{H}^n << \mathcal{L}_n$ *and* $\mathcal{L}_n << \mathcal{H}^n$.

5.4. **Hints. Showing that $\mathcal{H}^n << \mathcal{L}_n$:** First, we consider $I = \prod \mathrm{I} a_i, b_i \mathrm{I}$, $l_i := b_i - a_i$. If any $l_i = 0$ let's WLOG assume that $l_i, ..., l_k$ are all non-zero and $l_{k+1} = ... = l_n = 0$. Then $\forall \varepsilon > 0$, we can cover an interval of length L by $\frac{L}{\varepsilon}$ balls (one-dimensional) of radius ε. Similarly, we can cover I by $\prod\limits_{i=1}^{k} \frac{l_i}{\varepsilon}$ balls of radius ε. It follows that

$$\forall \delta \leq \varepsilon, \quad \mathcal{H}^n_\delta(I) \leq \prod_{i=1}^{k} \frac{l_i}{\delta}(2\delta)^n = \delta^{n-k} 2^n \prod_{i=1}^{k} l_i,$$

$$\delta \to 0 \Rightarrow \mathcal{H}^n(I) = 0.$$

If $l_i = 0$ for all i, then I is either a point or the empty set which both have $\mathcal{H}^n = 0$. For a point, this is because for any $\delta > 0$, we can cover a point by a ball of radius $\delta/2$, so that $\mathcal{H}^n_\delta(p) \leq \delta^n$ holds for all δ, which letting $\delta \downarrow 0$ gives $\mathcal{H}^n(p) = 0$.

Finally, if for all i, $l_i \neq 0$, then we can cover I by $\prod\limits_{i=1}^{n} \frac{l_i}{\varepsilon}$ balls of radius ε. Then

$$\forall \delta \geq \varepsilon, \quad H_{n,\delta}(I) \leq \prod_{i=1}^{n} \frac{l_i}{\delta}(2\delta)^n = 2^n \mathcal{L}_n(I).$$

Consequently, if $\mathcal{L}_n(I) = 0$, then $\mathcal{H}^n_\delta(I) = 0$ which implies that $\mathcal{H}^n(I) = 0$.

If $\mathcal{L}_n(A) = 0$, then $\exists \{I_j\}_{j\geq 1}$ such that $A \subset \bigcup\limits_{j\geq 1} I_j$ and, for a fixed $\varepsilon > 0$, $\sum\limits_{j\geq 1} \mathcal{L}_n(I_j) < \frac{\varepsilon}{2^n}$. Then

$$H_n(A) \leq \sum_{j\geq 1} H_n(I_j) \leq 2^n \sum_{j\geq 1} \mathcal{L}_n(I_j) < \varepsilon$$

Hence $H_n(A) = 0$. Therefore $H_n << \mathcal{L}_n$.

Showing that $\mathcal{L}_n << \mathcal{H}^n$: Let $A \subset \mathbb{R}^n$ such that $\mathcal{H}^n(A) = 0$, where $A \in \mathcal{B}$. Then, since $\mathcal{H}^n_\delta \leq \mathcal{H}_n$,

$$\mathcal{H}^n_\delta(A) = 0 \, \forall \delta > 0$$

$\Rightarrow \exists$ a sequence $\{B_j\}_{j\geq 1}$, which is closed in $\mathbb{R}^n$, such that $A \subset \bigcup\limits_{j=1}^{\infty} B_j$ and $\sum\limits_{j\geq 1} (\mathrm{diam}(B_j))^n < \varepsilon$, where $\varepsilon > 0$. Note that for $x \in B_j$, $\rho(x, y) \leq \delta_j = \mathrm{diam}(B_j) \forall x \in B_j$. So we can fix $x_j \in B_j$,

and we get $B_j \subseteq \bar{B}_{\delta_j}(x_j)$.
So we have

$$\mathcal{L}_n(B_j) \leq \mathcal{L}_n(B_{\delta_j}(x_j)) = w_n \delta_j^n$$

where $w_n = \text{Vol}\,(B_1(0))$ denotes the volume of the unit ball with radius 1 (around zero). Alltogether, we get

$$\varepsilon > \sum_{j\geq 1} \text{diam}(B_j)^n = \sum_{j\geq 1} \frac{\mathcal{L}_n(B_{\delta_j}(x_j))}{w_n} \geq \frac{1}{w_n} \sum_{j\geq 1} \mathcal{L}_n(B_j)$$

and since $A \subset \bigcup_{j=1}^{\infty} B_j$ we get

$$\varepsilon > \frac{1}{w_n} \sum_{j\geq 1} \mathcal{L}_n(B_j) \geq \frac{1}{w_n} \mathcal{L}_n(A)$$

Letting $\varepsilon \downarrow 0 \Rightarrow \mathcal{L}_n(A) = 0$.

6. Hausdorff dimension

If the notion of Hausdorff dimension is to be well-defined, then it should be invariant under isometries. We prove that the Hausdorff measure is indeed invariant under isometries, and therefore the Hausdorff dimension, which we shall define using the Hausdorff measure, will similarly enjoy this invariance. Let $\mathcal{H}^p$ denote p-dimensional Hausdorff measure. We first prove a more general fact. Before proceeding to that proof, there is an exercise which will allow us to be a little sloppy (or for a more positive connotation, allow us to be a little more mellow and groovy).

Exercise 9. *Change the definition of C_ϵ covers to require diameters less than or equal to ϵ. Show that the corresponding μ_0^* remains unchanged. Thus, in the definition of Hausdorff outer measure (and Hausdorff measure), it does not require if our $\mathcal{H}^p_\delta = \mu^*_{\delta,\nu}$ for $\nu(A) = \text{diam}(A)^p$ is for covers with diameter $< \delta$ or $\leq \delta$. Either way one obtains the same outer measure $\mathcal{H}^p_\delta$. Therefore, either way one also obtains the same $\mathcal{H}^p$.*

Proposition 6.1. *Let (X, d) be a metric space, and f, g be maps from some set Y into X. If $f, g : Y \to X$ satisfy $d(f(y), f(z)) \leq C d(g(y), g(z))$ $\forall y, z \in Y$, then $\mathcal{H}^p(f(A)) \leq C^p \mathcal{H}^p(g(A))$.*

Proof: Let $\varepsilon > 0$, $A \subset Y$. Then $g(A) \subset X$. If $\mathcal{H}^p(g(A)) = \infty$, there is nothing to prove. So, we assume this is not the case. Then, for all $\delta > 0$ small, we can find $\{B_j\}_{j\geq 1} \subset X$ such that

$$g(A) \subset \bigcup_{j=1}^{\infty} B_j, \quad \text{diam}(B_j) < \frac{\delta}{C},$$

and

$$\sum_{j\geq 1} \text{diam}(B_j)^p \leq \mathcal{H}^p(g(A)) + \frac{\varepsilon}{C^p}.$$

Of course, the particular collection B_j does depend on the particular small value of δ, but we shall suppress this dependence for notational convenience.
Let us define

$$\tilde{B}_j := f(g^{-1}(B_j)).$$

We claim that these are going to cover $f(A)$. Let $y \in A$ so that $f(y) \in f(A)$. Then, since $y \in A$, we also have $g(y) \in g(A) \subset \cup B_j$. So, in particular, $g(y) \in B_j$ for some j. Hence

$$y \in g^{-1}(B_j) = \{z \in Y : g(z) \in B_j\}.$$

Therefore $f(y) \in f(g^{-1}(B_j)) = \tilde{B}_j$.
We therefore have

$$f(A) \subset \cup_{j=1}^{\infty} \tilde{B}_j.$$

Now, if $f(y)$ and $f(z)$ are both in $\tilde{B}_j = f(g^{-1}(B_j))$, this means that y and z are both in $g^{-1}(B_j)$, so there exist x and x' in B_j with $g(y) = x \in B_j$ and $g(z) = x' \in B_j$. Then

$$d(f(y), f(z)) \le Cd(g(y), g(z)) \le C\mathrm{diam}(B_j) < C\frac{\delta}{C} = \delta.$$

Consequently $\mathrm{diam}(\tilde{B}_j) < \delta$. So,

$$\mathcal{H}^p_\delta(f(A)) \le \sum_{j\ge1} \mathrm{diam}(\tilde{B}_j)^p.$$

Moreover, by the same calculation as above, we also see that

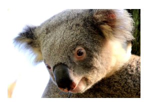
$$\mathrm{diam}(\tilde{B}_j) \le C\mathrm{diam}(B_j) \implies \mathrm{diam}(\tilde{B}_j)^p \le C^p\mathrm{diam}(B_j)^p.$$

Consequently,

$$\mathcal{H}^p_\delta(f(A)) \le \sum_{j\ge1} \mathrm{diam}(\tilde{B}_j)^p \le C^p \sum_{j\ge1} (\mathrm{diam}(B_j))^p \le C^p\mathcal{H}^p(g(A)) + \epsilon.$$

This holds for any $\epsilon > 0$, so we obtain the desired result:

$$\mathcal{H}^p(f(A)) \le C^p\mathcal{H}^p(g(A)).$$



Corollary 6.2. *$\mathcal{H}^p$ is invariant under isometries.*

Proof: Let $I : (X, d) \to X$ be an isometry. Let $\mathrm{id} : X \to X$ be the identity map. Then since I is an isometry, we have

$$d(I(x), I(z)) = d(x, z) = d(\mathrm{id}(x), \mathrm{id}(z)), \quad \forall x, z \in X.$$

Hence the hypotheses of the proposition hold true taking $X = Y$, $f = I$, $g = \mathrm{id}$, and $C = 1$. So, we obtain

$$\mathcal{H}^p(I(A)) \le \mathcal{H}^p(\mathrm{id}(A)) = \mathcal{H}^p(A).$$

On the other hand, we also have

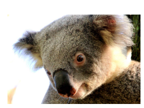
$$d(\mathrm{id}(x), \mathrm{id}(z)) = d(x, z) = d(I(x), I(z)) \le d(I(x), I(z)).$$

So, we apply the same proposition taking $X = Y$, $f = \mathrm{id}$, $g = I$, and $C = 1$. We therefore obtain

$$\mathcal{H}^p(A) = \mathcal{H}^p(\mathrm{id}(A)) \le \mathcal{H}^p(I(A)).$$

Thus, the inequality goes in both directions, and we have in fact an equality,

$$\mathcal{H}^p(A) = \mathcal{H}^p(I(A)).$$



Proposition 6.3 (Hausdorf dimension)**.** *If $\mathcal{H}^p(A) < \infty$, then $\mathcal{H}^q(A) = 0$ $\forall q > p$. If $\mathcal{H}^q(A) > 0$, then $\mathcal{H}^p(A) = \infty$ $\forall p < q$.*

Proof: For the first statement, assume $\mathcal{H}^p(A) < \infty$. Then, for any sufficiently small $\delta > 0$, we can find a cover of A by $\{B_j\}_{j\ge1}$ with $\mathrm{diam}(B_j) < \delta$, and

$$\mathcal{H}^p_\delta(A) \le \sum_{j\ge1} \mathrm{diam}(B_j)^p \le \mathcal{H}^p(A) + 1.$$

If $q > p$, then

$$\mathcal{H}^q_\delta(A) \le \sum_{j\ge1} \mathrm{diam}(B_j)^q = \sum_{j\ge1} \mathrm{diam}(B_j)^{p+q-p} \le \sum_{j\ge1} \mathrm{diam}(B_j)^p\delta^{q-p}$$

$$= \delta^{q-p} \sum_{j \geq 1} \operatorname{diam}(B_j)^p \leq \delta^{q-p}(\mathcal{H}^p(A) + 1),$$

which tends to zero as $\delta \to 0$. Hence we can show that $\mathcal{H}^q_\delta(A)$ tends to zero as $\delta \to 0$, thus it follows that $\mathcal{H}^q(A) = 0$.

The second statement is the contrapositive. To see this let us first fix $q > p$. We shall write $\star$ to denote the statement $\mathcal{H}^p(A) < \infty$, and $\heartsuit$ to denote the statement $\mathcal{H}^q(A) = 0$. We have proven: if $\star$ then $\heartsuit$. The contrapositive says: if not $\heartsuit$ then not $\star$. It is well known from elementary logic that a statement is true if and only if its contrapositive is true. In this case, not $\heartsuit$ says that $\mathcal{H}^q(A) \neq 0$. Since $\mathcal{H}^q(A) \geq 0$, we have $\mathcal{H}^q(A) > 0$. This should imply not $\star$. Not $\star$ is the statement that $\mathcal{H}^p(A) = \infty$. Since the $q > p$ was arbitrary, we have shown that if $\mathcal{H}^q(A) > 0$, then $\mathcal{H}^p(A) = \infty$ for any $p < q$. 

Corollary 6.4 (Definition of Hausdorff dimension)**.** *Let $A \subset X$, where (X, d) is a metric space. Then the following infimum and supremum are equal*

$$\delta = \inf\{p \geq 0 \mid \mathcal{H}^p(A) = 0\} = \sup\{p \geq 0 \mid \mathcal{H}^p(A) = \infty\}$$

This is how we define the Hausdorff dimension *of A, δ, denoted by* $\dim(A)$*. If for some p we have*

$$\mathcal{H}^p(A) \in (0, \infty)$$

then $p = \dim(A)$.

Proof: Let $\{p_n\}$ be a sequence which converges to the infimum on the left. Then, $\mathcal{H}^{p_n}(A) = 0$ for all n. Let $\{q_n\}$ be a sequence which converges to the supremum on the right. Then, $\mathcal{H}^{q_n}(A) = \infty$ for all n. By the second statement of the preceding proposition, since $\mathcal{H}^{q_n}(A) > 0$, $\mathcal{H}^p(A) = \infty$ for all $p < q$. This shows that $p_n \geq q_m$ for all n and m. Therefore

$$\liminf p_n \geq \limsup q_m.$$

Since in these cases the limits exist, we have

$$\liminf p_n = \lim p_n, \quad \limsup q_m = \lim q_m.$$

This shows that

$$\inf\{p \geq 0 \mid \mathcal{H}^p(A) = 0\} \geq \sup\{p \geq 0 \mid \mathcal{H}^p(A) = \infty\}.$$

For the sake of contradiction, let us assume that this inequality is strict, so that

$$\inf\{p \geq 0 \mid \mathcal{H}^p(A) = 0\} > \sup\{p \geq 0 \mid \mathcal{H}^p(A) = \infty\}.$$

Then, there is some number, x which lies precisely between these two values,

$$\inf\{p \geq 0 \mid \mathcal{H}^p(A) = 0\} > x > \sup\{p \geq 0 \mid \mathcal{H}^p(A) = \infty\}.$$

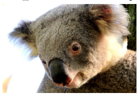
Since x is less than the infimum, we cannot have $\mathcal{H}^x(A) = 0$, (because then x would be included in the infimum, so the infimum would be $\leq x$ which by assumption it is not). So we must have $\mathcal{H}^x(A) > 0$. By the proposition, it follows that $\mathcal{H}^p(A) = \infty$ for all $p < x$. Hence, the supremum on the right side is taken over a set of p which contains all $p < x$. Therefore, by definition of the supremum, the supremum is greater than or equal to x. This is a contradiction. Hence, we cannot have

$$\inf\{p \geq 0 \mid \mathcal{H}^p(A) = 0\} > \sup\{p \geq 0 \mid \mathcal{H}^p(A) = \infty\},$$

as it leads to a contradiction. Thus, since the infimum is greater than or equal to the supremum, both sides must be equal.

Finally, assume that for some p we have $\mathcal{H}^p(A) \in (0, \infty)$. Then, by the proposition, $\mathcal{H}^q(A) = 0$ for all $q > p$. This shows that all $q > p$ are considered in the infimum, hence the infimum must be less than or equal to p. On the other hand, by the same proposition, $\mathcal{H}^q(A) = \infty$ for all $q < p$. Hence, the supremum is taken over a set which includes all $q < p$, hence the supremum must be greater than or equal to p. So, we get $\inf \leq p \leq \sup$, but since the infimum and supremum are equal, we have an equality all the way across.

This shows that the supremum here is less than or equal to p. Since the supremum and infimum equivalently define $\dim(A)$, we have $\dim(A) \geq p$ and $\dim(A) \leq p$. Hence we have $\dim(A) = p$.

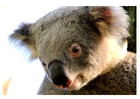


If our notion of dimension is a good one, then it ought to be monotone. We see below that this is the case.

Lemma 6.5 (Monotonicity of Hausdorff dimension)**.** *If* $A \subset B$*, then* $\dim(A) \leq \dim(B)$.

Proof:
If $A \subset B$, and $\mathcal{H}^p(B) = 0$, then $\mathcal{H}^p(A) = 0$. This is because $\mathcal{H}^p$ is an outer measure, which we proved, and outer measures are by definition monotone.
Therefore

$$\dim(B) = \inf\{p \geq 0 | \mathcal{H}^p(B) = 0\} \geq \inf\{p \geq 0 | \mathcal{H}^p(A) = 0\} = \dim(A).$$



If our definition of dimension is a good one, then we know what the dimension of $\mathbb{R}^n$ should be... To prove this, we shall prove a general fact about Hausdorff dimension.

Lemma 6.6. *The dimension of a countable union of sets,* E_j*,*

$$E = \cup E_j$$

is equal to

$$\dim(E) = \sup\{\dim(E_j\}.$$

Proof: We note that

$$E_j \subset E \forall j \implies \dim(E_j) \leq \dim(E) \quad \forall j,$$

so

$$\sup\{\dim(E_j)\} \leq \dim(E).$$

If the supremum on the left is infinite, there is nothing to prove, because both sides are therefore infinite and equal. Let us assume that it is not infinite. So, let us call this supremum δ. By the definition of $\dim(E_j) \leq \delta$, we have

$$\mathcal{H}^p(E_j) = 0 \quad \forall p > \delta.$$

Consequently, for all $p > \delta$, we have by countable subadditivity of Hausdorff outer measure

$$0 \leq \mathcal{H}^p(E) \leq \sum_j \mathcal{H}^p(E_j) = 0.$$

Thus

$$\mathcal{H}^p(E) = 0.$$

Since

$$\dim(E) = \inf\{p \geq 0 | \mathcal{H}^p(E) = 0\},$$

and $\mathcal{H}^p(E) = 0$ for all $p > \delta$, we have

$$\dim(E) \leq \delta.$$

Since $\dim(E) \geq \sup\{\dim(E_j)\} = \delta$, we obtain the equality. 

6.1. **Homework.**

(1) Compute the Hausdorff measure of the curve $\{(x, \sin(1/x)) : 0 < x < 1\} \subset \mathbb{R}^2$.
(2) Compute the Hausdorff measure of the curve $\{(x, \sin(1/x)) : 1/2 < x < 1\} \subset \mathbb{R}^2$.
(3) Compute the Hausdorff measure of the unit sphere sitting in $\mathbb{R}^3$.
(4) We shall see that a set whose Hausdorff dimension is positive is uncountable. Is the converse true, that is if the Hausdorff dimension of s set is zero, then is that set necessarily countable? Prove or give a counter example.
(5) Is it always true that $\mathcal{H}^{\dim(A)}(A) \in (0, \infty)$? Prove or a give a counter example. What if you assume that $\dim(A) \in (0, \infty)$, then is it always true that $\mathcal{H}^{\dim(A)}(A) \in (0, \infty)$?
(6) How should one define the Hausdorff dimension of the empty set? Philosophically and mathematically justify your answer.
(7) What is the Hausdorff dimension of a product of sets? How should this work? Figure it out and rigorize your answer.

7. Properties of Hausdorff dimension

Any set with positive Hausdorff dimension is uncountable!

Corollary 7.1. *Let $E \subset X$. If $\dim(E) > 0$, then E is uncountable.*

Proof: If E is countable, then $E = \bigcup_j e_j$, where $e_j \in X$ is a point. Therefore, we have proven that

$$0 \le \dim(E) = \sup \dim(\{e_j\}).$$

Now let $p > 0$. Note that a single point is contained in a ball of radius δ for any $\delta > 0$. Thus by definition

$$\mathcal{H}^p_\delta(e_j) \le 2^p \delta^p.$$

Letting $\delta \to 0$, we obtain

$$\mathcal{H}^p(e_j) = 0.$$

Therefore the Hausdorff dimension of a point is equal to $\inf\{p : p > 0\} = 0$. By the result we proved, the dimension of E is the supremum over the dimension of e_j, and this is the supremum over zero, hence it is zero. 

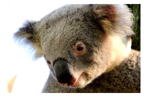
Corollary 7.2 (Hausdorff dimension of $\mathbb{R}^n$)**.** *The Hausdorff dimension of $\mathbb{R}^n$ is n.*

Proof: We can write the euclidian space $\mathbb{R}^n$ as $\mathbb{R}^n = \bigcup_{m \ge 1} B_m$, where B_m are balls of radius m centered at the origin. Here is where we are going to use some teamwork. In the exercises, you have proven that

$$\mathcal{H}^n(B_m) = c_n \mathcal{L}^n(B_m) = c_n m^n w_n,$$

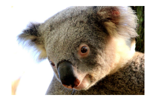
where c_n is a constant that depends only on n, and w_n is the volume of the unit ball in $\mathbb{R}^n$, and $\mathcal{L}^n$ is n-dimensional Lebesgue measure. (i.e. our usual human notion of n-dimensional volume). By a corollary proven today, the Hausdorff dimension of a ball in $\mathbb{R}^n$ is equal to n, since the Hausdorff measure of a ball of radius m is a positive, finite number. Moreover, a ball is an open set, so it is therefore contained in the Borel sigma algebra which is contained in the Hausdorff sigma algebra. So, since $\mathbb{R}^n$ is the union of these balls, and these balls are all Hausdorff measurable sets, the dimension of $\mathbb{R}^n$ is equal to the supremum of the dimensions of the balls. That is the supremum over the constant number n. Hence the supremum is n which gives the dimension of $\mathbb{R}^n$. 

Corollary 7.3. *For any $A \subset \mathbb{R}^n$, we have $\dim(A) \le n$.*

Proof: This follows immediately taking $B = \mathbb{R}^n$ in the lemma showing monotonicity of Hausdorff dimension. 

Lemma 7.4. *Let $E \subset \mathbb{R}^n$ such that* $\dim(E) < n$. *Then* $\mathring{E} = \emptyset$.

Proof: If $\mathring{E} \neq \emptyset$, then there $\exists r > 0$ and $x \in E$ such that $B_r(x) \subset E$. $\Rightarrow \dim(E) \geq \dim(B_r(x)) = n$

So we get $n \geq \dim E \geq n \Rightarrow \dim E = n$. 

Remark 6. The Hausdorff Dimension of a subset $E \subset \mathbb{R}^n$ is the same if we consider E as a subset of $\mathbb{R}^m$ for any $m \geq n$ via the canonical embedding, $\mathbb{R}^n \mapsto \mathbb{R}^n \times \{0\}$. In this sense, if we have a set E which naturally lives in k-dimensions, if we view the set E as living in 10 zillion dimensions, the Hausdorff dimension of E remains the same. This is simply because the Hausdorff dimension, which is determined by the Hausdorff (outer) measure is defined in terms of *diameter,* and the diameter of sets does not change if we embed the sets into higher dimensional Euclidean space. That is another reason the Hausdorff dimension is "a good notion of dimension," because it is invariant of the ambient space.

7.1. **Similitudes.** To study the relationship between fractals and Hausdorff dimension, we shall use a notion of a *similitude.*

Definition 7.5. For $r > 0$, a similitude with scaling factor r is a map $S : \mathbb{R}^n \to \mathbb{R}^n$ of the form

$$S(x) = r\mathcal{O}(x) + b,$$

where $\mathcal{O}$ is an orthogonal transformation (rotation, reflection, or composition of these), and $b \in \mathbb{R}^n$. If $S = (S_1, \cdots S_m)$ is a family of similitudes with common scaling factor $r < 1$, for $E \subset \mathbb{R}^n$ we define

$$S^0(E) = E, \qquad S(E) = \bigcup_{j=1}^{m} S_j(E), \qquad S^k(E) = S(S^{k-1}(E)) \text{ for } k > 1.$$

We say that E is invariant under S if $S(E) = E$.

Why is such a thing called a *similitude?* Indeed, this is aptly named if we ponder what a similitude *does.* If we apply S to a set E, then first E undergoes some composition of rotations and reflections. Next, it is scaled by the factor r. Finally, it is translated by b. So, the image under S, that is $S(E)$ is *similar* to E. It has just been reflected and/or rotated, shrunken or stretched, depending if $r < 1$ or $r > 1$, and then translated.
Similitudes are maps of the form $r \cdot O(x) + b$, where $O(x)$ is an orthogonal transformation, and b is a vector in $\mathbb{R}^n$. These are therefore affine linear maps. We would like to understand how similitudes and invariant sets under similitudes relate to Hausdorff measure which motivates the following.

Proposition 7.6. *If $k \leq n$, $A \subset \mathbb{R}^k$ and $T\colon \mathbb{R}^k \to \mathbb{R}^n$ is an affine linear map, then* $\mathcal{H}^k(T(A)) = \sqrt{\det(M^T M)}\mathcal{H}^k(A)$, *where $Tx = Mx + b$.*

Proof: First note that $\mathcal{H}^k$ is translation invariant because $\mathcal{H}^k(A + b) = \mathcal{H}^k(A)$ since

$$A \subset \bigcup_j E_j \iff A + b \subset \bigcup (E_j + b),$$

and $\operatorname{diam}(E_j) = \operatorname{diam}(E_j + b)$. So, without loss of generality, we shall assume $b = 0$. First, we consider the case $n = k$. Then, $Tx = Mx$, where M is an $n \times n$ matrix. Therefore, using the relationship between Hausdorff and Lebesgue measures,

$$\mathcal{H}^n(T(A)) = c_n \mathcal{L}_n(T(A)) = c_n \int_{T(A)} d\mathcal{L}_n = c_n \int_A \sqrt{\det(M^T M)} d\mathcal{L}_n = c_n \sqrt{\det(M^T M)} \mathcal{L}_n(A) = \sqrt{\det(M^T M)} \mathcal{H}^n(A).$$

If $k < n$, since $T : \mathbb{R}^k \to \mathbb{R}^n$, the matrix M must have k columns and n rows, so the span of the columns has dimension at most k, and therefore the image $M\mathbb{R}^k$ has dimension at most k. For this reason there exists an isometry R of $\mathbb{R}^n$ (a change of coordinates composed with a translation) which maps $T(\mathbb{R}^k)$ to the canonical embedding of $\mathbb{R}^k$ in $\mathbb{R}^n$ (where the last $n-k$ components are taken to be zero). Let us call this isometry R, with

$$R\colon T(\mathbb{R}^k) \to \{y \in \mathbb{R}^n | y = \sum < y_j e_j, y_j = 0 \forall j > k\}$$

Now, to reduce to the case in which we map between the same dimensional Euclidean space, let $\Phi\colon \mathbb{R}^n \to \mathbb{R}^k$ be the orthogonal projection,

$$\Phi(\sum_i^n y_i e_i) = \sum_i^k y_i e_i.$$

Let

$$S := \Phi \circ R \circ T \colon \mathbb{R}^k \to \mathbb{R}^k.$$

Note that the action of S is given by multiplication with a matrix, and so it is an affine linear map. Indeed, each of these maps is given by matrix multiplication, so we abuse notation slightly by identifying the maps with their matrices. We can therefore apply the first case:

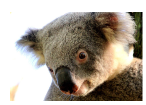
$$\mathcal{H}^k(S(A)) = \sqrt{\det(S^T S)}\mathcal{H}^k(A).$$

Then we have that

$$\sqrt{\det(S^T S)} = \sqrt{\det(\Phi R T)^T (\Phi R T)}.$$

Since R is an isometry, and Φ is projection, all that remains is

$$\sqrt{\det(T^T T)} = \sqrt{\det(M^T M)},$$

since M is the matrix giving the action of T. 

7.2. **Exercises.**

(1) Prove that if $f : X \to f(X)$ is a Lipschitz map between metric spaces then the Hausdorff dimension of $f(X)$ does not exceed that of X.

(2) Prove that if the Hausdorff dimension of X is d, and the Hausdorff dimension of Y is d', then the Hausdorff dimension of the Cartesian product $X \times Y$ is at least $d + d'$. Can it ever happen that the Hausdorff dimension of the product actually *exceeds* $d + d'$? Prove or give a counter-example.

(3) Prove that any connected set (in a metric space) which contains more than one point has Hausdorff dimension greater than or equal to one.

8. Similitudes, Hausdorff and Lebesgue measures, and Urysohn's Lemma

Let us nail down the relationship between Hausdorff and Lebesgue measures once and for all. First, let us define

$$\mathcal{H}^0(Z) = \#Z = \text{ the number of elements of the set, } Z.$$

Theorem 8.1 (Hausdorff and Lebesgue measures)**.** *For all $n \in \mathbb{N}$ we have*

$$\mathcal{H}^n = \frac{2^n}{w_n}\mathcal{L}^n,$$

where w_n is the n-dimensional volume of a unit ball in $\mathbb{R}^n$.

Proof: Let B_r be a ball of radius $r > 0$. Fix $\epsilon > 0$. Then, by the definition of the Lebesgue measure (and outer measure), there exist countably many hypercubes, denoted by R_j such that

$$B_r \subset \cup_j R_j,$$

and

$$\mathcal{L}^n(B_r) + \epsilon \geq \sum_j \mathcal{L}^n(R_j).$$

Next, fix $\delta > 0$.

Claim 1. *There exist countably many open balls $\{B_j^k\}$ which are disjoint, and satisfy*

$$\mathcal{L}^n(R_j \setminus \cup_k B_j^k) = 0.$$

Moreover, given $\delta > 0$, we may choose these balls to have diameters at most equal to δ.

The proof of the claim is an exercise! From the claim it follows that $\mathcal{L}^n(R_j) = \mathcal{L}^n(\cup B_j^k)$. Therefore we have the inequality

$$\mathcal{L}^n(B_r) + \epsilon \geq \sum_j \mathcal{L}^n(R_j) = \sum_{j,k} \mathcal{L}^n(B_j^k) = \frac{w_n}{2^n} \sum_{j,k} \text{diam}(B_j^k)^n.$$

By the absolute continuity of Lebesgue and Hausdorff measures with respect to each other,

$$\mathcal{H}^n(R_j \setminus \cup_k B_j^k) = 0 \implies \mathcal{H}_\delta^n(R_j \setminus \cup_k B_j^k = 0) \quad \forall \delta > 0.$$

This shows that

$$\mathcal{H}_\delta^n(R_j) = \mathcal{H}_\delta^n(\cup_k B_j^k),$$

and

$$\mathcal{H}_\delta^n(\cup R_j) = \mathcal{H}_\delta^n(\cup_{j,k} B_j^k).$$

Then, we also have by monotonicity, since $B_r \subset \cup_j R_j$,

$$\mathcal{H}_\delta^n(B_r) \leq \mathcal{H}_\delta^n(\cup R_j) = \mathcal{H}_\delta^n(\cup_{j,k} B_j^k).$$

Since $\cup B_j^k$ covers itself, by definition of Hausdorff measure

$$\mathcal{H}_\delta^n(\cup_{j,k} B_j^k) \leq \sum_{j,k} \text{diam}(B_j^k)^n.$$

Thus we get

$$\mathcal{H}_\delta^n(B_r) \leq \sum_{j,k} \text{diam}(B_j^k)^n \implies \frac{w_n}{2^n} \mathcal{H}_\delta^n(B_r) \leq \frac{w_n}{2^n} \sum_{j,k} \text{diam}(B_j^k)^n \leq \mathcal{L}^n(B_r) + \epsilon.$$

Letting $\delta \to 0$, we get

$$\frac{w_n}{2^n} \mathcal{H}^n(B_r) \leq \mathcal{L}^n(B_r) + \epsilon,$$

and then letting $\epsilon \to 0$, we get

$$\frac{w_n}{2^n} \mathcal{H}^n(B_r) \leq \mathcal{L}^n(B_r).$$

To complete the proof, we just need to get a lower bound for the Hausdorff measure in terms of the Lebesgue measure.
There is a nifty shortcut one can use here:

Proposition 8.2 (Isodiametric Inequality)**.** *For any $A \subset \mathbb{R}^n$, one has*

$$\mathcal{L}^n(A) \leq \frac{w_n \text{diam}(A)^n}{2^n}.$$

Exercise 10. *Locate a proof of this fact! Note that when $A = B_r$ the ball of radius r and hence diameter $2r$, the isodiametric inequality states that*

$$\mathcal{L}^n(B_r) = w_n \frac{\mathrm{diam}(B_r)^n}{2^n}.$$

Thus in that case, equality holds. This is a geometric fact which says that the ball of a specified diameter contains the largest volume amongst all sets of the same diameter. A proof can be found in Lawrence Evans & Ronald Gariepy's Measure theory and fine properties of functions, *or even earlier on p. 32 in* Littlewood's miscellany.

So, now let $\epsilon > 0$. Then, there exists a cover of B_r by $\{B_j\}$ of diameter at most δ such that

$$\mathcal{H}^n(B_r) + \epsilon \geq \sum_j \mathrm{diam}(B_j)^n.$$

Then, by the isodiametric inequality,

$$\mathrm{diam}(B_j)^n \geq \mathcal{L}^n(B_j) \frac{2^n}{w_n}.$$

So, we have

$$\mathcal{H}^n(B_r) + \epsilon \geq \frac{2^n}{w_n} \sum \mathcal{L}^n(B_j) \geq \frac{2^n}{w_n} \mathcal{L}^n(B_r),$$

where we have used in the last inequality the countable sub-additivity of the Lebesgue outer measure, since the B_j cover B_r. Since this can be done for any $\epsilon > 0$, we obtain

$$\mathcal{H}^n(B_r) \geq \frac{2^n}{w_n} \mathcal{L}^n(B_r).$$

Combining with the reverse inequality, we get

$$\mathcal{H}^n(B_r) = 2^n w_n \mathcal{L}^n(B_r).$$

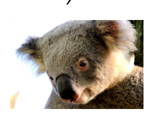
Since this holds for all balls which generate the Borel sigma algebra, it holds for all Borel sets. Then, the completion is the same in both cases, so we obtain both the equality of the Hausdorff and Lebesgue sigma algebras, as well as the equality of the Hausdorff and Lebesgue measures.

8.1. **Similitudes and Cantor sets.** Let S be a set of similitudes.

Lemma 8.3. *If $S(E) = E$, then $S^k(E) = E$ for all $k \geq 0$.*

Proof:
It is $S(E) = \bigcup_{j=1}^m S_j(E) = E$ and also

$$S^2(E) = \bigcup_{j=1}^m S_j \left(\bigcup_{j=1}^m S_j(E) \right) = \bigcup_{j=1}^m S_j(E) = E.$$

By induction we have $S^k(E) = E$ for $k \geq 2$. 

What does that mean if $S(E) = E$? Especially, in the case that $E \neq \mathbb{R}^n$ and $E \neq \emptyset$? Well, the scaling factor r is less than one, so applying each S_j spins/flips/shrinks and slides E. Hence E looks like, for each k, m^k copies of itself which are scaled down by a factor of r^k. If these copies are disjoint or have little (negligible) overlap, E is "self-similar." So, in particular, if

$$S(E) = E \implies E = S_1 E \cup S_2 E \cup \ldots \cup S_m E =: \cup_{k=1}^m E_k.$$

Each E_k is geometrically the same shape as E, it has just been shrunken by a factor of r, orthogonally translated (i.e. rotated and or reflected), and then translated (i.e. slid to be sitting in some other spot in $\mathbb{R}^n$). None of these procedures changes the *shape* of E. For this reason, when $S(E) = E$, we may call E a *self-similar* set. Moreover, we can do this again, with

$$S^2(E) = E \implies E = \cup_{i_1,i_2=1}^m S_{i_1} S_{i_2}(E) = \cup_{|J|=2} E_J,$$

where J is a multi-index of length two which each element in $\{1, 2, \ldots, m\}$. Similarly, we can write

$$E = \cup_{|J|=N} E_J,$$

for any $N \in \mathbb{N}$. Note that when we do this, there are m^N elements in the union, and each E_J is a copy of E at scale $r^{|J|}$. Let's recall a well-known example: generalized Cantor sets!

Example 8.4 (Generalized Cantor sets)**.** Let $\beta \in (0, 1)$, and $I_0 = [a, b]$ for some $a < b$. Define

$$\beta(a, b) = \left(\frac{a + b}{2} - \beta \left(\frac{b - a}{2} \right), \frac{a + b}{2} + \beta \left(\frac{b - a}{2} \right) \right).$$

Let $I_1 := I_0 \setminus \beta \mathring{I}_0$. This is closed and the union of two intervals, written $I_1 = \bigcup_{j=1}^2 I_j^1$. Then we define

$$I_2 := \bigcup_{j=1}^{2} I_j^1 \setminus \beta \mathring{I}_j^1,$$

which is a union of two disjoint unions of two closed intervals. Again we write $I_2 = \bigcup_{j=1}^4 I_j^2$. In general we write and define

$$I_k = \bigcup_{j=1}^{2^k} I_j^k \quad \text{and} \quad I_{k+1} := \bigcup_{j=1}^{2^k} I_j^k \setminus \beta \mathring{I}_j^k$$

As defined note that

$$I_0 \supset I_1 \supset \ldots \supset I_k \supset I_{k+1}$$

are a sequence of nested compact sets in $\mathbb{R}$ which is complete. Consequently,

$$\bigcap I_k = \lim_{k \to \infty} I_k =: C_\beta$$

is a compact subset of $\mathbb{R}$. Note that

$$\mathcal{L}_1(I_0) = b - a, \quad \mathcal{L}_1(I_1) = (b-a) - \beta(b-a) = (1-\beta)(b-a) = (1-\beta)\mathcal{L}_1(I_0), \quad \mathcal{L}_1(I_{k+1}) = (1-\beta)\mathcal{L}_1(I_k),$$

and so

$$\mathcal{L}_1(C_\beta) = (b - a) \lim_{k \to \infty} (1 - \beta)^k = 0,$$

since $\beta \in (0, 1)$. Note that more generally, one can let β vary at each step, so that

$$I_1 = I_0 \setminus \beta_0 I_0 = \bigcup_{j=1}^{2} I_j^k,$$

and in general

$$I_{k+1} = \bigcup_{j=1}^{2^k} I_j^k \setminus \beta_k \mathring{I}_j^k.$$

Similarly we have nested compact sets and so

$$C := \lim_{k \to \infty} I_k \text{ is a compact subset of } \mathbb{R}.$$

This is known as a *generalized Cantor set.* The Lebesgue measure

$$\mathcal{L}_1(C) = (b - a) \prod_{k \geq 0} (1 - \beta_k).$$

Hence, if $\beta_k = \beta$ is fixed and lies in the open interval $(0, 1)$, then

$$\mathcal{L}_1(C) = (b - a) \lim_{n \to \infty} (1 - \beta)^n = 0.$$

So, we also have $\mathcal{H}^1(C) = 0$. However, we shall see that the Hausdorff *dimension* of such a Cantor set is non-zero.
This is the usual way in which Cantor sets are described: by a procedure of cutting out the middle bit of each remaining interval at each step. It is perhaps not totally obvious that we can describe the Cantor set using the notion of similitudes and invariance under similitudes. However, we can indeed do this.
Now, fix $\beta \in (0, 1/2)$.

$$S := (S_1, S_2), \quad S_1(x) := \beta x, \quad S_2(x) = \beta x + (1 - \beta).$$

For the sake of simplicity, let us set $I_0 = [0, 1]$, that is take $a = 0$, $b = 1$, so that $\frac{b+a}{2} = \frac{1}{2} = \frac{b-a}{2}$. We compute

$$S(I_0) = S_1(I_0) \cup S_2(I_0) = [0, \beta] \cup [1 - \beta, 1] = I_1.$$

Similarly, we see that

$$S(I_1) = I_2 = S^2(I_0), \quad I_{k+1} = S^{k+1}(I_0).$$

So, since each S_i is continuous we have

$$S\left(\lim_{k\to\infty} S^k(I_0)\right) = S(C_\beta) = \lim_{k\to\infty} S^{k+1}(I_0) = C_\beta.$$

Consequently we see that C_β is invariant under the family of similitudes $S = (S_1, S_2)$.

8.2. **Urysohn's Lemma.** The following lemma will be required to prove our results about the dimension of iterated function system fractals as well as construct the invariant measure on said fractals.

Lemma 8.5 (Urysohn-light). *Let (X, d) be a complete metric space and $A, B \subset X$ non-empty, closed sets with $A \cap B = \emptyset$. Assume that either A and B are both compact or that A and B are at a positive distance apart. Then $\exists f \in C(X)$ s.t.*

$$f|_A = 0 \quad f|_B = 1.$$

Proof: First we know that the distance between A and B is finite because $\exists a \in A$, $b \in B \quad d(A, B) \leqslant d(a, b) < \infty$.
In the case that A and B are compact, if they were at a distance of zero, then we would have at least one sequence $\{a_n, b_n\}$ with $a_n \in A$, $b_n \in B$, and

$$\lim_{n\to\infty} d(a_n, b_n) = 0.$$

Since A is compact, the sequence $\{a_n\}$ has a convergent subsequence. Let us pass to that subsequence, but rename it $\{a_n\}$ because we may as well have started the argument with it. We then also rename the corresponding $\{b_n\}$ as well, so that we still have

$$d(a_n, b_n) \to 0.$$

Now, however, we also have

$$a_n \to a \in A.$$

Next, let us look at the sequence $\{b_n\} \subset B$. Since B is compact, there exists a subsequence of b_n which converges to some $b \in B$. Oh, the abuse of notation, as we shall still call this subsequence b_n, and the corresponding terms a_n. Then, we still have $a_n \to a \in A$, since these are a subsequence. Now, however, we also have $b_n \to b \in B$. Then, we have

$$d(a, b) \leq d(a, a_n) + d(a_n, b_n) + d(b_n, b) \to 0 \text{ as } n \to \infty.$$

Thus,

$$d(a, b) = 0 \implies a = b \implies a = b \in A \cap B,$$

which contradicts $A \cap B = \emptyset$. Thus it turns out that the first assumption, that A and B are both compact, actually implies that they are at a positive distance apart. Hence, we only need to consider the case in which A and B are closed (but not necessarily compact), and at a positive distance apart, because it covers all the possibilities. Let

$$\delta = d(A, B) > 0.$$

Let

$$U_r := \{x \in X|\ d(x, B) \geqslant (1-r)\delta\}, \quad r \in (0,1), \quad U_1 := X,$$

and

$$f(x) := \inf\{r \in (0,1]|\ x \in U_r\}.$$

Note that $f(x)$ is well defined because it's an infimum and defined $\forall x \in X$ since every $x \in U_1$. If $x \in B$, then $d(x, B) = 0$, so we shall not be able to obtain $x \in U_r$ for any $r > 0$ except for $x \in U_1$. Thus,

$$f(x) = 1 \quad \forall x \in B.$$

If $x \in A$, then $d(x, B) \geq d(A, B) = \delta$. This shows that for every $r > 0$, we have $d(x, B) \geq \delta \geq (1-r)\delta$. So, $x \in U_r$ for all $r \in (0, 1]$, which shows that

$$f(x) = 0.$$

Since $x \in A$ was arbitrary, we get

$$f(x) = 0 \quad \forall x \in A.$$

The last thing to show is the continuity of the function f. Let $x \in X$, and $x_n \to x$. This is equivalent to saying that $d(x_n, x) \to 0$ as $n \to \infty$. Then, for any $b \in B$ we have

$$d(x, b) \leq d(x, x_n) + d(x_n, b).$$

Exercise 11. *Show that f is continuous.*

Now with our teamwork, the proof is done!
We require Urysohn's Lemma (at least on metric spaces; it holds in the more general setting of a normal topological space under the assumption that the sets are closed and disjoint) to prove one of Riesz's Representation Theorems. For this, we recall the definition of the dual space for the continuous functions with compact support.

Definition 8.6. Let X be a Banach space. Let $\mathcal{C}_c(X)$ denote the set of functions from $X \to \mathbb{R}$ which are continuous and compactly supported. Compactly supported means that there exists a compact set $K \subset X$ such that

$$f(x) = 0 \forall x \notin K.$$

The *dual* of $\mathcal{C}_c(X)$ is the set of all bounded, (and thus continuous) linear functions from $\mathcal{C}(X)$ to $\mathbb{R}$. This dual space is denoted by $\mathcal{C}_c(X)'$. In particular $L \in \mathcal{C}_c(X)')$ if and only if L satisfies:

$$L(af + bg) = aL(f) + bL(g), \quad \forall a, b \in \mathbb{R}, \quad f, g \in \mathcal{C}_c(X),$$

and

$$|L(f)| \leq ||L||||f||_\infty, \quad \forall f \in \mathcal{C}_c(X),$$

for a finite, fixed constant $||L||$. Above $||f||_\infty$ is the $\mathcal{L}^\infty$ or supremum norm of f, which is defined by

$$||f||_\infty = \sup_{x \in X} |f(x)|.$$

Theorem 8.7 (Riesz Representation for $C_c(X)'$)**.** *If $0 \leq L \in C_c(X)' \ \Rightarrow \exists$ measure μ on X s.t.*

$$I(f) = \int_X f \ d\mu$$

and Borel sets are μ measurable. Here by $L \geq 0$ we mean that for all functions $f \geq 0$ we have $L(f) \geq 0$.

8.3. **Exercises.**

(1) Determine the Hausdorff measure and Hausdorff dimension of the standard Cantor set.
(2) Compute the Hausdorff dimension of a generalized Cantor set.
(3) Compute the Hausdorff dimension of the product of two Cantor sets.
(4) Show that every subset of $\mathbb{R}^n$ is measurable with respect to the 0-dimensional Hausdorff outer measure.
(5) Construct a subset of $\mathbb{R}$ which has Hausdorff dimension one but has zero Lebesgue measure. Note that since this will imply the set also has $\mathcal{H}^1$ measure zero, yet has Hausdorff dimension equal to one, so it will be an example as we discussed in class.

8.4. **Hints: continuity in Urysohn's Lemma.** Let $x \in X$ with $f(x) = r$. First, consider when $f(x) = 1$. This means that $x \in U_1$ but not in U_r for any $r < 1$. Consequently, $d(x, B) = 0$. (Why?) Therefore, since B is closed, $x \in B$. Assume that $x_n \to x$. Then, $d(x_n, B) \to 0$ as $n \to \infty$. Consequently, letting $r_n = f(x_n)$, we must have $r_n \to 1$ as $n \to \infty$. Thus $f(x_n) \to f(x)$.
Next, consider when $f(x) < 1$. Write $r = f(x)$. Then, for $r < r' < 1$, we have $x \in U_{r'}$ so that $d(x, B) \geq (1 - r')\delta$. If $x_n \to x$, then $d(x_n, x) \to 0$. Then, for any $b \in B$, we have

$$d(x, b) - d(x, x_n) \leq d(x_n, b).$$

Since for all $b \in B$,

$$d(x, B) \leq d(x, b)$$

we have

$$d(x, B) - d(x, x_n) \leq d(x, b) - d(x, x_n) \leq d(x_n, b).$$

Taking the infimum now over all b on the right side, we get

$$d(x, B) - d(x, x_n) \leq d(x_n, B).$$

Similarly,

$$d(x_n, b) - d(x_n, x) \leq d(x, b).$$

So, taking the inf over all $b \in B$ on the left (but not on the right), we first get that for any particular $b \in B$,

$$d(x_n, B) - d(x_n, x) \leq d(x, b).$$

Next, taking the infimum over all b on the right we get

$$d(x_n, B) - d(x_n, x) \leq d(x, B).$$

Thus,

$$d(x, B) \leq d(x_n, B) + d(x, x_n), \quad d(x_n, B) \leq d(x, B) + d(x, x_n) \implies |d(x_n, B) - d(x, B)| \leq d(x, x_n).$$

Since $d(x, x_n) \to 0$ as $n \to \infty$, we get that $d(x_n, B) \to d(x, B)$ as $n \to \infty$. It follows that $f(x_n) \to f(x)$.

9. Similitudes and Iterated Function System Fractals

We shall begin by determining a sufficient condition to guarantee that a set of similitudes has an associated non-empty, compact, invariant set. When such an invariant set exists, it is also unique. It is known in this contact as an *iterated function system fractal.*

Proposition 9.1. *Let S be a family of similarities with common scaling factor $r \in (0, 1)$. If there exists U open, non-empty and bounded such that $S(U) \subset U$, then S is said to satisfy the* open set condition. *(OSC) Equivalently, one may say that S admits a* separating set. *When this is the case, then there exists a unique $X \subset\subset \mathbb{R}^n$ such that $S(X) = X \neq \emptyset$. More generally, if there exists $X \subset\subset \mathbb{R}^n$ such that $S(X) = X$, $X \neq \emptyset$, then the set, X, is unique.*

Proof: First we note that since U is non-empty and bounded, then $\bar{U}$ is closed and bounded inside $\mathbb{R}^n$ and therefore compact. Moreover, we note that each similitude is an affine linear function from $\mathbb{R}^n \to \mathbb{R}^n$, and is therefore continuous. Consequently, the image of $\bar{U}$ under each S_i is compact. We also obtain by continuity that

$$S_i(\bar{U}) = \overline{S_i(U)}, \quad i = 1, \ldots, m, \quad S(\bar{U}) = \overline{S(U)}.$$

Consequently, since

$$S(U) \subset U \implies \overline{S(U)} = S(\bar{U}) \subset \bar{U}.$$

Therefore, the sets

$$S^k(\bar{U}), \quad S^{k+1}(\bar{U}) \subset S^k(\bar{U}), \quad k \geq 0.$$

Moreover, these are each compact and non-empty since U is nonempty and open, which guarantees that $S_i(U)$ is non-empty and open for each i since S_i is an affine linear transformation. Thus $S(\bar{U}) \supset S(U)$ is also nonempty, and repeating the argument, since $S^k(U)$ is non-empty and open for each $k \geq 1$, we have that

$$S^k(\bar{U})$$

is non-empty and compact for each k. It therefore follows that

$$X = \cap_{k \geq 0} S^k(\bar{U}) \neq \emptyset$$

and is compact. Then, since the $S^k(\bar{U})$ are nested, we have

$$X = \lim_{k \to \infty} S^k(\bar{U}).$$

Since all the similitudes are continuous, we also have

$$\lim_{k \to \infty} S(S^k(\bar{U})) = S\left(\lim_{k \to \infty} S^k(\bar{U})\right) = S(X).$$

On the other hand, it is always true that

$$\lim_{k \to \infty} S(S^k(\bar{U})) = \lim_{k \to \infty} S^{k+1}(\bar{U}) = \lim_{k \to \infty} S^k(\bar{U}) = X.$$

Thus X is compact, non-empty, and invariant under S.
Now let us show that X is the unique compact set which has this property. So, if $Y \neq \emptyset$ is compact, and $S(Y) = Y$, we wish to show that $Y = X$. For this purpose we define

$$D(Y, X) := \sup_{y \in Y} d(y, X) = \sup_{y \in Y} \inf_{x \in X} d(y, x).$$

Similarly, we define

$$D(S_i(Y), S_i(X)) = \sup_{y \in Y} d(S_i(y), S_i(X)) = \sup_{y \in Y} \inf_{x \in X} d(S_i(y), S_i(x)).$$

Now, recalling that $S_i(y) = rO_i(y) + b_i$, we have that

$$D(S_i(Y), S_i(X)) = \sup_{y \in Y} \inf_{x \in X} d(rO_i(y), rO_i(x)) = \sup_{y \in Y} \inf_{x \in X} rd(y, x),$$

since O_i is an orthogonal transformation, and so it does not change the distance between points, and r is simply the scaling factor. So, in fact we see that

$$D(S_i(Y), S_i(X)) = rD(Y, X).$$

Now we shall use the invariance of Y to eventually reach a contradiction. By the invariance of Y,

$$Y = \cup_{i=1}^m S_i(Y) = S(Y),$$

so we have

$$D(Y, X) = \max_{1 \leq i \leq m} D(S_i Y, X) = D(S_j Y, X),$$

for some specific j (or perhaps it is achieved by more than one j, we do not care). Now, for fixed

$$y \in Y,\ d(S_j y, X) = \inf_{x \in X, 1 \leq k \leq m} d(S_j y, S_k x) \leq \inf_{x \in X} d(S_j y, S_j x) = d(S_j y, S_j X).$$

Here we use that $X = S(X) = \cup S_k X$. Taking the supremum over $y \in Y$, we have

$$d(S_j Y, X) \leq d(S_j Y, S_j X) = r d(Y, X).$$

Since $r < 1$ this is only possible if

$$d(Y, X) = 0 \Rightarrow \sup_{y \in Y} d(y, X) = 0 \implies y \in X \quad \forall y \in Y.$$

The last statement above follows because X is a compact, and therefore closed, set. Consequently, we see that $Y \subset X$. We can repeat the exact same argument, swapping places with X and Y, and we obtain that $X \subset Y$. Hence they are equal. 

Theorem 9.2 (Riesz Representation for $C_c(X)'$)**.** *If $0 \leq L \in C_c(X)' \;\; \Rightarrow \exists$ a unique measure μ on X s.t.*

$$I(f) = \int_X f \; d\mu$$

and Borel sets are μ measurable. Here by $L \geq 0$ we mean that for all functions $f \geq 0$ we have $L(f) \geq 0$. For the sake of simplicity, we may take $X = \mathbb{R}^n$.

Proof: Write $f \prec U$ if U is open, $f \in C_c(X)$, $0 \leqslant f \leqslant 1$, and $\text{supp}(f) \subset U$. Recall that the support of a function is

$$\overline{\{x \in X : f(x) \neq 0\}}.$$

We shall define

$$\mu(\emptyset) = 0,$$

and for U open,

$$\mu(U) := \sup\{L(f) | \; f \prec U\}.$$

Since $I \geq 0$, $\mu(U) \geq 0$. Note that if U is open and non-empty, then there exists a point $p \in U$ and a ball $B_r(p)$ such that the closure of $B_{2r}(p)$ is contained in U. Let B be the closure of $B_r(p)$. Let

$$A = X \setminus B_{2r}(p).$$

First, we note that since $\overline{B_{2r}(p)} \subset U$, we have $A \supset U^c$. Next, let $q \in B$ and $x \in A$. Then, so defined

$$d(q, p) \leq r, \quad d(x, p) \geq 2r \implies d(q, x) \geq d(x, p) - d(p, q) = r.$$

Thus, we see that B and A are at a positive distance apart. By Urysohn's Lemma and its proof, there is a function f which is 0 on A and 1 on B and takes values between 0 and 1 in general. Since f must vanish identically on A, the support of f is contained in the closure of $B_{2r}(p)$ which is contained in U. Here is where it is convenient to take $X = \mathbb{R}^n$, because this implies that closed balls are compact, so the support of f, being a closed subset of $\overline{B_{2r}(p)}$ is also compact.

Therefore, μ is well defined for all open sets. Now we use it to make an outer measure. Let

$$\mu^*(E) = \inf\{\mu(U) \mid E \subset U, \quad U \text{ open } \}.$$

So defined, this vanishes on the empty set. Moreover, if $A \subset B$, then every U which covers B also covers A, so we obtain

$$\mu^*(A) \leq \mu^*(B).$$

Now note that if $U \subset V$ are two open sets, then it is more restrictive to require $f \prec U$ as compared with requiring $f \prec V$. Thus, the supremum taken for V can include more elements, so we have

$$\mu(U) \leq \mu(V).$$

By similar considerations, we also see that $\mu^*(U) = \mu(U)$ if U is open. Now, let $\{U_j\}$ be open sets. Define

$$U = \cup U_j,$$

and note that this set is also open. If $f \prec U$, let $K = \text{supp}(f)$. Since K is compact, by definition of compactness and since it is covered by the U_j (open cover), we have a finite open cover,

$$\cup_{j=1}^n U_j \supset K.$$

Exercise 12. *Show that you can find a so-called "partition of unity" that is $\{\phi_j\}_{j=1}^n$ non-negative functions, which have $\phi_j \prec U_j$ and $\sum_{j=1}^n \phi_j = 1$ on K. Hint: This is done in Folland's Real Analysis, Prop. 4.41.*

So, since f is supported on K, we have that $\sum_{j=1}^n \phi_j f = f$ on K, and it is also true off K because f is zero over there. So, we have that always $f = \sum_1^n \phi_j f$, and moreover, $f\phi_j \prec U_j$. So, by the linearity of linear functionals,

$$L(f) = L(\sum_1^n f\phi_j) = \sum_1^n L(f\phi_j) \leq \sum_1^n \mu(U_j) \leq \sum_1^\infty \mu(U_j).$$

This holds for all $f \prec U$, so we obtain

$$\mu(U) \leq \sum_1^\infty \mu(U_j).$$

Consequently, countable sub-additivity holds for all open sets. Now, if we have some other sets, with

$$E = \cup_j E_j,$$

if $\sum \mu^*(E_j) = \infty$, then we of course get

$$\mu^*(E) \leq \sum \mu^*(E_j).$$

So, assume all these guys on the right are finite. Let $\epsilon > 0$. Then, for each j there is an open set $U_j \supset E_j$ with

$$\mu^*(E_j) + \frac{\varepsilon}{2^j} \geq \mu(U_j).$$

Then, we also have

$$E \subset \cup_j U_j = U.$$

By the countable subadditivity for open sets (and note that U is open) we have

$$\mu(U) \leq \sum_j \mu(U_j).$$

So,

$$\mu(U) \leq \sum \mu(U_j) \leq \sum \mu^*(E_j) + \frac{\varepsilon}{2^j} = \varepsilon + \sum \mu^*(E_j).$$

Now we take the infimum over all open sets U on the left. This is a bit subtle, so let us write it out

$$\inf\{\mu(U) : E \subset U = \cup_j U_j, \quad E_j \subset U_j \text{ open}\} \leq \varepsilon + \sum \mu^*(E_j).$$

When we now take the infimum on the left of $\mu(U)$ for *all* open covers of E, it could possibly be smaller, so we get

$$\mu^*(E) \leq \varepsilon + \sum \mu^*(E_j).$$

Letting $\varepsilon \to 0$ we obtain countable sub-additivity for all sets. Next, we shall show that this is a metric outer measure.

If $d(A, B) > 0$, then by its definition

$$\mu^*(A \cup B) = \inf\{\mu(U) | A \cup B \subset U, \ U \text{ is open}\}.$$

Observe that if $f \prec V$, and $A \subset V$, and $g \prec W$ with $B \subset W$, and $V \cap W = \emptyset$, then $U = V \cup W \supset A \cup B$. Moreover, $f + g \prec U$. By the linearity of linear functionals

$$L(f + g) = L(f) + L(g).$$

Consequently,

$$\mu(U) \geq \sup\{L(f+g) = L(f) + L(g) | f \prec V \supset A, g \prec W \supset B, V \cap W = \emptyset\}.$$

Now, take the supremum of L over such f and g for fixed V and W. Since the V and W are disjoint, and we are supremuming over non-negative elements, the supremum of the sum is the sum of the suprema, so we have

$$\sup\{L(f+g) = L(f) + L(g) | f \prec V \supset A, g \prec W \supset B, V \cap W = \emptyset\} = \mu(V) + \mu(W).$$

Basically, the f and g above are independent of each other, so one simply maximizes for f and for g independently, which is why the supremum is equal to the sum of the suprema.
Next, we take the infimum over V and W which contain A and B respectively,

$$\begin{aligned} \mu(U) &\geq \inf\{\mu(V) + \mu(W) | V \text{ open, } A \subset V, W \text{ open } B \subset W, V \cap W = \emptyset\} \\ &\geq \inf\{\mu(V) | V \text{ open, } A \subset V\} + \inf\{\mu(W) | W \text{ open, } B \subset W\} \\ &= \mu^*(A) + \mu^*(B). \end{aligned}$$

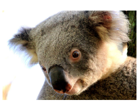
Next taking the infimum over U, which is open and contains $A \cup B$ to obtain

$$\mu^*(A \cup B) \geqslant \mu^*(A) + \mu^*(B) \geqslant \mu^*(A \cup B).$$

The right side followed from countable (and thus also finite) subadditivity which we already established. So, we conclude that μ^* is a metric outer measure.

Exercise 13. *Show that $L(f) = \int f d\mu$ for all $f \in \mathcal{C}_c(X)$. This is mostly aimed towards those who have taken integration theory already!*

Exercise 14. *Show that the measure obtained in this way is unique.*



Definition 9.3. For $x \in \mathbb{R}^n$, $E \subset \mathbb{R}^n$, a measure μ, $\{i_1, ..., i_k\} \subset \{1, ..., m\}$ we define

(1) $x_{i_1..i_k} := S_{i_1} \circ ... \circ S_{i_k}(x)$,
(2) $E_{i_1..i_k} := S_{i_1} \circ ... \circ S_{i_k}(E)$, and
(3) $\mu_{i_1..i_k} := \mu((S_{i_1} \circ ... \circ S_{i_k})^{-1}(E)$.

Theorem 9.4 (The invariant measure for an IFS fractal)**.** *Assume that $S = (S_1, ..., S_m)$ is a family of similitudes with common scaling factor $r \in (0,1)$, $X \subset\subset \mathbb{R}^n$, $X \neq \emptyset$, and $S(X) = X$. Then there exists a (non-negative) Borel measure μ on $\mathbb{R}^n$ such that $\mu(\mathbb{R}^n) = 1$, $\operatorname{supp}(\mu) = X$, and*

$$\forall k \in \mathbb{N}, \quad \mu = \frac{1}{m^k} \sum_{i_1..i_k=1}^{m} \mu_{i_1..i_k}.$$

Here we mean by the statement that $\operatorname{supp}(\mu) = X$ that for any $A \subset \mathbb{R}^n$ which is μ measurable, then

$$\mu(A) > 0 \iff A \cap X \neq \emptyset.$$

Proof: We will construct μ on X and extend it to $\mathbb{R}^n \setminus X$ to be identically zero. In this way the last statement will automatically hold true (although we shall also double-check for good measure). Let $x \in X$, and define

$$\delta_x(E) := \begin{cases} 1, & x \in E \\ 0, & x \notin E \end{cases}.$$

For $\{E_j\}_{j \geq 1}$ disjoint then either there exists i, j such that

$$x \in E_j \Rightarrow \delta_x(\underset{j \geq 1}{\cup} E_j) = 1 = \sum_{j \geq 1} \delta_x(E_j),$$

or not; in which case

$$\delta_x(\underset{j\geq 1}{\cup} E_j) = 0 = \sum_{j\geq 1} \delta_x(E_j).$$

Consequently, we have for any $A, B \subset \mathbb{R}^n$, $\delta_x(A) = \delta_x(A \cap B) + \delta_x(A \setminus B)$. This shows that every set in $\mathbb{R}^n$ is measurable for δ_x.
We define

$$\mu^k := \frac{1}{m^k} \sum_{i_1..i_k=1}^{m} [\delta_x]_{i_1..i_k}.$$

Then note that

$$[\delta_x]_{i_1..i_k}(E) = \delta_x(S_{i_1} \circ ... \circ S_{i_k}(E)) = \begin{cases} 1, & x \in (S_{i_1} \circ ... \circ S_{i_k})^{-1}(E) \Leftrightarrow S_{i_1} \circ ... \circ S_{i_k}(x) \in E \\ 0, & otherwise \end{cases}$$

The idea is that we want to show that letting $k \to \infty$, $\lim_{k\to\infty} \mu^k$ defines a bounded, linear functional, that is an element of $\mathcal{C}_c(\mathbb{R}^n)$. Then, we will use the Riesz representation theorem to obtain a measure. Finally, we will show that the measure which we obtain in this way satisfies the desired properties.
So, to begin, since bounded linear functionals are defined through their action on continuous functions with compact support, let f be such a function. Then, by definition

$$\int_{\mathbb{R}^n} f d\mu^k = \frac{1}{m^k} \sum_{i_1..i_k=1}^{m} f(x_{i_1..i_k}),$$

and

$$\mu^k(\mathbb{R}^n) = \frac{1}{m^k} \sum_{i_1..i_k=1}^{m} 1 = \frac{m^k}{m^k} = 1.$$

To Complete the Proof:

(1) Show that for any continuous f,

$$\{\int_{\mathbb{R}^n} f d\mu^k\}_{k\geq 1}$$

is a Cauchy sequence. Consequently we can conclude that it converges to a well-defined limit for each f. Call the limit

$$L(f) := \lim_{k\to\infty} \int_{\mathbb{R}^n} f d\mu^k.$$

(2) Observe that linearity is inherited by L from the linearity of the integral. Moreover, by definition

$$\left| \int_{\mathbb{R}^n} f d\mu^k \right| \leq ||f||_\infty \int_{\mathbb{R}^n} d\mu^k = ||f||_\infty.$$

Here we have used that

$$\int_{\mathbb{R}^n} d\mu^k = 1 \forall k.$$

(3) Show that the support of μ is precisely X.
(4) Show that μ enjoys the invariance property given in the theorem.

We shall complete items 1, 3, and 4 next time!

9.1. **Homework.**

Exercise 15. *Compute the Hausdorff dimension of the Koch snowflake curve.*

Exercise 16. *Let* Δ *be the closed triangular region in* $\mathbb{R}^2$ *with vertices* $(0,0)$, $(1,0)$, *and* $(\frac{1}{2},1)$. *Let*

$$S_j(\mathbf{x}) = \frac{1}{2}\mathbf{x} + b_j, \quad b_1 = (0,0), \quad b_2 = \left(\frac{1}{2},0\right), \quad b_3 = \left(\frac{1}{4},\frac{1}{2}\right).$$

Prove that $\cap_{k=0}^{\infty} S^k(\Delta)$ *is the unique compact non-empty invariant set under* $S = (S_1, S_2, S_3)$. *Determine its Hausdorff dimension. What is the name of this set?*

We shall be entering the realm of complex analysis and complex dynamics, that is the iteration of holomorphic (and meromorphic) functions. With this in mind, the exercises are now intended to refresh your memory of basic, fundamental facts in complex analysis.

(1) Prove that f is holomorphic on $D_r(z_0) \Leftrightarrow f$ is $\mathbb{R}^2$ differentiable and $u = \Re(f)$, $v = \Im(f)$ satisfy $u_x = v_y$ and $u_y = -v_x$. These are the Cauchy-Riemann equations. $(\Leftrightarrow \bar{\partial} f) = 0$.
(2) Prove that $f(z) = z$ and $f(z) \equiv c$ are holomorphic as in $\mathbb{R}$. Prove that f, g holomorphic $\Rightarrow fg, f+g, f/g (g \neq 0)$ also just as in $\mathbb{R}$.
(3) Not like in $\mathbb{R}$: Given $f\colon \mathbb{R} \to \mathbb{R}$ continuous. $\exists F\colon \mathbb{R} \to \mathbb{R}$ such that $F' = f$? Yes. $F(x) = \int_a^x f(t)dt$.
This is not necessarily true in $\mathbb{C}$. Give a counterexample.
(4) Prove that if $f\colon \Omega \to \mathbb{C}$, where Ω is a domain, is continuous, and if $\exists F\colon \Omega \to \mathbb{C}$ such that $F' = f$ then $\int_\gamma f(z)dz = 0 \forall$ closed curve $\gamma \subset \Omega$.
(5) Prove Goursat's theorem: if f is holomorphic on Ω, then $\int_{\partial T} f = 0 \,\forall$ triangle $T \subset\subset \Omega$ where $\mathring{T} \subset \Omega$.
(6) Recall that a domain is called *star-shaped* if there exists a point in the domain such that the line segment connecting this point and any other point of the domain lies entirely within the domain. This really looks like a star. Examples include all convex domains. Prove that if Ω is star-shaped, f holomorphic, f has primitive $F(z) = \int_a^z f$, and $\int_\gamma f = 0 \,\forall$ closed γ.
(7) Prove that if f holomorphic on $G \setminus z_0$ and continuous on G, we also get $\int_\gamma f = 0 \,\forall \gamma$ with $\gamma \cup \mathring{\gamma} \subset\subset G$.
(8) Prove that the converse is also true: If $\int\limits_{\partial T} f = 0 \,\forall T$ satisfying the hypothesis, then f is homolomorphic on G.
(9) Prove that if f is holomorphic on $T \setminus z$, where z denotes a point, then $\int\limits_{\partial T} = 0$.
(10) Prove the Cauchy Integral Formula: Let f be holomorphic on $D = D_r(z_0) \ni z$. Then

$$f(z) = \frac{1}{2\pi\imath} \int\limits_{\partial D} \frac{f(w)}{w-z} dw$$

9.2. **Hints.**

(1) Assume that f is holomorphic. Near z, $f(w) = f(z) + (w-z)A_z(w)$. For the coordinates $(z, \bar{z}) \in \mathbb{C} \cong \mathbb{R}^2$, we get $x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2\imath}$ and $\frac{\partial f}{\partial \bar{z}} = 0$. Therefore we get

$$\begin{aligned}\frac{\partial f}{\partial \bar{z}} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \bar{z}} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}f_x - \frac{1}{2\imath}f_y = \frac{1}{2}(f_x - \imath f_y) \\ &= \frac{1}{2}(u_x + \imath v_x + \imath(u_y + \imath v_y)) = \frac{1}{2}(u_x - v_y + \imath(v_x + u_y)) = 0\end{aligned}$$

$\Leftrightarrow u_x = v_y$ and $u_y = -v_x$.
On the other hand, assume f is $\mathbb{R}^2$ differentiable and

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z}.$$

Since $\bar\partial f = \frac{\partial f}{\partial \bar z} = 0$, we have near z_0:

$$f(z) = f(z_0) + M \cdot \begin{bmatrix} z - z_0 \\ \overline{z - z_0} \end{bmatrix} + B(z)$$

where $\lim\limits_{z\to z_0} \left\| \frac{B(z)}{z-z_0} \right\| \to 0$.

Since $\bar\partial f = 0 \Rightarrow M = \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix}$ and therefore

$$f(z) = f(z_0) + (z - z_0)\left((a+b) + \frac{B(z)}{z - z_0}\right)$$

$$= f(z_0) + (z - z_0)A(z), \quad A(z) = (a+b) + \frac{B(z)}{z - z_0},$$

and $A(z)$ is continuous because $\frac{B(z)}{z-z_0} \to 0$ as $z \to z_0$. Consequently

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = a + b$$

exists.

(2) $f(z) = \frac{1}{z}$ has no primitive since

$$\int_{\partial D_r} f(z)dz = \int_0^{2\pi} f(\gamma(t))\gamma'(t)dt = \int_0^{2\pi} \frac{1}{re^{it}} rie^{it} = 2\pi i \neq 0$$

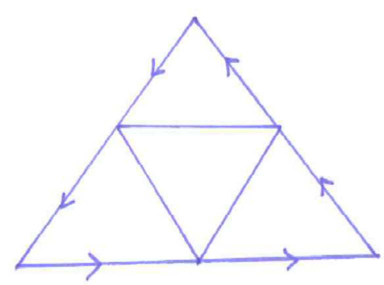


Figure 1.

Goursat First, we split the triangle into four triangles by joining the midpoints of each of the sides of T. Then integration along the interior edges cancel and so

$$|\int_{\partial T} f| \leq \sum_{i=1}^{4} |\int_{\partial T_i^1} f| \leq 4 \max_{1\leq i\leq 4} |\int_{\partial T_i^1} f|$$

We define T_1 to be any T_i^1 such that the integral achieves the maximum. We repeat this process with T_1, defining T_i^2 for $i = 1,2,3,4$, such that the integral over the boundary of T_1 is equal to the sum of the integrals over the boundaries of the T_i^2. The triangle whose integral is maximal is defined as T_2. This triangle is again split into four, and so forth, defining a nested sequence of triangles

$$T \supset T_1 \supset T_2 \supset \dots$$

Note that the length of the boundary $|\partial T_1| = \frac{1}{2}|\partial T|$ and therefore $|\partial T_k| = 2^{-k}|\partial T|$. Furthermore, we have $\text{diam}(T_1) = \frac{1}{2}\text{diam}(T)$ and therefore $\text{diam}(T_k) = 2^{-k}\text{diam}(T)$.

Since the triangles are compact and nested, and their diameters converge to zero, the intersection

$$\bigcap T_k = \{z_0\} = \lim_{k\to\infty} T_k.$$

Since f is holomorphic at z_0 which is in the interior of Ω,

$$\begin{aligned} f(z) &= f(z_0) + (z-z_0)f'(z_0) + (z-z_0)(A(z) - A(z_0)) \\ &= f(z_0) + (z-z_0)(A(z)). \end{aligned}$$

Note that $B(z) := A(z) - A(z_0)$ is continuous at z_0 because A is, and that $B(z_0) = 0$. Since the function

$$f(z_0) + (z-z_0)f'(z_0)$$

has a primitive, namely

$$F(z) = z(f(z_0) - z_0 f'(z_0)) + \frac{z^2}{2}f'(z_0) \implies F'(z) = f(z_0) + (z-z_0)f'(z_0),$$

the integral

$$\int_{\partial T} (f(z_0) + (z-z_0)f'(z_0))dz = 0, \quad \int_{\partial T_k} (f(z_0) + (z-z_0)f'(z_0))dz = 0, \, \forall\, k.$$

Consequently by linearity of the integral

$$\int_{\partial T_k} f(z)dz = \int_{\partial T_k} (z-z_0)B(z)dz$$

$$\Rightarrow \left| \int_{\partial T_k} f(z)dz \right| \le |\partial T_k| \max_{\partial T_k} |z-z_0||B(z)| \le |\partial T_k| \mathrm{diam}(T_k) \max_{\partial T_k} |B(z)| = 2^{-k}\mathrm{diam}(T) \max_{\partial T_k|} |B(z)| \cdot 2^{-k}|\partial T|$$

$$\Rightarrow \left| \int_{\partial T} f(z)dz \right| \le 4^k \cdot 4^{-k}\mathrm{diam}(T)|\partial T| \max_{\partial T_k} |B(z)|$$

Since $T_k \to z_0$ and $B(z) \to B(z_0) = 0$ as $z \to z_0$, it follows that the maximum over ∂T_k of $|B(z)|$ tends to 0 as $k \to \infty$. Consequently the integral on the left above must vanish. 

(3) For the proof of the Cauchy integral formula, let

$$g(w) := \begin{cases} \frac{f(w)-f(z)}{w-z} & w \neq z \\ f'(z) & w = z \end{cases}$$

Then g is holomorphic on $D \setminus z$ and it is continous at z.
Therefore since D is convex and hence star-shaped

$$\begin{aligned} &\int_{\partial D} g(w)dw = 0 \\ \Rightarrow &\int_{\partial D} \frac{f(w)}{w-z}dw = \int_{\partial D} \frac{f(z)}{w-z}dw = f(z)\int_{\partial D}\frac{dw}{w-z} \end{aligned}$$

Compute $\int\limits_{\partial D} \frac{dw}{w-z_0}dw = 2\pi i$ and prove that the function $h(z) := \int\limits_{\partial D} \frac{dw}{w-z}$ is constant on D.

9.3. **Hints: the integral in Riesz's representation.** To show $L(f) = \int f\ d\mu\ \forall f \in C_c(X)$, we first show

$$\mu(K) = \inf\{L(f)|\ f \in C_c(X),\ f \geqslant \chi_K\} \quad \forall K \Subset X.$$

(<u>Note:</u> $\int \chi_K\ d\mu = \mu(K)$ by def.)

Let $U_\varepsilon := \{x|\ f(x) > 1-\varepsilon\}$ for such an $f \in C_c(X)$, $f \geqslant \chi_K$. U_ε is open.
If $g \prec U_\varepsilon\ \Rightarrow (1-\varepsilon)^{-1}f - g \geqslant 0\ \Rightarrow I\left((1-\varepsilon)^{-1}f - g\right) \geqslant 0$
$\Rightarrow (1-\varepsilon)^{-1}I(f) \geqslant I(g)$

$\Rightarrow \mu(K) \underset{K\subset U_\varepsilon}{\leqslant} \mu(U_\varepsilon) \underset{\inf\ over\ g}{\leqslant} (1-\varepsilon)^{-1} I(f)$

$\overset{\varepsilon\downarrow 0}{\Rightarrow} \mu(K) \leqslant I(f)$

On the other hand for U open with $U \supset K$, by Urysohn

$$\exists f \in C_c(X) \text{ s.t. } f \geqslant \chi_K \text{ and } f \prec U$$

$\Rightarrow L(f) \leqslant \mu(U)$ (by def. of μ).

$\mu(K) = \inf\{\mu(U) |\ U \supset K, U \text{ open}\}$

$\Rightarrow \mu(K) \leqslant L(f) \leqslant \mu(U) \quad \forall U \text{ open } U \supset K$

$\overset{\inf\ on\ RHS}{\Rightarrow} \mu(K) \leqslant L(f) \leqslant \mu(K)$

$\Rightarrow \mu(K) = \inf\{L(f) |\ f \in C_c(X),\ f \geqslant \chi_K\} \quad \forall K \subset X.$

It is therefore enough to show

$$L(f) = \int f\ d\mu \text{ for } f \in C_c(X, [0,1))$$

since C_c is the linear span of such f, and both L and the integral $\int\ d\mu$ are linear functionals on C_c.

For $N \in \mathbb{N}$, $1 \leqslant j \leqslant N$ let $K_j := \{x |\ f(x) \geqslant \frac{j}{N}\}$ and $K_0 := supp(f)$.
Then note that

$$K_0 \supset K_1 \supset K_2 \supset \ldots.$$

Define

$$f_j(x) := \begin{cases} 0 & \text{if } x \notin K_{j-1} \\ f(x) - \frac{(j-1)}{N} & \text{if } x \in K_{j-1} \setminus K_j \\ \frac{1}{N} & \text{if } x \in K_j \end{cases}$$

So defined, f_j vanishes on K_{j-1}^c, and on K_j, $f_j = \frac{1}{N}$, whereas on $K_{j-1} \setminus K_j$, since

$$\frac{j-1}{N} \leq f < \frac{j}{N} \implies 0 < f_j < 1/N.$$

Consequently,

$$N^{-1}\chi_{K_j} \leqslant f_j \leqslant N^{-1}\chi_{K_{j-1}} \tag{9.1}$$

$$\Rightarrow \frac{1}{N}\mu(K_j) \leqslant \int f_j\ d\mu \leqslant \frac{1}{N}\mu(K_{j-1}). \tag{9.2}$$

If U is open and $U \supset K_{j-1}$, then

$$N f_j \prec U,$$

because the support of f_j is K_{j-1} which is compactly contained in U. Therefore, by the definition of $\mu(U)$ as the supremum over all such f_j, we have

$$L(f_j) \leqslant N^{-1}\mu(U).$$

Now since for a compact set (which we note K_j is) we showed that $\mu(K_j)$ is the infimum over $L(f)$ for all $f \in C_c$ with $f \geq \chi_{K_j}$, by (9.1)

$$\frac{1}{N}\mu(K_j) \leq I(f_j) \leq N^{-1}\mu(U).$$

Taking the infimum over all open U which contain K_{j-1} as in the definition of μ we then have

$$\frac{1}{N}\mu(K_j) \leq L(f_j) \leq \frac{1}{N}\mu(K_{j-1}). \tag{9.3}$$

Note that so defined

$$f = \sum_{j=1}^{N} f_j,$$

so summing over (9.2) by linearity of the integral,

$$\Rightarrow \frac{1}{N}\sum_{j=1}^{N}\mu(K_j) \leqslant \sum_{j=1}^{N} L(f_j) \leqslant \frac{1}{N}\sum_{j=0}^{N-1}\mu(K_j).$$

Next we sum over (9.3) using the linearity of the functional I,

$$\frac{1}{N}\sum_{j=1}^{N}\mu(K_j) \leqslant \int f \; d\mu \leqslant \frac{1}{N}\sum_{j=0}^{N-1}\mu(K_j).$$

Finally, we subtract these inequalities which leaves only the first and last terms, and so

$$\Rightarrow |L(f) - \int f \; d\mu| \leqslant \frac{\mu(K_0)-\mu(K_N)}{N} \leq \frac{\mu(supp(f))}{N} \to 0, \text{ as } N \to \infty.$$

Note that the measure of the support of f is finite because the support is compact, and for compact sets, $\mu(K)$ is defined as the infimum of $L(f)$, and L is a linear functional (which implies L is continuous and hence has bounded norm). Therefore we have $L(f) = \int f \; d\mu$.

10. The invariant measure associated to an IFS

Just so that we don't forget what is going on, recall:

Definition 10.1. For $x \in \mathbb{R}^n$, $E \subset \mathbb{R}^n$, a measure μ, $\{i_1, ..., i_k\} \subset \{1, ..., m\}$ we define

(1) $x_{i_1..i_k} := S_{i_1} \circ ... \circ S_{i_k}(x)$,
(2) $E_{i_1..i_k} := S_{i_1} \circ ... \circ S_{i_k}(E)$, and
(3) $\mu_{i_1..i_k} := \mu((S_{i_1} \circ ... \circ S_{i_k})^{-1}(E)$.

We have nearly finished the proof of this awesome result.

Theorem 10.2 (The invariant measure for an IFS fractal)**.** *Assume that $S = (S_1, ..., S_m)$ is a family of similitudes with common scaling factor $r \in (0,1)$, $X \subset\subset \mathbb{R}^n$, $X \neq \emptyset$, and $S(X) = X$. Then there exists a (non-negative) Borel measure μ on $\mathbb{R}^n$ such that $\mu(\mathbb{R}^n) = 1$, $\operatorname{supp}(\mu) = X$, and*

$$\forall k \in \mathbb{N}, \quad \mu = \frac{1}{m^k}\sum_{i_1..i_k=1}^{m}\mu_{i_1..i_k}.$$

Here we mean by the statement that $\operatorname{supp}(\mu) = X$ that for any $A \subset \mathbb{R}^n$ which is μ measurable, then

$$\mu(A) > 0 \iff A \cap X \neq \emptyset.$$

To Complete the Proof: Recall how we defined

$$\mu^k := \frac{1}{m^k}\sum_{i_1..i_k=1}^{m}[\delta_x]_{i_1..i_k}.$$

Then note that

$$[\delta_x]_{i_1..i_k}(E) = \delta_x(S_{i_1} \circ ... \circ S_{i_k}(E)) = \begin{cases} 1, & x \in (S_{i_1} \circ ... \circ S_{i_k})^{-1}(E) \Leftrightarrow S_{i_1} \circ ... \circ S_{i_k}(x) \in E \\ 0, & otherwise \end{cases}$$

$$\int_{\mathbb{R}^n} f d\mu^k = \frac{1}{m^k}\sum_{i_1..i_k=1}^{m} f(x_{i_1..i_k}),$$

and

$$\mu^k(\mathbb{R}^n) = \frac{1}{m^k}\sum_{i_1..i_k=1}^{m} 1 = \frac{m^k}{m^k} = 1.$$

(1) Show that for any continuous f,

$$\{\int_{\mathbb{R}^n} f d\mu^k\}_{k\geq 1}$$

is a Cauchy sequence. Consequently we can conclude that it converges to a well-defined limit for each f. Call the limit

$$L(f) := \lim_{k\to\infty} \int_{\mathbb{R}^n} f d\mu^k.$$

(2) Observe that linearity is inherited by L from the linearity of the integral. Moreover, by definition

$$\left| \int_{\mathbb{R}^n} f d\mu^k \right| \leq ||f||_\infty \int_{\mathbb{R}^n} d\mu^k = ||f||_\infty.$$

Here we have used that

$$\int_{\mathbb{R}^n} d\mu^k = 1 \forall k.$$

(3) Show that the support of μ is precisely X.
(4) Show that μ enjoys the invariance property given in the theorem.

Let $\varepsilon > 0$. That X is compact, and f is continuous implies $\exists k > 0$ such that

$$|x - y| \leq r^k \text{diam}(X), \quad x, y \in X \implies |f(x) - f(y)| < \varepsilon.$$

Above we have used the fact that $r < 1$ hence $r^k \to 0$ as $k \to \infty$.
If $l > k \geq K$, then since

$$x_{i_1..i_l} \in X_{i_1..i_l} = S_{i_1} \circ ... \circ S_{i_l}(X) = S_{i_1} \circ ... \circ S_{i_k} ... \circ S_{i_l}(X)$$

$$S_{i_{k+1}...i_l}(X) \subset X, \implies S_{i_1} \circ ... \circ S_{i_k} ... \circ S_{i_l}(X) \subset S_{i_1} \circ ... \circ S_{i_k}(X),$$

and

$$\text{diam} X_{i_1..i_k} = r^k \text{diam} X,$$

we have

$$|f(x_{i_1..i_k}) - f(x_{i_1..i_l})| < \varepsilon$$

which follows because $x_{i_1..i_k}$ and $x_{i_1..i_l}$ are both in $X_{i_1..i_k}$, so

$$|x_{i_1..i_k} - x_{i_1..i_l}| \leq \text{diam}(X_{i_1..i_k}) = r^k \text{diam}(X).$$

Summing over $i_{k+1}..i_l$, and using the trick

$$f(x_{i_1..i_k}) = \frac{1}{m^{l-k}} \sum_{i_{k+1}..i_l=1}^{m} f(x_{i_1..i_k}),$$

because the sum on the right is simply $f(x_{i_1..i_k})$ repeated m^{l-k} times, we have

$$\begin{aligned}
\left| f(x_{i_1..i_k}) - \frac{1}{m^{l-k}} \sum_{i_{k+1}..i_l=1}^{m} f(x_{i_1..i_l}) \right| &= \left| \left(\sum_{i_{k+1}..i_l=1}^{m} f(x_{i_1..i_k}) - f(x_{i_1..i_l}) \right) \frac{1}{m^{l-k}} \right| \\
&\leq \frac{1}{m^{l-k}} \sum_{i_{k+1}..i_l=1}^{m} |f(x_{i_1..i_k}) - f(x_{i_1..i_l})| \\
&< \frac{m^{l-k}\varepsilon}{m^{l-k}} = \varepsilon.
\end{aligned}$$

Next we sum over $i_1..i_k$ and use the estimate above

$$\begin{aligned}
&\left|\frac{1}{m^k}\sum_{i_1..i_k=1}^{m} f(x_{i_1..i_k}) - \frac{1}{m^k}\sum_{i_1..i_k=1}^{m}\left(\sum_{i_{k+1}..i_l=1}^{m} f(x_{i_1..i_l})\right)\frac{1}{m^{l-k}}\right|\\
=&\left|m^{-k}\left(\sum_{i_1..i_k=1}^{m} f(x_{i_1..i_k}) - \frac{1}{m^{l-k}}\sum_{i_{k+1}..i_l=1}^{m} f(x_{i_1..i_l})\right)\right|\\
\leq& m^{-k}\sum_{i_1..i_k=1}^{m}\left|f(x_{i_1..i_k}) - \frac{1}{m^{l-k}}\sum_{i_{k+1}..i_l=1}^{m} f(x_{i_1..i_l})\right|\\
<& m^{-k}m^k\varepsilon = \varepsilon.
\end{aligned}$$

Since

$$\int_{\mathbb{R}^n} f d\mu^l = \frac{1}{m^l}\sum_{i_1..i_l=1}^{m} f(x_{i_1..i_l}), \quad \int f d\mu^k = \frac{1}{m^k}\sum_{i_1..i_k=1}^{m} f(x_{i_1..i_k}),$$

we have

$$\left|\int_{\mathbb{R}^n} f d\mu^k - \int_{\mathbb{R}^n} f d\mu^l\right| < \varepsilon, \quad l > k \geq K.$$

We have therefore shown that for any $\varepsilon > 0$ there exists $K \in \mathbb{N}$ such that for $l > k \geq K$,

$$\left|\int_{\mathbb{R}^n} f d\mu^k - \int_{\mathbb{R}^n} f d\mu^l\right| < \varepsilon \Rightarrow \left\{\int_{\mathbb{R}^n} f d\mu^k\right\}_{k\geq 1}$$

is a Cauchy sequence in $\mathbb{R}$, which is complete, so the sequence converges. Consequently we define a bounded linear functional on $\mathcal{C}_c(\mathbb{R}^n)$ by

$$I(f) := \lim_{k\to\infty}\int_{\mathbb{R}^n} f d\mu^k.$$

Now, we have called it a bounded linear functional, but let us indeed verify that it is, and that it is also non-negative. For this, note that if

$$f \geq 0 \Rightarrow \int f d\mu^k \geq 0 \forall k \Rightarrow I(f) \geq 0.$$

So, I is non-negative. For $g \in \mathcal{C}_c(\mathbb{R}^n)$,

$$I(f+g) = \lim_{k\to\infty}\int (f+g)d\mu^k = \lim_{k\to\infty}\int f d\mu^k + \lim_{k\to\infty}\int g d\mu^k = I(f) + I(g).$$

Similarly, for $\lambda \in \mathbb{R}$,

$$I(\lambda f) = \lim_{k\to\infty}\int \lambda f d\mu^k = \lambda I(f).$$

Therefore I is linear and non-negative. The functional is bounded because

$$\left|\int f d\mu^k\right| \leq ||f||_\infty \mu^k(\mathbb{R}^n) = ||f||_\infty,$$

which implies

$$|I(f)| \leq ||f||_\infty \quad \forall f \in \mathcal{C}_c(\mathbb{R}^n).$$

By Reisz Representation Theorem there exists a Borel measure μ such that

$$I(f) = \int f d\mu, \quad \forall f \in \mathcal{C}_c(\mathbb{R}^n).$$

Note since

$$\mu^k(\mathbb{R}^n \setminus X) = 0 \quad \forall k,$$

if a function f has support in $\mathbb{R}^n \setminus X$, then

$$\int f d\mu^k = 0 \,\forall\, k \implies \int f d\mu = 0.$$

Since we can approximate the characteristic function of any compact subset of $\mathbb{R}^n$ by continuous, non-negative functions, it follows that

$$\mu^k(E) \to \mu(E) \quad \text{for any } E \subset\subset \mathbb{R}^n \implies \mu(E) = 0 \,\forall\, E \subset \mathbb{R}^n \setminus X.$$

Therefore we have

$$\mathrm{supp}(\mu)^c = \cup G, \quad G \subset \mathbb{R}^n \text{ open, such that } \mu(G) = 0,$$

$$\mathrm{supp}(\mu)^c \supset \mathbb{R}^n \setminus X \implies \mathrm{supp}(\mu) \subset X.$$

By the Lebesgue dominated convergence theorem,

$$\int 1 d\mu = \mu(\mathbb{R}^n) = \lim_{k\to\infty} \int 1 d\mu^k = \mu^k(\mathbb{R}^n) = 1.$$

By definition,

$$x_{i_1..i_k} \in X_{i_1..i_k}, \quad \text{for each } k \in \mathbb{N}.$$

We also have

$$\mathrm{diam}(X_{i_1..i_k}) = r^k \mathrm{diam}(X) \to 0 \text{ as } k \to \infty.$$

By the invariance of X under the family S, we have

$$X = \cup_{i_1...i_k=1}^m X_{i_1..i_k}.$$

Then note that for any $\varepsilon > 0$ there exists $k \in \mathbb{N}$ such that

$$\mathrm{diam}(X_{i_1..i_k}) = r^k \mathrm{diam}(X) < \varepsilon.$$

This means that for any point $y \in X$, since

$$y \in X = \cup_{i_1...i_k=1}^m X_{i_1..i_k},$$

the point y lies in at least one of the elements in the union,

$$y \in X_{i_1..i_k} \implies |y - x_{i_1...i_k}| \le \mathrm{diam}(X_{i_1..i_k}) = r^k \mathrm{diam}(X) < \varepsilon.$$

This shows that the collection of points

$$\{\{x_{i_1...i_k}\}_{i_1...i_k=1}^m\}_{k\ge 1}$$

is dense in X, and hence the closure of this collection of points is X. By the definition of μ^k,

$$\mathrm{supp}(\mu^k) = \{x_{i_1...i_k}\}_{i_1...i_k=1}^m.$$

Let p be one of these points, and let f be a compactly supported continuous function with $f(p) = 1$, and $0 \le f \le 1$. Then there exists $\epsilon > 0$ and $N \in \mathbb{N}$ such that

$$|y - p| < \epsilon \implies f(y) > 1/2, \quad k \ge N \implies r^k \mathrm{diam}(X) < \epsilon, \quad p \in X_{i_1...i_N}.$$

Note that we have already seen

$$X_{i_1...i_k...i_l} \subset X_{i_1...i_k} \implies \cup_{i_1...i_k...i_l} \subset \cup X_{i_1...i_k}.$$

Consequently for any $l \ge N$ we know that $p \in X_{i_1...i_N}$ and consequently

$$x_{i_1...i_N} \in X_{i_1...i_N} \implies f(x_{i_1...i_N}) \ge 1/2.$$

Similarly, we also have

$$f(x_{i_1..i_N...i_l}) \ge 1/2 \quad \forall i_{N+1}...i_l.$$

Then we also have for any $k \le l$,

$$\int f d\mu^l = \frac{1}{m^{l-k}} \sum_{i_{k+1}...i_l=1}^m \frac{1}{m^k} \sum_{i_1...i_k=1}^m f(x_{i_1....i_l})$$

and in the second sum taking the specific choice $i_1...i_N$ we have

$$\ge \frac{1}{m^{l-N}} \sum_{i_{N+1}...i_l=1}^m \frac{1}{m^N} f(x_{i_1...i_N...i_l}) \ge \frac{m^{l-N}}{2m^{l-N}m^N} = \frac{1}{2m^N}.$$

Keeping N fixed and letting $l \to \infty$, this shows that
$$\int f d\mu = \lim_{l\to\infty} \int f d\mu^l \geq \frac{1}{2m^N}.$$
If we had $p \in \text{supp}(\mu)^c$, then since by definition this is an open set, there would be an open neighborhood of this point contained in $\text{supp}(\mu)^c$, and so for such an f with support contained in this neighborhood we'd have
$$\int f d\mu \leq \mu(\text{supp}(f)) = 0.$$
That is a contradiction. Hence the entire set of points
$$\{\{x_{i_1\dots i_k}\}_{i_1\dots i_k=1}^m\}_{k\geq 1} \subset \text{supp}(\mu),$$
and by definition $\text{supp}(\mu)$ is closed so $\text{supp}(\mu)$ contains the closure of these points which is X. We have already seen that $\text{supp}(\mu) \subset X$, so this shows that we have equality.
Finally, we will show the invariance property. By definition,
$$\mu^{k+l} = \frac{1}{m^{k+l}} \sum_{i_1..i_{k+l}=1}^m [\delta_x]_{i_1..i_{k+l}},$$
and
$$\mu^l = \frac{1}{m^l} \sum_{i_1..i_l=1}^m [\delta_x]_{i_1..i_l}$$
$$\Rightarrow [\mu^l]_{i_1..i_k} = \frac{1}{m^l} \sum_{j_1..j_l=1}^m [[\delta_x]_{j_1..j_l}]_{i_1..i_k}$$
First, we compute that
$$[[\delta_x]_{j_1\dots j_l}]_{i_1\dots i_k}(E) = \delta_x((S_{j_1}\dots S_{j_l})^{-1}(S_{i_1}\dots S_{i_k})^{-1})(E).$$
Here, note that
$$(S_i S_j)^{-1} = S_j^{-1} S_i^{-1}.$$
To see this, just compute
$$(S_j^{-1} S_i^{-1})(S_i S_j) = \text{ the identity map.}$$
So, we have
$$\begin{aligned}(S_{j_1}\dots S_{j_l})^{-1}(S_{i_1}\dots S_{i_k})^{-1} &= (S_{i_1}\dots S_{i_k})^{-1}(S_{j_1}\dots S_{j_l})^{-1} \\ &= (S_{i_1}\dots S_{i_k} S_{j_1}\dots S_{j_l})^{-1}\end{aligned}$$
Thus,
$$[[\delta_x]_{j_1..j_l}]_{i_1..i_k} = [\delta_x]_{i_1\dots i_k j_1\dots j_l}.$$
Let us therefore define
$$i_{k+1} = j_1, \dots, i_{k+l} = j_l.$$
Then we have
$$[[\delta_x]_{j_1..j_l}]_{i_1..i_k} = [\delta_x]_{i_1\dots i_k j_1\dots j_l} = [\delta_x]_{i_1\dots i_{k+l}}.$$
Consequently,
$$\begin{aligned}\Rightarrow [\mu^l]_{i_1..i_k} &= \frac{1}{m^l} \sum_{j_1..j_l=1}^m [[\delta_x]_{j_1..j_l}]_{i_1..i_k} \\ &= \frac{1}{m^l} \sum_{i_{k+1}\dots i_{k+l}=1}^m [\delta_x]_{i_1\dots i_k\dots i_{k+l}}.\end{aligned}$$
So, now summing over all k combinations
$$\sum_{i_1\dots i_k=1}^m [\mu^l]_{i_1..i_k} = \frac{1}{m^l} \sum_{i_1\dots i_k=1}^m \sum_{i_{k+1}\dots i_{k+l}=1}^m [\delta_x]_{i_1\dots i_k\dots i_{k+l}}$$

$$= \frac{1}{m^l} \sum_{i_1 \ldots i_{k+l}=1}^{m} [\delta_x]_{i_1 \ldots i_k \ldots i_{k+l}}$$

$$= m^k \mu^{k+l} \implies \frac{1}{m^k} \sum_{i_1 \ldots i_k=1}^{m} [\mu^l]_{i_1 .. i_k} = \mu^{k+l}.$$

Now, let f be continuous. Taking the limit on both sides as $l \to \infty$ we have

$$\lim_{l \to \infty} \int_{\mathbb{R}^n} f d\mu^{k+l} = \lim_{l \to \infty} \frac{1}{m^k} \sum_{i_1 \ldots i_k=1}^{m} \int_{\mathbb{R}^n} f d[\mu^l]_{i_1 \ldots i_k}.$$

Basically, the dominated convergence theorem allows us to move the limit inside everywhere, obtaining

$$\int_{\mathbb{R}^n} f d\mu = \frac{1}{m^k} \sum_{i_1 \ldots i_k=1}^{m} \int_{\mathbb{R}^n} f d[\mu]_{i_1 \ldots i_k}.$$

For more details, note that

$$\chi_{\varphi^{-1}(E)}(x) = \begin{cases} 1, x \in \varphi^{-1}(E) \\ 0, \text{else} \end{cases}$$

and

$$\chi_E \circ \varphi(x) = \begin{cases} 1, \varphi(x) \in E \Leftrightarrow x \in \varphi^{-1}(E) \\ 0, \text{else} \end{cases}$$

Therefore,

$$\chi_{\varphi^{-1}(E)} = \chi_E \circ \varphi.$$

Analogously, (integration is the limit over simple functions i.e sums) and using the definition of μ,

$$\begin{aligned} \int f[d\mu^l]_{i_1 .. i_k} &= \int f \circ S_{i_1} \circ .. \circ S_{i_k} d\mu^l \\ &\underset{l \to \infty}{\longrightarrow} \int f \circ S_{i_1} \circ .. \circ S_{i_k} d\mu \\ &= \int f[d\mu]_{i_1 .. i_k} \end{aligned}$$

Let us now assume k is fixed. By the above calculation relating μ^{k+l} and μ^l and the linearity of the integral,

$$\begin{aligned} \int f d\mu^{k+l} &= \frac{1}{m^k} \sum_{i_1 .. i_k=1}^{m} \int f[d\mu^l]_{i_1 .. i_k} \\ &\underset{l \to \infty}{\longrightarrow} \frac{1}{m^k} \sum_{i_1 .. i_k=1}^{m} \int f[d\mu]_{i_1 .. i_k} \end{aligned}$$

Since

$$\lim_{l \to \infty} \int f d\mu^{k+l} = \int f d\mu$$

by definition, this shows that

$$\int f d\mu = \frac{1}{m^k} \sum_{i_1 .. i_k=1}^{m} \int f[d\mu]_{i_1 .. i_k}.$$

This means that on the right side, we also have a linear functional, namely

$$f \mapsto \frac{1}{m^k} \sum_{i_1 .. i_k=1}^{m} \int f[d\mu]_{i_1 .. i_k},$$

which coinicides with our linear functional I. By the proof of the Riesz representation theorem assuming the measure associated with our functional above is constructed in the same way, these measures are therefore the same, and so

$$\mu = \frac{1}{m^k} \sum_{i_1..i_k=1}^{m} [\mu]_{i_1..i_k}.$$

The $k \in \mathbb{N}$ was arbitrary and fixed, hence this holds for all $k \in \mathbb{N}$. 

10.1. **Ball counting Lemma.** To compute the dimension of IFS fractals, it will be important to be able to estimate how much certain sets intersect with our self-similar X such that $SX = X$. The following lemma is key.

Lemma 10.3 (Ball counting Lemma)**.** *Let $c, C, \delta > 0$. Let $\{U_\alpha\}$ be a collection of open, disjoint sets such that. a ball of radius $c\delta \subset U_\alpha \subset$ a ball of radius $C\delta$. Then no ball of radius δ intersects more than $(1+2C)^n c^{-n}$ of the sets $\overline{U}_\alpha$ (note: we are in $\mathbb{R}^n$).*

Proof: If B is a ball of radius δ, and $B \cap \overline{U}_\alpha \neq \emptyset$, then let p be the center of B, so that

$$B = B_\delta(p).$$

Then, by the assumptions of the lemma, there is some $q \in U_\alpha$ such that

$$U_\alpha \subset B_{C\delta}(q).$$

Next, we wish to show that $U_\alpha \subset B_{(1+2C)\delta}(p)$. For this, let

$$x \in U_\alpha.$$

Then for $z \in B \cap \overline{U}_\alpha$, by definition of the ball, we have

$$|z - p| < \delta.$$

By the triangle inequality,

$$|x - p| \leqslant |x - z| + |z - p| < \operatorname{diam}(B_{C\delta}(q)) + \delta = (1 + 2C)\delta.$$

Here we have used that $U_\alpha \subset B_{C\delta}(q)$, to get that $|x - z| \leq \operatorname{diam}(B_{C\delta}(q))$. So, since this holds $\overset{for\ any\ x \in U_\alpha}{\Rightarrow}$ $U_\alpha \subset B_{(1+2C)\delta}(p)$. So, in conclusion, any U_α whose closure has non-empty intersection with B is contained in

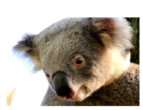
$$B_{(1+2C)\delta}(p).$$

So, the rest of the argument is all about counting. If N of the $\overline{U}_\alpha$'s intersect B (i.e. have $\neq \emptyset$ intersection), then since they are disjoint, and each contains a ball of radius $c\delta$, and they are all contained in $B_{(1+2C)\delta}(p)$, $\Rightarrow$ adding up the Lebesgue measures of all these N disjoint balls of radius $c\delta$ which are contained in the one ball of radius $(1 + 2C)\delta$ we have the inequality:

$$\Rightarrow N(c\delta)^n \omega_n \leqslant \mathcal{L}_n \left(B_{(1+2C)\delta}(p)\right) = (1 + 2C)^n \delta^n \omega_n.$$

Simplifying:

$$\Rightarrow N \leqslant (1 + 2C)^n c^{-n}.$$



10.2. **Homework.** The exercises shall continue with the complex analysis fundamentals to prepare for the second part of the course.

(1) **Super-Mega-Differentiability:** Prove that the derivative of a holomorphic function is holomorphic as are *all* derivatives. Holomorphic functions are infinitely differentiable (and in fact much better than merely $\mathcal{C}^\infty$).
(2) **Maximum Principle:** Prove that $|f|$ has its maximum on the boundary. Otherwise, f is constant.
(3) **Identity Theorem** Prove that TFAE
1. $f \equiv g$
2. $f^k(z_0) = g^k(z_0)\,\forall k$ and some z_0
3. $f(z_n) = g(z_n)\,\forall n, z_n \neq z_0, z_n \to z_0 \in G$.
(4) **Liouville:** Let $f\colon \mathbb{C} \to \mathbb{C}$ be holomorphic. If f is bounded, then it is constant.
(5) **Fundamental theorem of Algebra:** $p(z)$ is a polynomial with coefficients in $\mathbb{C}$, degree of p is $k \geq 1$. Then $\exists!$ (up to rearrangement) $\{r_j\}_{j=0}^k$ in $\mathbb{C}$ such that $p(z) = r_0 \prod\limits_{j=0}^{k} (z - r_j)$.
(6) **Riemann's Removable Singularity Theorem:** Let $f\colon D_r(z_0) \setminus z_0 \to \mathbb{C}$ be holomorphic and bounded. Then z_0 is removable. In case you have forgotten, here is the classification of singularities:

Definition 10.4. If f is holomorphic on $D_r(z_0)\setminus\{z_0\}$, then z_0 is an isolated singularity.
(i) Removable $\Leftrightarrow \exists!$ holomorphic extension to z_0.
(ii) $f(z) \to \infty$ as $z \to z_0 \Leftrightarrow \exists! g(z)$ holomorphic on $D_p(z_0)$ where $p \leq r$ such that $g(z_0) = 0$ and $f(z) = \frac{1}{g(z)}$ on $D_p(z_0) \setminus \{z_0\}$. z_0 is a pole.
(iii) Neither 1 nor 2. "Essential singularity". If f only has a finite set of singularities on $G \subset \mathbb{C}$ of type 1 and/or type 2, f is called "meromorphic".

10.3. **Hints.**

(1) Expanding $\frac{1}{w-z}$ in a geometric series one can prove that f has a power series expansion.
$$f(z) = \sum_{k\geq 0} a_k (z - z_0)^k.$$
It follows from the Lebesgue Dominated Convergence Theorem that
$$f^{(k)}(z) = \frac{k!}{2\pi i} \int_{\partial D} \frac{f(w)}{(w-z)^{k+1}} dw.$$
The coefficients in the power series expansion are therefore
$$a_k = \frac{f^{(k)}(z_0)}{k!}.$$
(2) One way to prove the Identity Theorem is to show that $3 \implies 2$ by considering $h = f - g$ and the power series expansion at z_0. By continuity $h(z_0) = 0$. So, using the power series expansion of h at z_0, assume all coefficients up to a_j vanish (we know this is true for $j \geq 1$ some j, because $a_0 = h(z_0) = 0$. Then use the assumption to show that $a_j = 0$ also. By induction this shows 2. To show the first statement follows from 2, show that the set of points where $f = g$ is clopen (closed and open). Since the set is non-empty, this means that the set is the entire domain.
(3) Assume $|f| \leq M$ on $\mathbb{C}$. The Cauchy Ingegral Formula implies
$$f(z) = \frac{1}{2\pi i} \int\limits_{\partial D_R} \frac{f(w)}{w-z} dw$$

Therefore, we have

$$f'(z) = \frac{1}{2\pi i}\int_{\partial D_R(z_0)} \frac{f(w)}{(w-z)^2}dw$$

and

$$f^{(k)}(0) = \frac{k!}{2\pi i}\int\limits_{\partial D_R} \frac{f(w)}{w^{k+1}}dw$$

Therefore, we get the estimation

$$|f^{(k)}(0)| \leq \frac{k!}{2\pi}\frac{2\pi RM}{R^{k+1}} \forall R > 0$$

Letting $R \to \infty$, we get $f^{(k)}(0) = 0 \forall k \geq 1$. Using the Identity Theorem, we get since $f^{(k)}(0) = g^{(k)}(0) \forall k \geq 0$, $g(z) \equiv f(0) \Rightarrow f \equiv g \Rightarrow f \equiv f(0)$ is constant.

FTA If degree of p is 1, then $p(z) = az + b$ and $a \neq 0 \Rightarrow r_0 = a$ and $r_1 = -\frac{b}{a}$. finish.
By induction on K. If $p\big|_{\mathbb{C}} \neq 0$ then $\frac{1}{p}$ is entire and $\to 0$ at ∞. $\Rightarrow$ bounded $\Rightarrow$ constant $\Rightarrow p$ constant $\lightning$
p has at least one zero $r_k \Rightarrow p$ is polynomial, $\frac{p(z)}{z-r_k}$ is a rational funcion without poles $\Rightarrow$ polynomial.
$p(z) = (z - r_k)q(z)$ where q has degree $k - 1 < k$. $\Rightarrow$ by induction $\exists!\{r_j\}_{j=0}^{k-1}$ such that $q(z) = r_0 \prod_{j=1}^{k-1}(z - r_j)$. $\Rightarrow p(z) = r_0 \prod_{j=0}^{k}(z - r_j)$.

RRT $g(z) := (z - z_0)f(z)$, $z \neq z_0$. g is holomorphic on $D_r(z_0) \setminus z_0$ $\lim_{z\to z_0} g(z) = 0 \Rightarrow$ define $g(z_0) = 0 \Rightarrow g$ is continuous on $D_r(z_0)$. $\Rightarrow g$ is holomorphic on $D_r(z_0)$ and so $\lim_{z\to z_0} \frac{g(z)-g(z_0)}{z-z_0} = g'(z_0)$ exists, and $\lim_{z\to z_0} \frac{(z-z_0)f(z)}{z-z_0} =: f(z_0)$. Consequently this limit exists, is unique, and defining $f(z_0)$ by this limit is unique and makes f continuous at z_0. Moreover, any holomorphic function on a punctured disk which is continuous on the whole disk is in fact holomorphic, which follows from the fact that the integral of such a function over any triangle in the disk vanishes, hence the function has a well-defined primitive. By super-mega differentiability the original function, that is the derivative of the primitive, is also holomorphic.

11. The dimension of IFS fractals and an introduction to complex dynamics

We shall now prove the major goal of the geometric measure theory part of this course!

Theorem 11.1 (Dimension of IFS Fractals!!!)**.** *Let $S = (S_1, ..., S_m)$ be a family of similitudes with common scale factor $r \in (0, 1)$. Let U be a separating set, that is an open, bounded, non-empty set with $S(U) \subset U$, and $S_i(U) \cap S_j(U) = \emptyset$ if $i \neq j$. Let X be the unique, non-empty, compact set s.t. $S(X) = X$. Let $p := \log_{\frac{1}{r}}(m)$. Then we have*

i) $\mathcal{H}^p(X) \in (0, \infty)$, so we may conclude that $p = \dim(X)$.
ii) Moreover, $\mathcal{H}^p(S_i(X) \cap S_j(X)) = 0$ for all $i \neq j$.

Proof: For any $k \in \mathbb{N}$, by the invariance of X, we can write

$$X = S^k(X) = \bigcup_{i_1,\ldots,i_k=1}^{m} S_{i_1} \circ \ldots \circ S_{i_k}(X) = \bigcup_{i_1,\ldots,i_k=1}^{m} X_{i_1,\ldots,i_k}.$$

Each of these $X_{i_1,\ldots,i_k}$ has diameter $= r^k \mathrm{diam}(X)$. So, if $\delta_k = r^k \mathrm{diam}(X)$, then

$$\mathcal{H}^p_{\delta_k}(X) \leqslant \sum_{i_1,\ldots,i_k=1}^{m} (\mathrm{diam}\,(X_{i_1,\ldots,i_k}))^p = m^k r^{pk} \mathrm{diam}(X)^p$$

By definition $p = \log_{\frac{1}{r}}(m) \;\Rightarrow \left(\frac{1}{r}\right)^p = m \Rightarrow m^k = r^{-pk} \Rightarrow \mathcal{H}^p_{\delta_k}(X) \leqslant \operatorname{diam}(X)^p$. Letting $\overset{\delta_k \downarrow 0}{\Rightarrow} \mathcal{H}^p(X) \leqslant \operatorname{diam}(X)^p < \infty$, because X is compact and thus also bounded.
Next, we wish to show that p-dimensional Hausdorff measure of X is positive. For this purpose, let $0 < c < C$ be chose such that U contains a ball of radius $\frac{c}{r}$ and is contained in a ball of radius $C \left(= \frac{Cr}{r}\right)$. Since U is a non-empty bounded set, clearly it is possible to find such a c and C.
Let

$$N = (1+2C)^n c^{-n}.$$

We will prove that

$$\mathcal{H}^p(X) \geqslant \frac{1}{2^p N}$$

by showing that if $\{E_j\}_{j\geqslant 1}$ cover X with $\operatorname{diam}(E_j) \leqslant 1 \forall j$, then

$$\sum \operatorname{diam}(E_j)^p \geqslant \frac{1}{N2^p}.$$

In this way, we shall obtain that

$$\mathcal{H}^p_1(X) \geq \frac{1}{N2^p}.$$

Since $\mathcal{H}^p_\delta(X)$ is monotonically increasing as $\delta \downarrow 0$, it follows that

$$\mathcal{H}^p(X) \geq \mathcal{H}^p_1(X) \geq \frac{1}{N2^p} > 0.$$

Now, let us make some further reductions. Any (non-empty) set E of diameter δ is contained in a closed ball of radius δ because the distance between any two points of E is at most δ. So it suffices to pick any old point $p \in E$, and then $d(p,e) \leq \operatorname{diam}(E) = \delta$ for all $e \in E$, so by definition $E \subset \overline{B_\delta(p)}$. With this observation we note that

$$\operatorname{diam}(E) = \frac{\operatorname{diam}(B_\delta)}{2} \;\Rightarrow \sum \operatorname{diam}(E_j)^p = \sum \left(\frac{\operatorname{diam}(B_\delta)}{2}\right)^p = \frac{1}{2^p} \sum \operatorname{diam}(B_\delta)^p.$$

Hence, it is enough to show that if

$$X \subset \cup B_j = \cup B_{\delta_j}, \quad \delta_j \leqslant 1 \forall j,$$

then

$$\sum_{j=1}^{\infty} \delta_j^p \geqslant \frac{1}{N},$$

because

$$\sum_{j=1}^{\infty} \delta_j^p = 2^p \sum_{j=1}^{\infty} \operatorname{diam}(E_j)^p \implies \sum_{j=1}^{\infty} \operatorname{diam}(E_j)^p \geq \frac{1}{2^p N}.$$

To prove this, we will prove:
$\star$: if the radius of B is $\delta \leqslant 1$ then $\mu(B) \leqslant N\delta^p$. Here μ is our special measure associated to X which has that cool invariance property.
This shows that

$$1 = \mu(X) \leqslant \sum \mu(B_j) \leqslant N \sum \delta_j^p.$$

Above, we are using that X is contained in the balls, and μ is a Borel measure, so we have countable sub-additivity. Then, note that this shows that

$$\frac{1}{N} \leq \sum_j \delta_j^p,$$

which is what we want.
To prove $\star$ let $k \in \mathbb{N}$ s.t. $r^k < \delta \leqslant r^{k-1}$. Then by the invariance property enjoyed by μ we have

$$\mu(B) = \frac{1}{m^k} \sum_{i_1,\ldots,i_k=1}^{m} \mu_{i_1,\ldots,i_k}(B).$$

Since

$$X \subset \overline{U}, \quad \operatorname{supp}(\mu_{i_1,\dots,i_k}) = X_{i_1,\dots,i_k} \subset \overline{U}_{i_1,\dots,i_k}.$$

Thus,

$$\text{if we have } \mu_{i_1,\dots,i_k}(B) \neq 0 \text{ then we have } B \cap \overline{U}_{i_1,\dots,i_k} \neq \emptyset.$$

Next we use the fact that

$$S_i(U) \cap S_j(U) = \emptyset \; i \neq j,$$

together with

$$S(U) \subset U \implies S_i(U) \subset U \forall i$$

to conclude

$$\begin{matrix} S_k(S_i(U)) \subset S_k(U) \\ S_l(S_i(U)) \subset S_l(U) \end{matrix} \Rightarrow S_k(S_i(U)) \cap S_l(S_i(U)) = \emptyset \text{ if } k \neq l$$

Moreover, we also have

$$S_k(S_i(U)) \cap S_k(S_j(U)) = \emptyset, \text{ for } i \neq j,$$

because

$$S_i(U) \cap S_j(U) = \emptyset \text{ for } i \neq j, \text{ and } S_k \text{ is injective.}$$

This shows that if $i_1, \dots, i_k \neq j_1, \dots, j_k$, then $U_{i_1,\dots,i_k} \cap U_{j_1,\dots,j_k} = \emptyset$.

Now we use the fact that U contains a ball of radius $\frac{c}{r} \Rightarrow U_{i_1,\dots,i_k}$ contains a ball of radius $\frac{c}{r} r^k = cr^{k-1}$. Note: that $cr^{k-1} \geqslant c\delta$ and $Cr^k < C\delta$.
Thus, $U_{i_1,\dots,i_k}$ contains a ball of radius $cr^{k-1} \geqslant c\delta$, and is contained in a ball of radius $Cr^k < C\delta$.
$\overset{\textit{Ball counting Lemma}}{\Rightarrow}$ B can intersect at most $N = (1+2C)^n c^{-n}$ of the $\{\overline{U_{i_1,\dots,i_k}}\}_{i_1,\dots,i_k=1}^m$.

$$\Rightarrow \mu(B) = \frac{1}{m^k} \sum_{i_1,\dots,i_k=1}^{m} \mu_{i_1,\dots,i_k}(B) \leqslant N m^{-k}.$$

Note that for the last inequality we have used the fact that $\mu_{i_1,..,i_k}$ is supported in $X_{i_1,\dots,i_k} \subset \overline{U_{i_1,\dots,i_k}}$, and the mass of each of these is at most 1 because the total mass is one. Since B intersects at most N of them, the right side of the inequality $m^{-k}N$ follows. Now, recalling that
$p = \log_{\frac{1}{r}}(m) \Rightarrow m^{-k} = r^{kp} \Rightarrow \mu(B) \leqslant Nr^{kp} \leqslant N\delta^p$, since $r^k < \delta \leq 1$. This is $\star$. So, we have proven that $\mathcal{H}^p(X)$ is positive and finite, thus by our previous results, this shows that p is precisely the Hausdorff dimension of X. Pretty cool!
Finally, we show that the copies of X have zero-measure $\mathcal{H}^p$ intersection. Since S_j scales by r, we have proven that $\mathcal{H}^p(S_j(X)) = r^p \mathcal{H}^p(X) = m^{-1}\mathcal{H}^p(X)$, using the definition of p. Consequently,

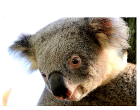
$$\Rightarrow \mathcal{H}^p(X) = \sum_{j=1}^{m} \mathcal{H}^p(S_j(X)).$$

Since

$$X = \bigcup_{j=1}^{m} S_j(X),$$

this holds iff $\mathcal{H}^p(S_i(X) \cap S_j(X)) = 0$ whenever $i \neq j$. More generally, for any measure ν, measurable sets A and B

$$\nu(A \cup B) = \nu(A) + \nu(B) \Leftrightarrow \nu(A \cap B) = 0$$



11.1. **Introducing complex dynamics.** Let us begin with a famous set, the Mandelbrot set.

Definition 11.2. The Mandelbrot set is the set of $c \in \mathbb{C}$ such that the function

$$f_c(z) := z^2 + c,$$

satisfies

$$\{f_c^n(0)\}_{n\in\mathbb{N}} \text{ is a bounded subset of } \mathbb{C}.$$

Exercise 6. *Play around with this definition. Take specific values of c, pop them in, and see what happens to the sequence you are obtaining.*

To study this set and its mysteries, we shall require some notions from complex analysis and the iteration of complex functions. For a function $f(z) : \mathbb{C} \to \mathbb{C}$, the function

$$f^n(z) := f \circ f \circ f \circ \ldots \circ f(z), \text{ is } f \text{ composed with itself } n \text{ times.}$$

This clearly makes sense when n is a positive integer. Complex dynamics is the study of the family of functions $\{f^n\}$ for certain choices of the function f. This is exactly how the Mandelbrot set is defined! It consists of the complex numbers, c, such that the quadratic function $f_c(z) = z^2 + c$ satisfies

$$\{f_c^n(0)\}_{n\in\mathbb{N}} \text{ is bounded.}$$

So, to dig deeper into the Mandelbrot set, we need to understand fundamental facts about iterating functions. To begin, we define what it means for a family of holomorphic functions to be *normal* on a domain in $\mathbb{C}$. For this, we recall a definition that you really ought to already know.

Definition 11.3. A function f is holomorphic in a neighbourhood $D_r(z_0)$ of z_0, iff $\forall z \in D_r(z_0)$ the following limit exists

$$\lim_{w\to z} \frac{f(w) - f(z)}{w - z} =: f'(z).$$

Proposition 11.4. *This definition is equivalent to requiring that for all $z \in D_r(z_0)$ there exists a function A_z which is continuous at z, and such that*

$$f(w) = f(z) + (w - z)A_z(w), \quad \forall w \text{ in a neighborhood of } z.$$

Proof: First, assume that f is holomorphic. Then for w near z we can define

$$A_z(w) = \frac{f(w) - f(z)}{w - z}, \quad w \neq z, \quad A_z(z) = f'(z).$$

Then, for all w near z, we have

$$A_z(w)(w - z) = f(w) - f(z) \implies f(w) = f(z) + (w - z)A_z(w),$$

and so defined

$$\lim_{w\to z} A_z(w) = A_z(z).$$

Next assume that we have such a continuous $A_z(w)$. Then

$$\lim_{w\to z} \frac{f(w) - f(z)}{w - z} = \lim_{w\to z} \frac{f(z) + (w - z)A_z(w) - f(z)}{w - z} = \lim_{w\to z} A_z(w) = A_z(z).$$

In the last step we used the fact that A_z is continuous at the point z.

Definition 11.5. A family of holomorphic functions $\mathcal{F}$ defined on a domain $G \subset \mathbb{C}$ is *normal* if for any sequence in $\mathcal{F}$, there exists a subsequence which converges locally uniformly (this means uniformly on compact subsets).

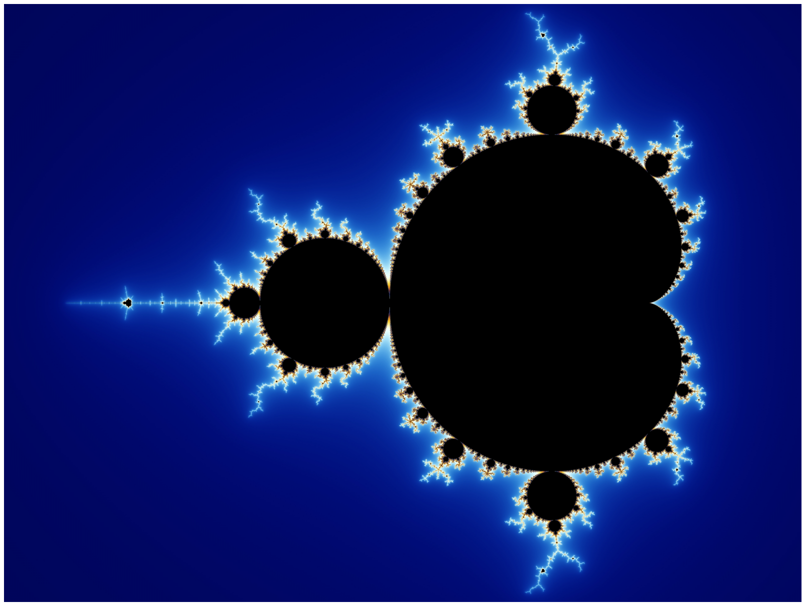


FIGURE 2. The Mandelbrot set.

Definition 11.6. Let $f : \mathbb{C} \to \mathbb{C}$. The *Fatou set* of f is defined to be

$$\{z \in \mathbb{C} : \exists r > 0 \text{ such that } \{f^n\} \text{ is a normal family on } D_r(z)\}.$$

The *Julia set* of f is the complement of the Fatou set.

By demonstrating and recalling a few theorems, we will prove that if we wish to study the complex dynamics (i.e. behavior of the family $\{f^n\}$) of $f : G \to G$ for a domain $G \subset \mathbb{C}$, it's actually completely equivalent to studying the dynamics on either the unit disk, the entire complex plane, or the Riemann sphere. In this way, we can simplify the problem by working on a simple set (disk, plane, or Riemann sphere) rather than working on some creepy wonky set, G.

Theorem 11.7 (Open Mapping Theorem)**.** *Let $f : G \to \mathbb{C}$ be holomorphic and non-constant. Then f is an open map, i. e. $f(G)$ is a domain.*

Proof: Since G is connected and f is continuous, $f(G)$ is also connected.
Let $w_0 = f(z_0)$ and $r > 0$ such that $\overline{D_r(z_0)} \subset\subset G$ and so that

$$f\big|_{\overline{D_r(z_0)}\setminus z_0} \neq w_0 \tag{11.1}$$

To see why we can do this, we use the Identity Theorem. If there are points $z_n \neq z_0$ such that $z_n \to z_0$ and $f(z_n) = f(z_0)$, then the function

$$f(z) - f(z_0)$$

is also holomorphic, and it has infinitely many zeros occurring at the points z_n which accumulate at z_0. The identity theorem then says that this function is identically zero, which would mean that $f(z) \equiv f(z_0)$ is constant. That is a contradiction $\lightning$.
Now, let us define

$$\delta := \min_{z \in \partial D_r(z_0)} |f(z) - w_0| > 0$$

Since $\partial D_r(z_0)$ is compact, and f is continuous, so therefore $|f(z) - w_0|$ is also continuous, the minimum is assumed at some point (that's why we called it a minimum not an infimum... every minimum is an infimum but not the other way around). Also, since $f(z) \neq w_0$ on the closure of $D_r(z_0)$, we know that $\delta > 0$.

Claim. $D_{\frac{\delta}{2}}(w_0) \subset f(D_r(z_0))$. If we can prove this claim, then we will have proven the theorem, because we will have proven that any point $w_0 \in f(G)$ has a neighborhood, $D_{\delta/2}(w_0)$ which is also contained in $f(G)$, thus $f(G)$ is an open set.
So, let us prove the claim!
Assume that w has $|w - w_0| < \frac{\delta}{2}$. Then let $z \in \partial D_r(z_0)$. The triangle inequality gives

$$|f(z) - w| \geq |f(z) - w_0| - |w - w_0|.$$

By definition of δ as the minimum, and since $|w - w_0| < \frac{\delta}{2}$, we therefore have

$$|f(z) - w| \geq \delta - \frac{\delta}{2} = \frac{\delta}{2}, \quad \forall z \in \partial D_r(z_0).$$

We would like to show that somewhere inside the disk, $|f(z) - w| = 0$. To achieve this, let us consider

$$g(z) := \frac{1}{f(z) - w}.$$

At least on the boundary $D_r(z_0)$ this function is well defined. For the sake of contradiction, let us assume that $f(z) \neq w$ for all $z \in D_r(z_0)$. Then g is holomorphic on all of $D_r(z_0)$. Note that

$$|f(z) - w| \geq \frac{\delta}{2} \quad \forall z \in \partial D_r(z_0) \implies |g(z)| \leq \frac{2}{\delta}, \quad \forall z \in \partial D_r(z_0).$$

However, we have

$$g(z_0) = \frac{1}{f(z_0) - w} = \frac{1}{w_0 - w} \implies |g(z_0)| = \frac{1}{|w_0 - w|} > \frac{2}{\delta}.$$

Yikes! Holomorphic functions assume their maximum on the boundary, not in the interior somewhere. So this is impossible. What lead to this impossibility was the assumption that $f(z) \neq w$ for all $z \in D_r(z_0)$. Thus, there must be some $z \in D_r(z_0)$ with $f(z) = w$. Hence, this shows that $w \in f(D_r(z_0))$. Since w was arbitrary with $|w - w_0| < \frac{\delta}{2}$, this shows that

$$D_{\delta/2}(w_0) \subset f(D_r(z_0)) \subset f(G),$$

since $D_r(z_0) \subset G$. 

11.2. **Homework.**

Exercise 17. *Review everything we have done thus far in the geometric measure theory part of the course. See if there are any lingering questions, gaps, and if so, make a list of these. Bring your list to class so that these lingering questions can get answered!*

Exercise 18. *Now review your complex analysis. If you can read German, you can check out my lecture notes from teaching complex analysis to physicists in Hannover... These are linked on the course webpage.*

Exercise 19. *Prove that the locally uniform limit of holomorphic functions is again holomorphic.*

Exercise 20. *Who is this guy*  *?*

12. Complex dynamics on the plane and Riemann sphere

We would like to understand when domains in the complex plane are equivalent in a certain sense. This sense is phrased in terms of biholomorphic maps.

Definition 12.1. A map $f : G \to f(G)$ is biholomorphic precisely when f is holomorphic, f^{-1} exists and is holomorphic on $f(G)$.

Exercise 21. *Prove that if f is holomorphic with $f' \neq 0$ on a domain G, then f is* locally *1:1, in the sense that for each $z \in G$ there exists $r > 0$ such that f is 1:1 on $D_r(z)$. Find an example of an f and a G which has f locally 1:1 but not 1:1 on all of G.*

Corollary 12.2. *Assume that a holomorphic map $f\colon G \to \Omega$. Then it is is biholomorphic $\iff$ $f'\big|_G \neq 0$, and f is 1:1.*

Proof: ($\Rightarrow$): Assume that f is biholomorphic. Then we differentiate the identity map, $f^{-1} \circ f : G \to G$:

$$(f^{-1})'(f(z))f'(z) = 1,$$

since the derivative of the identity map is 1. Consequently, we get that $f'(z)$ can never vanish. Moreover, by assumption that f is biholomorphic, f^{-1} exists (is defined!) thus f must be 1:1. ($\Leftarrow$): We assume that f is 1:1 and $f'\big|_G \neq 0$. Then f is not constant. Therefore $f(G) = \Omega$ is open. To see that f^{-1} is continuous, we use the characterization that requires the inverse image of open sets be open. So, let $U \subset G$ be open. Then $(f^{-1})^{-1}(U) = f(U)$ is open by the Open Mapping Theorem. Thus f^{-1} is continuous. We can therefore compute

$$\lim_{w \to z_0 = f(\xi_0)} \frac{f^{-1}(w) - f^{-1}(z_0)}{w - z_0} = \lim_{\xi = f^{-1}(w) \to \xi_0} \frac{\xi - \xi_0}{f(\xi) - f(\xi_0)} = \frac{1}{f'(\xi_0)}$$

The step where we changed the limit to $\xi \to \xi_0$ is legit because we proved that f^{-1} is continuous, so

$$w \to z_0 = f(\xi) \implies f^{-1}(w) = \xi \to f^{-1}(z_0) = \xi_0.$$

This exists because f is holomorphic and $f'\big|_G \neq 0$. Consequently, we have proven that f^{-1} is also holomorphic. 

Definition 12.3. If G, Ω are domains in $\mathbb{C}$ such that $\exists f\colon G \to \Omega$ biholomorphic, then G and Ω are biholomorphically equivalent. A map $f\colon G \to \mathbb{C}$ such that $f'\big|_G \neq 0$ is known as a conformal map, and G is conformally equivalent to $f(G)$.

Remark 7. "Conformal" means angle-preserving.

Theorem 12.4 (Uniformization Theorem)**.** *Let $G \subset \mathbb{C}$ be simply connected. Then G is conformally equivalent to one of the following:*

(1) $\mathbb{C}$
(2) $\mathbb{D}$
(3) $\hat{\mathbb{C}} = \mathbb{C} \cup \infty$.

Moreover, the same holds for any simply connected Riemann surface (2-dimensional Riemannian manifold with biholomorphic coordinate charts $\to \mathbb{C}$).

The following theorem shows that essential singularities are extremely special.

Theorem 12.5 (Big Picard Theorem)**.** *If f is holomorphic on $D_r(z_0)\backslash z_0$, and z_0 is an essential singularity, then $\forall \varepsilon \in (0, r)$, $\#\{\mathbb{C} \setminus f(D_\varepsilon(z_0) \setminus z_0)\} \leq 1$.*

Remark 8. This means that the image of any punctured disk, no matter how tiny, about the essential singularity gets mapped to *cover all of $\mathbb{C}$, except possibly one point!!!*

Definition 12.6. If f is entire and $\lim_{z\to\infty} f(z) = \infty$, then f has a pole at ∞.

Corollary 12.7. *Assume that f is entire. Then either*

(1) f is constant.
(2) f has a pole at ∞.
(3) f has an essential singularity at ∞.

Proof: If f is not constant, then something happens at infinity. By definition, if f has a pole at infinity, then $\frac{1}{f(\frac{1}{z})} =: g(z)$ is holomorphic near $z = 0$ and $g(0) = 0$. Consequently, f cannot have an essential singularity at infinity by Picard's theorem. On the other hand, if f has an essential singulariy at infinity, then $\frac{1}{f(\frac{1}{z})} =: h(z)$ has an essential singularity at 0. By Picard's theorem, it is impossible that f has a pole at infinity.

A useful result is Montel's Theorem, which allows us to conclude that if a family is bounded, then it is normal.

Theorem 12.8 (Montel's Little Theorem)**.** *If a family $\mathcal{F}$ is uniformly bounded, then it is normal.*

Proof: Let $M \geq ||f||_\infty$ for all $f \in \mathcal{F}$. Fix $z_0 \in G$ and $R > 0$ such that

$$D_R(z_0) \subset\subset G.$$

Then for any $z \in D_{R/2}(z_0)$ we have by the Cauchy Integral Formula for $f \in \mathcal{F}$,

$$f'(z) = \frac{1}{2\pi i}\int_{\partial D_R(z_0)} \frac{f(w)}{(w-z)^2}dw \implies$$

$$|f'(z)| \leq \frac{2\pi R}{2\pi}\frac{M}{(R-R/2)^2} =: c.$$

This holds for all $z \in D_{R/2}(z_0)$. It follows that the family $\mathcal{F}$ is equicontinuous on this disk. Recall that this means that given $\epsilon > 0$, the same δ "works" in the definition of continuity for all $f \in \mathcal{F}$. In particular, given $\epsilon > 0$, we can take

$$\delta = \frac{\epsilon}{1+c} \implies |f(z) - f(w)| \leq |z-w| \sup_{\zeta \in D_{R/2}(z_0)} |f'(\zeta)| \leq c|z-w|,$$

so when

$$|z-w| < \delta = \frac{\epsilon}{1+c} \implies |f(z)-f(w)| < \frac{\epsilon c}{1+c} < \epsilon.$$

Note that we have a sort of Fundamental Theorem of Calculus in complex analysis, in the sense that $f(z) = \int_{z_0}^z f'(w)dw$, which is why we get the estimate $|f(z) - f(w)| \leq |z - w| \sup_{\zeta \in D_{R/2}(z_0)} |f'(\zeta)|$. Now, since the family was assumed to be bounded, and we proved it was equicontinuous, the Arzela-Ascoli theorem implies that every sequence has a locally uniformly convergent subsequence.

By Montel's Theorem, if a function $f : \mathbb{D} \to \mathbb{D}$, then $\mathcal{F} := \{f^n\}$ is a normal family. So, we can already say something about the holomorphic dynamics on $\mathbb{D}$. In particular we have

Theorem 12.9 (Holomorphic dynamics on $\mathbb{D}$)**.** *Let $f : G \to G$ be holomorphic on the simply connected domain, G. Assume that G is conformally equivalent to the unit disk, $\mathbb{D}$. Then G belongs to the Fatou set of f.*

Proof: Let $\phi : G \to \mathbb{D}$ be the conformal (biholomorphic) map given from the Uniformization Theorem. Then the map

$$\varphi = \phi \circ f \circ \phi^{-1} : \mathbb{D} \to \mathbb{D}.$$

Hence, we also have $\varphi^n : \mathbb{D} \to \mathbb{D}$ for all $n \in \mathbb{N}$. Therefore the family

$$\{\varphi^n\}$$

is a normal family, by Montel's Theorem. Thus on any compact subset $K \subset \mathbb{D}$, we can find a uniformly convergent subsequence. Let φ^{n_k} be such a subsequence, with $\varphi^{n_k} \to g$ on K. In particular, let $B \subset G$ be compact. Then by the open mapping theorem $\phi(G \setminus B)$ is open, which shows that $\phi(B)$ is closed, and being contained in $\mathbb{D}$, it is therefore compact. Hence, taking $K = \phi(B)$ we have φ^{n_k} converging on K. Therefore, we have

$$\lim_{n_k \to \infty} \varphi^{n_k}(z) \text{ exists for all } z \in K.$$

Then, since

$$\varphi^{n_k} = \phi \circ f^{n_k} \circ \phi^{-1} \implies f^{n_k} = \phi^{-1} \circ \varphi^{n_k} \circ \phi.$$

Hence

$$\lim_{n_k \to \infty} f^{n_k}(w) = \lim_{n_k \to \infty} \phi^{-1} \circ \varphi^{n_k} \circ \phi(w) = \lim_{n_k \to \infty} \phi^{-1}\varphi^{n_k}(z),$$

for $z = \phi(w) \in \phi(B)$. Since

$$\lim_{n_k \to \infty} \varphi^{n_k}(z)$$

exists, and ϕ^{-1} is holomorphic and therefore continuous, we also have

$$\lim_{n_k \to \infty} f^{n_k}(w) = \lim_{n_k \to \infty} \phi^{-1}\varphi^{n_k}(z) \text{ exists and equals}$$

$$\phi^{-1}\left(\lim_{n_k \to \infty} \varphi^{n_k}(z)\right).$$

This shows that $\{f^n\}$ is a normal family on all of G. Hence G belongs to the Fatou set of f.

Corollary 12.10 (Conformal Sandwich)**.** *Assume that $f : G \to G$ for a domain G which is conformally equivalent to a domain Ω. Let $\phi : G \to \Omega$ be biholomorphic. Then $\{f^n\}$ is normal on G if and only if $\{\varphi^n\}$ is normal on Ω, where $\varphi = \phi \circ f \circ \phi^{-1}$.*

Proof: Since the two directions are the same by symmetry of the statement, it suffices to prove that if $\{f^n\}$ is normal then $\{\varphi^n\}$ is normal. For this purpose, let $K \subset \Omega$ be compact. Then $B = \phi^{-1}(K) \subset G$ is also compact, by the open mapping theorem as in the proof of the preceding theorem. Hence, we have a subsequence of f^n converging on B. Denote this subsequence by $\{f^{n_k}\}$. Then, we have

$$\lim_{n_k \to \infty} \varphi^{n_k}(z) = \lim_{n_k \to \infty} \phi \circ f^{n_k} \circ \phi^{-1}(z).$$

For $z \in K$, we have $\phi^{-1}(z) = w \in B$, and therefore

$$\lim_{n_k \to \infty} \phi \circ f^{n_k} \circ \phi^{-1}(z) = \lim_{n_k \to \infty} \phi \circ f^{n_k}(w) = \phi\left(\lim_{n_k \to \infty} f^{n_k}(w)\right).$$

Above, we have used the fact that f^{n_k} converges at $w \in B$, together with the fact that ϕ is holomorphic and thus continuous.

Remark 9. Two functions, f and g, are said to be conformally equivalent if there exists a conformal map ϕ such that we have $f \circ \phi = \phi \circ g$, i.e. $f = \phi \circ g \circ \phi^{-1}$, or equivalently, $g = \phi^{-1} \circ f \circ \phi$. By the preceding corollary, $\{f^n\}$ is normal on G if and only if $\{g^n\}$ is normal on $\phi^{-1}(G)$. Hence, when functions are conformally equivalent, we can always choose the simpler one to study.

Complex dynamics is all about determining when the family of iterates $\{f^n\}$ is normal. For f defined on a simply connected domain $G \subset \mathbb{C}$, assume that $f : G \to G$, so that $f \circ f$ and more generally f^n is well defined on all of G. Let

$$\phi^{-1} : E \mapsto G,$$

be the conformal map given by the Uniformization Theorem, where $E = \mathbb{D}, \mathbb{C}$, or $\hat{\mathbb{C}}$. Then let

$$\tilde{f} := \phi \circ f \circ \phi^{-1} : E \to E.$$

Note that $\tilde{f}^n = \phi \circ f^n \circ \phi^{-1}$. Therefore the preceding corollary shows that the study of holomorphic dynamics on any simply connected domain is reduced, by the Uniformization Theorem, to the study of holomorphic dynamics on $\mathbb{D}$, $\mathbb{C}$, and $\hat{\mathbb{C}}$. Moreover, we have proven that in case G is conformally equivalent to the unit disk, $\mathbb{D}$, then such a function f is normal on G. Hence, the more interesting cases shall be when G is conformally equivalent to $\mathbb{C}$ or $\hat{\mathbb{C}}$. Since we can reduce to the case of studying the iterates of the conformal sandwich function on $\mathbb{C}$ or $\hat{\mathbb{C}}$, it shall be much simpler to work over there. For this purpose, we shall classify all functions which are (1) entire and without essential singularity at infinity and (2) meromorphic on $\hat{\mathbb{C}}$. Recall that meromorphic means that there are only discrete poles of finite rank (i.e. no essential singularities), and elsewhere such a function is holomorphic.

12.1. **Homework.**

(1) Locate and read a proof of Picard's BIG theorem.
(2) Locate and read a proof of the Uniformization theorem.
(3) Prove that the Fatou set is always open.
(4) Prove that the Julia set is always closed.

13. Entire and meromorphic functions without essential singularities

We shall first prove that it is quite natural to focus on the Fatou and Julia sets of polynomials and rational functions.

Theorem 13.1. *If $f : \mathbb{C} \to \mathbb{C}$ is entire and without essential singularity at infinity, then f is a polynomial.*

Proof: First note that if f is bounded, then it is constant, and hence a polynomial of degree 0. How interesting (not). Let us assume that f is non-constant and therefore unbounded, then we must have $|f(z)| \to \infty$ as $|z| \to \infty$. Consequently the function

$$\frac{1}{f(1/z)} = g(z)$$

is holomorphic on a disk about 0 with $g(0) = 0$. Since $f \not\equiv \infty$, we cannot have $g \equiv 0$, and therefore there exists $k \in \mathbb{N}$ such that

$$g(z) = \sum_{j \geq k} a_j z^j, \quad a_k \neq 0.$$

Consequently,

$$f(z) = \frac{1}{g(1/z)} = \frac{1}{a_k z^{-k} + \dots} = \frac{z^k}{a_k + a_{k+1} z + \dots} \sim z^k \text{ as } k \to \infty.$$

Next since $|f| \to \infty$ as $|z| \to \infty$, there exists $R > 0$ such that for all $|z| > R$, $|f(z)| > 100000$. In particular for all such z, $f \neq 0$. So, the set of zeros of f is contained in a compact set. Since we assumed that f is not constant, by the identity theorem f can only have a finite set of zeros

(of finite order) because they are all contained in a compact set, and so any infinite set would accumulate there thus implying f vanishes identically (ID theorem) which it does not.
Let $\{z_k\}_1^n$ be the zeros of f of respective degrees d_k. Then consider

$$\frac{f(z)}{\prod_1^n (z-z_j)^{d_j}}.$$

We know that $|f(z)| \sim |z|^k$ as $|z| \to \infty$. If on the one hand $k < \sum d_j$, then this function tends to 0 at infinity and is entire, hence bounded, hence constant by Liouville's theorem. Since it tends to *zero* at infinity, this would imply the function is identically 0, hence so is f, which is a contradiction. So we must have $k \geq \sum d_j$. Now, on the other hand, we consider

$$\frac{\prod_1^n (z-z_j)^{d_j}}{f(z)}.$$

This function is also entire. If $k > \sum d_j$, then by the same argument we also get a contradiction. Hence $k = \sum d_j$, and so both of these functions are again bounded and entire, hence constant (and that constant cannot be zero), so there is $c \in \mathbb{C} \setminus \{0\}$ such that

$$\frac{f(z)}{\prod_1^n (z-z_j)^{d_j}} \equiv c \iff f(z) \equiv c \prod_1^n (z-z_j)^{d_j}$$

which is a polynomial. 

So, we now see that holomorphic dynamics for entire functions without essential singularity at ∞ is reduced to the study of iteration of polynomial functions. Moreover, we shall see that such functions which are non-constant are surjective. The advantage of this is that we can reduce the study of $\{f^n\}$ on some wonky G to the study of $\{p^n\}$ on $\mathbb{C}$. To see this, start with $f : G \to G$ is holomorphic on G, without essential singularity at ∂G. Assume that G is a simply connected domain which is conformally equivalent to the plane. Let $\phi : G \to \mathbb{C}$ be the conformal map obtained through the uniformization theorem. Then $p = \phi \circ f \circ \phi^{-1} : \mathbb{C} \to \mathbb{C}$ is entire and without essential singularity at infinity. Consequently p is a polynomial. By the theorem below, we shall see that as long as f is non-constant, then p is surjective. Thus the study of $\{f^n\}$ on G is in this case reduced to the study of $\{p^n\}$ on $\mathbb{C}$.

Proposition 13.2. *Entire functions without essential singularity at infinity which are non-constant are surjective.*

Proof: By the theorem, such a function is a polynomial $p(z)$ of degree $d \geq 1$. Proceeding by contradiction we assume there is $q \in \mathbb{C}$ such that $p(z) \neq q$ for all $z \, in \mathbb{C}$. Then the function

$$\frac{1}{p(z)-q}$$

is entire. Moreover, since $|p(z)| \sim |z|^d$ as $|z| \to \infty$, it follows that this function tends to zero at infinity and hence is bounded. By Liouville the function is constant, which furthermore implies that p is constant which it is not. Therefore the assumption that $p(z) \neq q$ for all $z \, in \mathbb{C}$ must be false, and hence p is surjective. 

Next let's consider holomorphic dynamics for meromorphic functions on $\hat{\mathbb{C}}$.

Theorem 13.3. *Any meromorphic function on $\hat{\mathbb{C}}$ is a rational function. If it is non-constant, then it is surjective.*

Proof: Let's assume $f(z)$ is non-constant and meromorphic. Let $\{p_k\}_1^n$ be the poles of f with corresponding degrees d_k. Then

$$F(z) := f(z) \prod_1^n (z - p_k)^{d_k}$$

is entire, and has at worst a pole at ∞. Therefore this function is a polynomial $q(z)$ and hence

$$f(z) = \frac{q(z)}{\prod_1^n (z - p_k)^{d_k}}$$

is a rational function. To show surjectivity first note that a meromorphic function defined on $\hat{\mathbb{C}}$ without pole is constant by Liouville's theorem (it is entire and bounded!) Therefore the value ∞ is assumed at a pole. For $p \neq \infty$, for the sake of contradiction we assume $f(z) \neq p$ for all $z \in \hat{\mathbb{C}}$. The function $f(z) - p$ may have poles, but it has no zeros, so

$$\frac{1}{f(z) - p} = g(z)$$

is entire. It has at worst a pole at infinity. If it has no pole at infinity, then it is constant and hence so is f which is a contradiction. So, this function has a pole at infinity and hence is a polynomial. Therefore

$$f(z) - p = \frac{1}{g(z)} \to 0 \text{ as } z \to \infty.$$

Since f is meromorphic, this shows that

$$f(z) \to p \text{ as } z \to \infty \implies f(\infty) = p.$$

Hence f does assume the value p since $\infty \in \hat{\mathbb{C}}$.
So, holomorphic dynamics for meromorphic functions on $\hat{\mathbb{C}}$ is reduced to the study of iteration of rational functions. We can also reduce holomorphic dynamics on any simply connected domain, G, which is conformally equivalent to $\hat{\mathbb{C}}$ to the study of the iterates of a rational function. To see this, assume that $f : G \to G$ is meromorphic. Assume that $\phi : G \to \hat{\mathbb{C}}$ is given by the Uniformization Theorem. Then, the study of the iterates of f is equivalent to the study of the interates of $r = \phi \circ f \circ \phi^{-1}$. Since f is meromorphic on $\hat{\mathbb{C}}$, it follows that $r : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is meromorphic. Hence it is a rational function. If it is non-constant, then it is surjective as well.

13.1. **What about essential singularities?** For examples, functions which have essential singularity at infinity, like e^z, or functions which have a discrete set of essential singularities, like exponentiating a rational function? Well, I did a bit of research into this and found a paper which has a nice historic introduction to the general field of complex dynamics, followed by some interesting results. Guess the year of the paper? It is an arxiv preprint from 2017 (!!!!!) Yeah, so this seems to be relatively unexplored territory.

Exercise 22. *Read this paper:* https://arxiv.org/pdf/1705.03960*.*

13.2. **Fixed points.** A likely candidate for the Fatou set is a point z_0 such that

$$f(z_0) = z_0.$$

Then, we at least know that $f^n(z_0)$ converges to z_0, because it is a rather monotonous convergent sequence (it is constant). However, to belong to the Fatou set, we need to know that f^n is normal in a *neighborhood* of the point z_0. To understand this, we need to understand different types of fixed points.

Definition 13.4. Let f be holomorphic in a neighborhood of z_0 and assume $f(z_0) = z_0$. The value $\lambda := f'(z_0)$ is known as the *multiplier* at the fixed point z_0.

(1) If $|\lambda| < 1$, then z_0 is an *attracting* fixed point. If $|\lambda| = 0$, then z_0 is a *super-attracting* fixed point. We shall see that in both of these cases, z_0 belongs to the Fatou set.
(2) If $|\lambda| > 1$, then z_0 is a *repelling* fixed point. We shall see that in this case, z_0 belongs to the Julia set.
(3) If there exists $n \in \mathbb{N}$ such that $\lambda^n = 1$, then z_0 is a *rationally neutral* fixed point. This is subtle, but in this case as well, one can show that z_0 belongs to the Julia set (this is an exercise).
(4) Otherwise z_0 is an *irrationally neutral* fixed point. This is quite subtle. We shall see that such a fixed point belongs to the Fatou set if and only if $\{f^n\}$ stay uniformly bounded in some neighborhood of z_0. When the multiplier is of the form $e^{2\pi i\theta}$ where θ is Diophantine (we shall define this later), then Siegel proved that a neighborhood of z_0 belongs to the Fatou set (so z_0 also belongs to the Fatou set). In the special case of quadratic polynomials, there is a necessary and sufficient condition on the multiplier to guarantee that the fixed point belongs to the Fatou set (this will be an exercise, but not for the faint hearted).

We recall here the definition of conformally conjugate functions.

Definition 13.5 (Conformally conjugate)**.** We say that $f : U \to U$ is *conformally conjugate* to $g : V \to V$, if there exists a conformal $\varphi : U \to V$ such that $g = \varphi \circ f \circ \varphi^{-1}$. (Schröder's equation)
(g and f are like the same, only in different coordinate systems).

Note 1. *If $g = \varphi \circ f \circ \varphi^{-1}$, then z_0 is a fixed point for f if and only if $\varphi(z_0)$ is a fixed point for g.*

Proposition 13.6. *If f and g are conformally conjugate, then the multiplier λ at a fixed point for f is the same for g. In words: The multiplier is* invariant *under conjugation by conformal maps.*

Proof: $f'(z_0) = \lambda$. $g = \varphi \circ f \circ \varphi^{-1}$ if and only if $g \circ \varphi = \varphi \circ f$. Therefore,

$$\underbrace{(g \circ \varphi)'(z_0)}_{g'(\varphi(z_0))\varphi'(z_0) = \lambda_g \varphi'(z_0)} = (\varphi \circ f)'(z_0) = \varphi'(\underbrace{f(z_0)}_{z_0})\lambda_f$$

With φ conformal it follows that $\varphi'(z_0) \neq 0$. Thus $\lambda_g = \lambda_f$.

Remark 10. At a fixed point, we have $f(z_0) = z_0$. Letting $T(z) = z + z_0$, then defining

$$\widetilde{f} = T^{-1} \circ f \circ T,$$

note that T is a conformal map! Hence, f and $\widetilde{f}$ are conformally conjugate. We have

$$\widetilde{f}(0) = 0, \widetilde{f}^n = T^{-1} \circ f^n \circ T.$$

So in the general study of fixed points, we lose no generality by assuming the fixed point is at zero!

Proposition 13.7. *Let z_0 be an attracting fixed point for an holomorphic function f on $D_r(z_0)$. Then there exists $0 < p \leq r$ such that*

$$f^n(z) \to z_0$$

on $D_p(z_0)$.

Proof: Since f is holomorphic on $D_r(z_0)$, we can write it as a power series

$$f(z) = \sum_{k\geq 0} a_k (z-z_0)^k = \underbrace{a_0}_{=z_0} + \lambda(z-z_0) + (z-z_0)^1 \underbrace{\sum_{k\geq 0} a_{k+2}(z-z_0)^{k+1}}_{\text{this is a convergent power series on } D_r(z_0)}$$

For $\Lambda \in (|\lambda|, 1)$, note that

$$|f(z) - f(z_0)| = \left|\lambda(z-z_0) + (z-z_0)\sum_{k\geq 0} a_{k+2}(z-z_0)^{k+1}\right| \leq |\lambda||z-z_0| + |z-z_0||z-z_0| \underbrace{\left|\sum_{k\geq 0} a_{k+2}(z-z_0)^k\right|}_{\text{convergent}}$$

Since $\sum_{k\geq 0} a_{k+2}(z-z_0)^k$ converges in $D_r(z_0)$, it follows that there is $M > 0$ such that

$$\left|\sum_{k\geq 0} a_{k+2}(z-z_0)^k\right| < M \text{ for all } z \in D_{r/2}(z_0).$$

Then, letting

$$p = \min\left\{\frac{r}{2}, \frac{\Lambda - |\lambda|}{M}\right\},$$

we also get the same inequality

$$\left|\sum_{k\geq 0} a_{k+2}(z-z_0)^k\right| < M \text{ for all } z \in D_p(z_0).$$

Therefore,

$$|f(z) - f(z_0)| \leq \underbrace{|\lambda||z-z_0| + \frac{\Lambda - |\lambda|}{M} M |z-z_0|}_{=\Lambda|z-z_0|}$$

on $D_p(z_0)$. Since $\Lambda < 1$ we have

$$|f(z) - f(z_0)| = |f(z) - z_0| \leq \Lambda|z-z_0| \leq |z-z_0|$$

which shows that

$$f(D_p(z_0)) \subset D_p(z_0).$$

Hence we can apply our estimate to $f(f(z))$ since $f(z) \in D_p(z_0)$ presuming $z \in D_p(z_0)$, and we have

$$|f^2(z) - f^2(z_0)| \leq \Lambda|f(z) - f(z_0)| \leq \Lambda^2|z-z_0|$$

and in general

$$|f^n(z) - f^n(z_0)| = |f^n(z) - z_0| \leq \Lambda^n|z-z_0| \to 0$$

as $n \to \infty$ because $\Lambda < 1$. This proves that $f^n(z) \to z_0$ for all $z \in D_p(z_0)$.

Definition 13.8 (Basin of attraction)**.** For an attracting fixed point z_0, the *basin of attraction* of z_0 is

$$A(z_0) := \{z | f^n(z) \text{ is defined for all z and } f^n(z) \to z_0, \text{ as } n \to \infty\}.$$

We have proven that $D_p(z_0) \subset A(z_0)$.

Proposition 13.9.

$$A(z_0) = \bigcup_{n \geq 1} f^{-n}(D_p(z_0)).$$

Proof: We do the standard argument, showing that the sets on the left and right are contained in each other. Hence we will conclude that they are equal.
"$\subseteq$:" Let $z \in A(z_0)$. Then, by definition $f^m(z) \to z_0$. Since $f^m(z) \to z_0 \in D_p(z_0)$, where p is the same as the radius of the disk in the preceding proposition, there is some such N such that for all $m \geq N$, $|f^m(z) - z_0| < p$. Hence, we have

$$f^m(z) \in D_p(z_0) \implies z \in f^{-m}(f^m(z)) \in f^{-m}(D_p(z_0)).$$

This means that

$$z \in f^{-m}(D_p(z_0)) \subset \cup_{n \geq 1} f^{-n}(D_p(z_0)).$$

Since $z \in A(z_0)$ was arbitrary, this shows that

$$A(z_0) \subseteq \bigcup_{n \geq 1} f^{-n}(D_p(z_0)).$$

"$\supseteq$:" If $z \in f^{-n}(D_p(z_0))$ for some $n \geq 1$, then

$$f^n(z) \in D_p(z_0).$$

We proved that $f^k(w) \to z_0$ for all $w \in D_p(z_0)$ in the preceding proposition. Hence taking

$$w = f^n(z) \implies f^k(w) = \underbrace{f^k(f^n(z))}_{f^{n+k}(z)} \to z_0.$$

By definition, we conclude that $z \in A(z_0)$. Since $z \in f^{-n}(D_p(z_0))$ was arbitrary, this shows the containment in this direction. 

Corollary 13.10. $A(z_0)$ *is open.*

Proof: Since f is continuous, $f^{-n}(D_p(z_0))$ is open for all $n \geq 1$. Arbitrary unions of open sets remain open. 

Definition 13.11 (Immediate basin of attraction)**.** The connected component of $A(z_0)$ which contains z_0 is the *immediate basin of attraction*, denoted $A^*(z_0)$.

13.3. **Homework.**

(1) Prove that every polynomial has a super attracting fixed point at ∞.
(2) Let $T_n(z) = \cos(n \arccos z)$ be the n^{th} Tchebycheff polynomial. Let $F_n(z) = 2T_n(z/2)$. Determine a conjugation of F_n and ζ^n.
(3) Schröder was motivated by Newton's method, to understand the question of conformal conjugation. Consider a quadratic polynomial which has simple zeros. Determine the basins of attraction for Newton's method.
(4) Assume that f has a fixed point at z_0 with multiplier $\lambda \neq 0$, such that λ is not a root of unity. Assume that there exists a conformal map ϕ which conjugates f to $g(\zeta) = \lambda\zeta$. Prove that the conjugation ϕ is unique up to a scale factor. What can be said in case λ is a root of unity?

(5) Show that $z = -i \tan \zeta$ conjugates
$$\frac{2z}{1+z^2}$$
to 2ζ.
(6) Compute the basin of attraction for the super-attracting fixed point at ∞ for the polynomial $P(z) = z^2 - 2$.

13.4. **Hints.**

(1) Just do it.
(2) Consider the conjugation $z = h(\zeta) = \zeta + 1/\zeta$. For $|\zeta| = 1$, use the fact that for $\zeta = e^{iw}$ then $h(\zeta) = 2\cos w$.
(3) Show that $f(z) = z - P(z)/P'(z)$ (this is the function you are iterating, which you ought to have figured out because you learned Newton's method in first year analysis) has superattracting fixed points at the zeros of P and a repelling fixed point at ∞. Use the Möbius transformation $\zeta = \phi(z)$ to send the two zeros of p to 0 and ∞, and send ∞ to 1. Show that this conjugates $f(z)$ to ζ^2. Show that the midpoint of the line segment joining the zeros of p is mapped by f to ∞, and that the midpoint is sent by ϕ to the preimage -1 of 1. Show that the perpendicular bisector of the line segment joining the zeros of P is mapped by ϕ to the unit circle. Use this to show that the basis of attraction for Newton's method are the respective open half-planes on either side of the bisector.
(4) Show that any conjugation of $f(z) = \lambda z$ is a constant multiple of z. Do this by considering the power series of the conjugating function and determining the coefficients.
(5) The double angle formula for tangent.
(6) The double angle formula for cosine. Consider the conformal map $h(\zeta) = \zeta + 1/\zeta$ for $\{|\zeta| > 1\} \mapsto \mathbb{C} \setminus [-2, 2]$. Use this to conjugate P to ζ^2. In this way show that the dynamics of P on $\mathbb{C} \setminus [-2, 2]$ are the same as those of ζ^2 on $\{|\zeta| > 1\}$.

14. Fixed points and conformal conjugation

Note that near a fixed point
$$f(z) = z_0 + \lambda(z - z_0) + \dots, \quad \lambda \neq 0,$$
or presuming f is non-constant, then if $\lambda = 0$ there is some $p \in \mathbb{N}$ such that
$$f(z_0) = z_0 + a_p(z - z_0)^p + \dots.$$
Since the dynamics of f, by which we mean the behavior of the iterates of f, are the same as the dynamics of $\widetilde{f} = \phi^{-1} \circ f \circ \phi$ with
$$\phi(z) = z + z_0,$$
and $\widetilde{f(0)} = 0$, let's assume $z_0 = 0$. Then near the fixed point
$$f(z) = \lambda z + \dots, \quad \text{or} \quad a_p z^p + \dots.$$
So, roughly speaking f looks like either λz or $a_p z^p$. Let's call that function g (either $g(z) = \lambda z$ if $\lambda \neq 0$ or $g(z) = a_p z^p$ if $\lambda = 0$). These functions are significantly more simple than f. Schröder asked the question:

Question 1. *Does there exist a neighborhood of the fixed point and a holomorphic map ψ which conjugates f to g? In other words, does there exist a solution ψ to*
$$\psi \circ f = g \circ \psi?$$
Assuming the fixed point $z_0 = 0$, by which no generality is lost, we also demand that
$$\psi(0) = 0, \quad \psi'(0) = 1.$$

This equation is known as Schröder's equation. Note that it immediately implies that ψ^{-1} is uniquely defined on $g(\psi(D_r))$ via $\psi^{-1}(g(\psi(x)) = f(g(\psi(x)))$ and hence any solution to Schröder's equation is a locally conformal map. Solving Schröder's equation turns out to be super important for understanding whether or not fixed points belong to the Fatou set.

Theorem 14.1 (Koenig's)**.** *Let f have an attracting fixed point z_0 with $0 < |\lambda| < 1$. Then there exists a conformal mapping $\varphi(z)$ that maps a neighborhood of z_0 onto a neighborhood $D_r(0)$ of zero, such that*

$$\lambda\varphi(z) = \varphi(f(z)), \quad \varphi'(z_0) = 1.$$

Moreover, φ is unique up to multiplication by $c \neq 0$. This shows that $\varphi \circ f \circ \varphi^{-1}(z) = \lambda z$, and therefore f is conformally conjugate to multiplication by λ in a neighborhood of the fixed point.

Proof: Without loss of generality, let $z_0 = 0$. Note that we can do this because we may just complete the proof for

$$\widetilde{f} := T \circ f \circ T^{-1}, \quad T(z) = z - z_0, \quad T^{-1}(z) = z + z_0.$$

Then 0 is a fixed point for $\widetilde{f}$, and we have proven that the multiplier for $\widetilde{f}$ is the same as that for f. So, proving the theorem for $\widetilde{f}$, we obtain $\widetilde{\varphi}$ for $\widetilde{f}$, so that

$$\lambda\widetilde{\varphi} = \widetilde{\varphi}\tilde{f} = \widetilde{\varphi}T \circ f \circ T^{-1} \implies \lambda\widetilde{\varphi} \circ T = \widetilde{\varphi}T \circ f.$$

Thus, we use

$$\varphi := \widetilde{\varphi} \circ T.$$

Then Schröder's equation is solved for f with φ. Moreover,

$$\varphi'(z_0) = \widetilde{\varphi}'(T(z_0)) = \widetilde{\varphi}'(0) = 1.$$

So, we may indeed lose no generality by assuming the fixed point is at 0. Now, since 0 is an attracting fixed point, let us assume for the rest of this argument that $z \in D_p(0)$, where p is from our preceding propositions. In particular, this guarantees that $f^n(z)$ is well defined, since $f : D_p(0)$ into itself. Hence, every time we compose f with itself, the result has a power series which converges in $D_p(0)$. Now, let

$$\varphi_n(z) := \lambda^{-n} f^n(z).$$

First, note that since λ is a non-zero complex number, by the preceding considerations, $\varphi_n(z)$ is holohomrphic in $D_p(0)$. Therefore, it has a convergent power series. We claim that this power series is of the form:

$$\varphi_n(z) = z + \sum_{k \geq 2} a_k z^k. \tag{14.1}$$

The proof is by induction. For $n = 1$, we have

$$\varphi_1(z) = \lambda^{-1} f(z) = z + \sum_{k \geq 2} \lambda^{-1} c_k z^k,$$

since f is holomorphic and therefore has a power series of the form

$$f(z) = \sum_{k \geq 1} c_k z^k, \quad c_1 = \lambda,$$

since $f(0) = 0$. So, the base case is true.
Continuing inductively, let us assume that the result holds for n. Then, for $n + 1$, we first note that φ_{n+1} has a convergent power series. Moreover, by its definition:

$$\varphi_{n+1}(z) = \lambda^{-1}\lambda^{-n} f(f^n(z)) = \lambda^{-1}\lambda^{-n} \sum_{k \geq 1} c_k (f^n(z))^k = \lambda^{-1}\lambda^{-n}\lambda f^n(z) + \lambda^{-1}\lambda^{-n} \sum_{k \geq 2} c_k (f^n(z))^k$$

$$= \lambda^{-n} f^n(z) + \sum_{k \geq 2} \lambda^{-1-n} c_k((f^n(z))^k = \varphi_n(z) + \sum_{k \geq 2} \lambda^{-1-n} c_k((f^n(z))^k.$$

By the induction assumption,

$$\varphi_n(z) = z + \sum_{k\geq 2} a_k z^k.$$

Hence,

$$\varphi_{n+1}(z) = z + \sum_{k\geq 2} a_k z^k + \lambda^{-1-n} c_k((f^n(z))^k.$$

We note that since

$$f(z) = z\left(\lambda + \sum_{k\geq 2} c_k z^k\right) \implies f^n(z) = z^n\left(\lambda + \sum_{k\geq 2} c_k z^k\right)^n.$$

Consequently, the terms in the series

$$\sum_{k\geq 2} a_k z^k + \lambda^{-1-n} c_k((f^n(z))^k$$

have powers of z starting with z^2 and increasing. Since φ_{n+1} has a convergent power series, it follows that its convergent power series is of the form

$$\varphi_{n+1}(z) = z + \sum_{n\geq 2} b_k z^k.$$

This proves the desired statement. (Note that the fact they were named a_k in (14.1) is irrelevant - the point is that the first term in the power series is z, and the rest of the terms have higher powers of z going from z^2 on upwards).

Now, we want to prove that the φ_n actually converge to something. By the convergence of the power series of f, there exists $c > 0$ fixed, and $\delta > 0$ (without loss of generality also assume that $\delta < p$) such that

$$(*)\ |f(z) - \lambda z| \leq c|z|^2 \text{ for } |z| < \delta.$$

Thus

$$|f(z)| \leq |\lambda||z| + c|z|^2 \leq (|\lambda| + c\delta)|z|.$$

We can now choose δ small enough such that $(|\lambda| + c\delta) < 1$ (which is possible because $|\lambda| < 1$). Then this implies two things. Since

$$|z| < \delta \implies |f(z)| \leq (|\lambda| + c\delta)|z| < |z| < \delta,$$

this shows that

$$(**) f(D_\delta(0)) \subset D_\delta(0), \quad f^n(D_\delta(0)) \subset D_\delta(0), \quad \text{and } |f^n(z)| \leq (|\lambda| + c\delta)^n|z|, \quad |z| \leq \delta.$$

Choose δ possibly smaller so that $(|\lambda + c\delta|)^2 \leq |\lambda|^2 + 2|\lambda|c\delta + c^2\delta^2 < |\lambda|$. Then

$$|\varphi_{n+1}(z) - \varphi_n(z)| = |\lambda^{-n-1} f^n(f(z)) - \lambda^{-n} f^n(z)| = \left|\frac{f(f^n(z)) - \lambda f^n(z)}{\lambda^{n+1}}\right|$$

$$\overset{(*)+(**)}{\leq} \frac{c|f^n(z)|^2}{|\lambda^{n+1}|} \overset{(**)}{\leq} \frac{c(|\lambda| + c\delta)^{2n}|z|^2}{|\lambda|^{n+1}} = \frac{c\rho^n|z|^2}{|\lambda|} \text{ where } \rho := \frac{(|\lambda| + c\delta)^2}{|\lambda|} < 1.$$

Now, we have shown that

$$|\varphi_{n+1}(z) - \varphi_n(z)| < \rho^n \frac{c|z|^2}{|\lambda|}, \quad |z| < \delta.$$

Without loss of generality, choose δ perhaps a bit smaller to guarantee that

$$\frac{c\delta^2}{\lambda} < 1.$$

Then we have

$$|\varphi_{n+1}(z) - \varphi_n(z)| < \rho^n \quad \forall z \in D_\delta(0).$$

Consequently, for all $k \geq 1$ we can estimate

$$|\varphi_{n+k}(z) - \varphi_n(z)| \leq \sum_{j=1}^{k} |\varphi_{n+j}(z) - \varphi_{n+j-1}(z)| \leq \sum_{j=1}^{k} \rho^{n+j-1} = \sum_{l=n}^{n+k} \rho^l.$$

Since $0 < \rho < 1$, the series

$$\sum_{l=1}^{\infty} \rho^l$$

converges. The tail of a convergent series can be made as small as we like. Hence, given any $\epsilon > 0$, we may choose n sufficiently large such that

$$\sum_{l=n}^{\infty} \rho^l < \epsilon.$$

We therefore have

$$|\varphi_{n+k}(z) - \varphi_n(z)| \leq \sum_{l=n}^{n+k} \rho^l \leq \sum_{l=n}^{\infty} \rho^l < \epsilon.$$

This shows that $\{\varphi_n(z)\}$ is Cauchy for all $|z| \leq \delta$, and moreover, converges uniformly there. We therefore have a holomorphic limit function, φ.
Let us see what happens with the limit function. The way we have defined it,

$$\varphi_n = \lambda^{-n} f^n(z) \implies \varphi_n \circ f = \lambda^{-n} f^{n+1} = \lambda(\lambda^{-n-1} f^{n+1}) = \lambda \varphi_{n+1}.$$

So, we have

$$\varphi_n \circ f = \lambda \varphi_{n+1}.$$

Since φ_n converges uniformly on $D_\delta(0)$, and $f : D_\delta(0)$ to itself, taking the limit on both sides we obtain

$$\varphi \circ f = \lambda \varphi.$$

By the locally uniform convergence of $\varphi_n \to \varphi$, using the Cauchy Integral Formula we obtain that all the derivatives converge as well.

Exercise 23. *Write up the details of this argument. In particular, prove that if $\varphi_n \to \varphi$ locally uniformly, then we also have $\varphi_n' \to \varphi'$ locally uniformly. Repeating, one obtains the locally uniform convergence of all derivatives.*

Since $\varphi_n'(0) = 1$ for all n, it follows that $\varphi'(0) = 1$ as well. We can therefore decrease the radius, δ, of our disk a bit more, to guarantee that φ is injective with $\varphi' \neq 0$ on $D_\delta(0)$. It is therefore a biholomorphic (conformal) map onto its image. So, we have obtained the
The last thing to prove is the uniqueness statement. On $D_\delta(0)$ assume that we have

$$\phi(f(z)) = \lambda \phi(z)$$

then

$$\phi(f(0)) = \phi(0) = \lambda \phi(0),$$

and $\lambda \neq 1$, which forces

$$\phi(0) = 0.$$

The same argument shows that $\varphi(0) = 0$. Near zero, we may therefore define

$$\psi = \varphi \circ \phi^{-1}.$$

Since ϕ and φ are both conformal, we have

$$f(z) = \phi^{-1} \circ \lambda \circ \phi = \varphi^{-1} \circ \lambda \circ \varphi.$$

So, in particular

$$\varphi \circ \phi^{-1} \circ \lambda \circ \phi = \lambda \circ \varphi \implies \varphi \circ \phi^{-1} \circ \lambda = \lambda \circ \varphi \circ \phi^{-1} \implies \psi \circ \lambda = \lambda \circ \psi.$$

Since $\psi(0) = 0$, the power series of ψ is of the form
$$\psi(z) = \sum_{k \geq 1} a_k z^k.$$
On the one hand then
$$\psi \circ \lambda(z) = \sum_{k \geq 1} a_k \lambda^k z^k.$$
On the other hand, this is equal to
$$\lambda \circ \psi(z) = \lambda \sum_{k \geq 1} a_k z^k.$$
By the uniqueness of coefficients in these expansions, we must have
$$a_k \lambda^k = \lambda a_k \quad \forall k \geq 1.$$
For $k = 1$ this is fine, but since $\lambda \neq 1$, the only way this equation is satisfied for $k \geq 2$ is when
$$a_k = 0 \quad \forall k \geq 2.$$
Thus
$$\psi(z) = \varphi \circ \phi^{-1}(z) = cz,$$
for some non-zero $c \in \mathbb{C}$. Consequently,
$$\varphi(z) = c \circ \mathrm{id} \circ \phi(z) = c\phi(z).$$

Here id is the identity map, $\mathrm{id}(z) = z$.

Remark 11. Note that an equivalent formulation, since φ is conformal if and only if φ^{-1} is conformal, is to require the existence of a conformal map h such that
$$f(h(z)) = h(\lambda z).$$
To see this, just let $h = \varphi^{-1}$. By what we proved, we have
$$\lambda \circ \varphi = \varphi \circ f \implies \lambda z = \varphi \circ f \circ \varphi^{-1}(z) \implies$$
$$\varphi^{-1}(\lambda z) = f(\varphi^{-1}(z)).$$
This is
$$h(\lambda z) = f(h(z)).$$

As a corollary, we can obtain the same result for repelling fixed points!

Corollary 14.2. *If z_0 (WLOG $= 0$) is repelling, then $\exists!$ (up to $\star$ by $c \neq 0$) conformal ϕ conjugating $f(z)$ to λz.*

Proof: By assumption, there is some $r > 0$ so that we can write
$$f(z) = \lambda z + \ldots$$
on $D_r(0)$. Since $|\lambda| > 1 > 0$, then $f'(0) \neq 0$. We can take $r > 0$ sufficiently small, so that and we may assume
$$|f'(z)| \geq \frac{\lambda}{2}$$
on $D_r(0)$. Moreover, we can also take r sufficiently small so that f is bijective from $D_r(0)$ to the image of this set under f. Therefore, we have proven that f^{-1} is holomorphic on $f(D_r(0))$. Furthermore, since $f(0) = 0$, by the chain rule,
$$f^{-1}(0) = 0 \text{ and } (f^{-1})'(0) = \lambda^{-1}.$$

Hence, 0 is an attracting fixed point for f^{-1}. We may therefore apply Koenig's Theorem to f^{-1}. This gives the existence which is unique, up to scaling by a non-zero constant, of ϕ conjugating f^{-1} to λ^{-1}. In particular, there is ϕ which is biholomorphic on some disk about zero, with

$$\phi(0) = 0, \quad \phi'(0) = 1$$

and

$$\phi(f^{-1}(z)) = \lambda^{-1}\phi(z) \implies \lambda \circ \phi = \phi \circ f.$$

The uniqueness follows from Koenig's Theorem.

14.1. **Homework.**

(1) Prove that all rationally netural fixed points belong to the Julia set.
(2) Locate a proof of Pfeiffer's theorem from 1917 and read it. The theorem states: there is $\lambda = e^{2\pi i \phi}$ so that the Schröder Equation has no solution for any polynomial f.
(3) A number ϕ is *Diophantine* (badly approximated by rational numbers) if there exists $c > 0, \mu < \infty$ so that $|\phi - \frac{p}{q}| \geq \frac{c}{q^\mu}$ for all $p, q \in \mathbb{Z}, q \neq 0$. This is equivalent to $|\lambda^n - 1| \geq cn^{1-\mu}$ for all $n \geq 1$. Which real numbers are Diophantine? Which are not? Quantify the set of Diophantine real numbers measure theoretically in terms of Hausdorff measure and dimension.
(4) We used an argument to obtain convergence in Koenig's theorem which is an example of a general analysis trick. Prove that if you can show that for a family of functions $\{f_n\}$ and for all $z \in K$ for some compact K that

$$|f_{m+1}(z) - f_m(z)| < c^m \quad \forall z \in K, \quad \text{for all } m \geq N \text{ for some } N,$$

where the constant $c < 1$, then the sequence $\{f_n\}$ converges uniformly on K.
(5) Locate a proof of Siegel's theorem and read it. The theorem states: if ϕ is Diophantine, $f(0) = 0,\ f'(0) = \exp 2\pi i \phi$, then there exists a solution h to Schröder's Equation.

14.2. **Hints.** Rationally neutral fixed points, cases:

(1) $\lambda = 1, p = 1$
(2) $\lambda = 1, p > 1$
(3) $\lambda^n = 1, \lambda \neq 1$

Write $f(z) = \lambda z + az^{p+1} + \ldots, a \neq 0$.
Case 1: Conjugate f by $\varphi(z) = az \to \tilde{f} = \varphi \circ f \circ \varphi^{-1} = a(f(\frac{z}{a})) = a(\frac{\lambda z}{a}) + a(\frac{z}{a})^2 + \ldots) = \lambda z + z^2 + \cdots \Rightarrow \text{WLOG} a = 1$. Move 0 to ∞ by $z \to \frac{-1}{z} \to g(z) = z + 1 + \frac{b}{z} + \ldots$. Fatou proved that φ conjugates g to $z \to z + 1$.
Case 2: Another conjugation.
Case 3: Reduce to case 1 or case 2 by considering f^n.
Conclude that at such a fixed point, there are both "repelling" and "attracting" directions. Thus all rationally neutral fixed points are in $\mathcal{J}$.

15. Super attracting fixed points, irrationally neutral fixed points and iteration of rational functions

We shall prove the classification theorem for fixed points of holomorphic functions. We begin by proving the Koenig's egg theorem.

Theorem 15.1 (Conformal conjugation at super-attracting fixed points)**.** *If f has a super-attracting fixed point, which without loss of generality is located at zero, then for $f \not\equiv 0\,\exists!$ (up to $p-1$ root of unity) conformal ϕ such that $\phi(0) = 0$, and*

$$f \circ \phi = g \circ \phi, \quad g(z) = z^p,$$

where $f(z) = az^p + \dots$ is a holomorphic function in a neighborhood of $z = 0$.

Proof: Fix

$$c > 1, \quad r > 0$$

such that

$$|f(z)| \le c|z|^p \quad \forall z \in D_r(0).$$

Note that we can do this because f has a convergent power series, as it is holomorphic in a neighborhood of $z = 0$,

$$f(z) = az^p + \sum_{k \ge p+1} a_k z^k.$$

Now, choose

$$\delta = \min\left\{\frac{1}{c^{\frac{1}{p-1}}}, \frac{c}{2}, 1, r\right\}.$$

Then, note that

$$|f(z)| \le c|z|^p \le c\delta^{p-1}\delta \le \delta.$$

Hence f maps the closed disk of radius δ into itself. Consequently, f^n also maps this disk into itself.

Claim 2.

$$|f^n(z)| \le cc^{p^{n-1}}|z|^{p^n} \quad \forall z \in D_\delta(0), \quad \forall n \ge 1.$$

The proof is by induction and teamwork. For the base case, we have the estimate

$$|f(z)| \le c|z|^p.$$

This is the statement of the claim when $n = 1$. Now, assume that the claim holds for some n. Since $f^n(z)$ is in the closed disk of radius δ, we get the estimate

$$|f^{n+1}(z)| = |f(f^n(z))| \le c|f^n(z)|^p \le c(c^{p^{n-1}}|z|^{p^n})^p = cc^{p^n}|z|^{p^{n+1}}.$$

This is the statement of the claim for $n+1$. So it's true. Next, note that by our choice of δ, we have

$$|f^n(z)| \le cc^{p^{n-1}}|z|^{p^n} \le cc^{p^n}\delta^{p^n} \le \frac{c}{2^n} \to 0 \text{ as } n \to \infty.$$

So f is converging to zero in $D_\delta(0)$.
We wish to obtain the conformal map. For this purpose, let

$$b := \left(\frac{1}{|a|}\right)^{\frac{1}{p-1}} \cdot e^{\frac{i(2\pi - \theta)}{p-1}},$$

where

$$a = |a|e^{i\theta}, \quad \theta \in [0, 2\pi).$$

We define first

$$\phi(z) := bz.$$

This is a conformal map since $b \ne 0$. (Why is this true? Think about it!) Then

$$f \cong \tilde{f} = \phi^{-1} \circ f \circ \phi,$$

and

$$\tilde{f} = b^{-1}(a(bz)^p + \dots) = ab^{p-1}z^p + \dots = z^p + \dots.$$

Since f and $\tilde{f}$ are conformally conjugate, we may simplify life a bit by assuming that

$$f(z) = z^p + \dots.$$

(That is, we may without loss of generality assume that $a = 1$).

Define
$$\phi_n(z) := (f^n(z))^{p^{-n}}.$$
This is tricky business, because of that fractional exponent. Fractional exponent are really properly defined in $\mathbb{C}$ via the complex logarithm, using
$$w^x = e^{\log(w)x}.$$
So, for this to be well-defined, it is very important to know that w is staying in some region of $\mathbb{C}$ where we can define the logarithm. Any small neighborhood away from zero will be fine. This looks a bit scary though, because our function f is converging to zero quite fast... However, note that
$$f(z) = z^p(1 + \sum_{k\geq p+1} a_k z^{k-p}).$$
So, we compute that
$$f^n(z) = z^{p^n}(1 + \sum_{k\geq p+1} a_k z^{k-p})^n.$$
Since f is holomorphic on D_δ and maps this disk to itself, f^n is holomorphic for all n, and therefore the term
$$(1 + \sum_{k\geq p+1} a_k z^{k-p})^n = 1 + \sum_{j\geq 1} \alpha_j z^j.$$
The series on the right is the convergent power series for the holomorphic function, f^n. Hence
$$\phi_n(z) = (f^n(z))^{p^{-n}} = z\left(1 + \sum_{j\geq 1} \alpha_j z^j\right)^{\frac{1}{p^n}}. \tag{15.1}$$
The first part, z is totally fine. It is well defined everywhere. Also holomorphic everywhere. The second term is holomorphic for $|z|$ small, because due to the convergence of the series, the series can be estimated from above by $|z|$ times a constant. Hence, the expression in the parentheses can be made very close to one, taking z small. Therefore this fractional power is well defined.

We then compute that
$$\phi_{n-1} \circ f = (f^{n-1}(f(z)))^{p^{-n+1}} = (f^n(z))^{p^{-n}\cdot p} = (\phi_n(z))^p$$
Consequently, if we can prove that ϕ_n converges locally uniformly to some nice holomorphic ϕ, we will obtain in the limit that
$$\phi \circ f = (\phi)^p.$$
We will prove that
$$\prod_{n=1}^{N} \frac{\phi_{n+1}}{\phi_n} \text{ converges.}$$
Since this is a telescoping product, with
$$\prod_{n=1}^{N} \frac{\phi_{n+1}}{\phi_n} = \frac{\phi_{N+1}}{\phi_1},$$
This shows that
$$\frac{\phi_{N+1}}{\phi_1} \to \phi \implies \phi_{N+1}(z) \to \phi_1(z)\phi(z).$$
Compute for this purpose
$$\frac{\phi_{n+1}}{\phi_n} = \frac{(f^{n+1})^{p^{-n-1}}}{(f^n)^{p^{-n}}} = \frac{(f(f^n))^{p^{-1}p^{-n}}}{(f^n)^{p^{-n}}}.$$
Above we recognize
$$f(f^n))^{p^{-1}} = \phi_1(f^n) = (f^{n+1})^{p^{-1}}.$$

So

$$\frac{\phi_{n+1}}{\phi_n} = \frac{(\phi_1(f^n))^{p^{-n}}}{(f^n)^{p^{-n}}}.$$

Then, also

$$(\phi_1(f^n))^{p^{-n}} = \left((f^{n+1})^{p^{-1}}\right)^{p^{-n}} = (f^{n+1})^{p^{-n-1}}.$$

We write this equivalently as

$$\left[(f(f^n))^{p^{-1}}\right]^{p^{-n}}.$$

On the inside we use (15.1) to get this equal to

$$\left[(f^n)^{p/p}\left(1+\sum_{k\geq p+1} a_k(f^n(z))^{k-p}\right)^{p^{-1}}\right]^{p^{-n}}.$$

The coefficients above, a_k, come from the power series expansion of our function, f (itself). So, we get

$$\begin{aligned}\frac{\phi_{n+1}}{\phi_n} &= \frac{(f^n)^{p^{-n}}\left(1+\sum_{k\geq p+1} a_k(f^n(z))^{k-p}\right)^{p^{-n-1}}}{(f^n)^{p^{-n}}} \\ &= \left(1+\sum_{k\geq p+1} a_k(f^n(z))^{k-p}\right)^{p^{-n-1}}\end{aligned}$$

It is sufficient to prove that the product

$$\prod_{n\geq 1}\left|\frac{\phi_{n+1}}{\phi_n}\right|$$

converges. For $|z|$ small, by the estimate

$$|f^n(z)| \leq cc^{p^{n-1}}|z|^{p^n},$$

we have that

$$\left|1+\sum_{k\geq p+1} a_k(f^n(z))^{k-p}\right| \geq 1-\mathcal{O}(|z|) > 0.$$

So, the following will be helpful.

Exercise 24. *Prove that if $\{a_n\}_{n\geq 1}$ are all positive, then*

$$\prod_{n\geq 1} a_n$$

converges if and only if

$$\sum_{n\geq 1}\log a_n$$

converges.

Therefore, it suffices to prove that

$$\sum_{n\geq 1}\log\left(\frac{|\phi_{n+1}|}{|\phi_n|}\right)$$

converges, because we can apply the preceding exercise with

$$a_n = \frac{|\phi_{n+1}|}{|\phi_n|} > 0$$

for all n for all $|z| < \delta$. By our calculations above,

$$\frac{\phi_{n+1}}{\phi_n} = \left(1 + \sum_{k \geq p+1} a_k (f^n(z))^{k-p}\right)^{p^{-n-1}}$$

we get

$$\log \left|\frac{\phi_{n+1}}{\phi_n}\right| = \frac{1}{p^{n+1}} \log \left|1 + \sum_{k \geq p+1} a_k (f^n(z))^{k-p}\right|.$$

Since we can estimate

$$\left|1 + \sum_{k \geq p+1} a_k (f^n(z))^{k-p}\right| \leq 2,$$

our estimate becomes

$$\sum_{n \geq 1} \log \left|\frac{\phi_{n+1}}{\phi_n}\right| \leq 2 \sum_{n \geq 1} \frac{1}{p^{n+1}}.$$

This converges because $p \geq 2 > 1$.
Hence, we have that the product converges, since it converges absolutely, which also shows that

$$\lim_{n \to \infty} \phi_n(z) =: \phi(z)$$

and the convergence is uniform on $D_\delta(0)$, taking δ possibly smaller, but really, it ought to be small enough already.
Since the uniform limit of holomorphic functions is holomorphic, we have that ϕ is holomorphic.
We recall that so defined

$$\phi_{n-1} \circ f = (\phi_n)^p.$$

The right side is well defined as $n \to \infty$, since $\phi_n \to \phi$. The left side is also well defined. So, we obtain in the limit

$$\phi \circ f = (\phi)^p.$$

Why is this a conformal map? Recall (15.1). Differentiate:

$$\left(z\left(1 + \sum_{j \geq 1} \alpha_j z^j\right)^{\frac{1}{p^n}}\right)' = \left(1 + \sum_{j \geq 1} \alpha_j z^j\right)^{\frac{1}{p^n}} + z\left(\left(1 + \sum_{j \geq 1} \alpha_j z^j\right)^{p^{-n}}\right)'.$$

Now set $z = 0$. We get just one from the first term. Hence we have $\phi_n'(0) = 1$ for all n. Therefore, since the ϕ_n converge uniformly to ϕ, we also get $\phi'(0) = 1$, since ϕ is holomorphic (why?). Therefore, ϕ is bijective onto its image in some neighborhood of 0. Without loss of generality, take δ perhaps a bit smaller, so that ϕ is bijective from $D_\delta(0)$ to $\phi(D_\delta(0))$.
Finally, we demonstrate the uniqueness up to roots of unity. If $\psi \circ f = \psi^p$, with $\psi(0) = 0$, we muck around a bit

$$\psi \circ f = z^p \circ \psi \implies f = \psi^{-1} \circ z^p \circ \psi.$$

We also have

$$f = \phi^{-1} z^p \circ \phi.$$

So

$$\psi^{-1} \circ z^p \circ \psi = \phi^{-1} z^p \circ \phi \implies \phi \circ \psi^{-1} \circ z^p = z^p \circ \phi \circ \psi^{-1}.$$

Let us define

$$\Phi := \phi \circ \psi^{-1}.$$

Then

$$\Phi(z^p) = (\Phi(z))^p.$$

We consider the power series of Φ near zero. Since $\phi(0) = \psi(0) = 0$, we also have

$$\Phi(0) = 0.$$

So the power series looks like

$$\Phi'(0)z^p + \ldots = \left(\sum_{j \geq 1} \frac{\Phi^{(j)}(0)}{j!} z^j \right)^p = \Phi'(0)^p z^p + \ldots .$$

Since both ϕ and ψ are conformal, their derivatives cannot vanish, so neither can the derivative of Φ (why?), so this forces

$$\Phi'(0) = \Phi'(0)^p \implies \Phi'(0)^{p-1} = 1.$$

Next, we write out both sides more carefully:

$$\sum_{j \geq 1} \frac{\Phi^{(j)}(0) z^{pj}}{j!} = \left(\sum_{j \geq 1} \frac{\Phi^{(j)}(0)}{j!} z^j \right)^p .$$

On the right side, the terms start with z^p and then z^{p+1} etc. On the left side the terms between z^p and z^{2p} do not appear. Consequently, the coefficient on the right side for eg z^{p+1} must vanish. Since we have the series multiplied by itself p times, the coefficient of z^{p+1} comes from taking the z^1 term $p-1$ times and the z^2 term the last time. There are p ways to do this, so the coefficient is

$$p\Phi'(0)\frac{\Phi''(0)}{2!}.$$

Since $\Phi'(0) \neq 0$, this forces $\Phi''(0) = 0$.

Exercise 25. *Continue (suggestion: by induction) to prove that $\Phi^{(j)}(0) = 0$ for all $j \geq 2$.*

So, this means that $\Phi(z)$ is just given by $\Phi(z) = \lambda z$, where λ is a $p-1$ root of unity (that is $\lambda^{p-1} = 1$). Recalling that

$$\Phi := \phi \circ \psi^{-1} = \lambda z \implies \phi(z) = \lambda \psi(z).$$

It remains to study what happens at neutral fixed points.

15.1. **Neutral fixed points.**

Proposition 15.2 (The neutral case)**.** *Let $\lambda = e^{2\pi \imath \theta}$ where $\theta \in \mathbb{R}$. Assume that f is holomorphic and has a fixed point at $z = 0$ with multiplier λ. Then, if a holomorphic function h to Schröder's equation exists, formulated as*

$$f(h(z)) = h(\lambda z), \quad h'(0) = 1,$$

this h is injective in D_r for some $r > 0$.

Proof: Since $h'(0) = 1 \neq 0$, this shows that $h' \neq 0$ in some disk about zero, because h' is also holomorphic and thus continuous. It was an exercise to prove that a holomorphic function

whose derivative is non-zero is locally injective. The exercise completes the proof.

Proposition 15.3. *Just for fun, it is nice to include the fact that if a holomorphic function has a non-zero derivative at a point, then it is locally injective, that is injective in a neighborhood of that point.*

Proof: Without loss of generality, let us assume that the point in question is $z = 0$. Then, f is holomorphic in $D_r(0)$ for some $r > 0$. Thus, we have

$$f(z) = \int_0^z f'(\zeta)d\zeta + f(0) \quad \forall z \in D_r(0).$$

This is because the segment from 0 to z lies entirely inside $D_r(0)$, and since f is holomorphic, one can directly prove that the derivative of the function on the right is the same as the derivative of the function on the left. Moreover, they have the same value at $z = 0$, so the left and right sides above agree. Then, by assumption $f'(0) \neq 0$. Let's say

$$f'(0) = \lambda.$$

For simplicity, we shall prove the result for the function

$$g(z) = \frac{1}{\lambda} f(z).$$

Then g too is holomorphic in $D_r(0)$ and $g'(0) = 1$. This is rather convenient. Moreover,

$$g(z) = \int_0^z g'(\zeta)d\zeta + g(0).$$

Since $g'(0) = 1$, there exists $\delta > 0$ with $\delta < r$ such that

$$|g'(z) - 1| < \frac{1}{2}.$$

Then note that for all z and w in $D_\delta(0)$,

$$|g(z) - g(w)| = \left| \int_0^z g'(\zeta)d\zeta - \int_0^w g'(\zeta)d\zeta \right| = \left| \int_{[z,w]} g'(\zeta)d\zeta \right|.$$

We estimate

$$\left| \int_{[z,w]} g'(\zeta)d\zeta - \int_{[z,w]} 1 d\zeta \right| \leq \left| \int_{[z,w]} |g'(\zeta) - 1| d\zeta \right| \leq \frac{1}{2}|z - w|.$$

This is because the segment from z to w lies entirely in $D_\delta(0)$ since z and w are both in this disk, and disks are extremely convex. So, we get

$$\left| \int_{[z,w]} g'(\zeta)d\zeta - \int_{[z,w]} 1 d\zeta \right| \leq \frac{1}{2}|z-w| \implies \left| \int_{[z,w]} g'(\zeta)d\zeta \right| \geq \left| \int_{[z,w]} 1 d\zeta \right| - \frac{1}{2}|z-w| = \frac{1}{2}|z-w|.$$

Recalling that

$$|g(z) - g(w)| = \left| \int_{[z,w]} g'(\zeta)d\zeta \right|,$$

we therefore get that for all $z, w \in D_\delta(0)$

$$|g(z) - g(w)| \geq \frac{1}{2}|z - w| \implies \forall z \neq w \in D_\delta(0) g(z) \neq g(w).$$

Consequently g is injective on $D_\delta(0)$. Now, note that

$$g(z) \neq g(w) \iff f(z) \neq f(w).$$

Hence we also get

$$f(z) \neq f(w) \quad \forall z, w \in D_\delta(0).$$

Corollary 15.4. *Assume that f is holomorphic in a neighborhood of 0, with $f'(0) \neq 0$, and $f(0) = 0$. Then there exists $r > 0$ such that both f and f^{-1} are holomorphic in $D_r(0)$.*

Proof: From the preceding proposition, we know that there exists $\delta > 0$ in which h is holomorphic and injective. Since $f'(0) \neq 0$, f is clearly non-constant. Therefore by the open mapping theorem $f(D_\delta(0))$ is open. Moreover it contains 0 since $f(0) = 0$. We therefore have some $p > 0$ which is contained in $f(D_\delta(0))$. Simply define

$$r := \min\{\delta, p\}.$$

Then both f and f^{-1} are defined and holomorphic in $D_r(0)$.

Proposition 15.5. *Assume that f is holomorphic in a neighborhood of 0, and has a fixed point at 0 with multiplier λ with $|\lambda| = 1$. Then a solution h to Schröder's equation exists if and only if $\{f^n\}$ are uniformly bounded on some $D_r(0)$ for some $r > 0$.*

Proof: If h exists, then since h and h^{-1} are both holomorphic on some $D_r(0)$, we have

$$f^n(z) = h(\lambda^n h^{-1}(z)).$$

We may assume h is continuous on the closed disk, $D_r(0)$, by possibly taking r smaller. Then we note that the image of a compact set, like the closed disk, under a continuous function, like h^{-1} is again a compact set. So, there is some $\rho > 0$ such that the image under h^{-1} of $D_r(0)$ is contained in $D_\rho(0)$. Then, note that $\lambda^n(D_\rho(0)) \subset D_\rho(0)$. Here we note that the fact that $|\lambda| = 1$ is used, and that this statement holds for all $n \in \mathbb{N}$. The closure of this set is again compact, so the image under h, also being continuous, is therefore also compact. Hence, we have $f^n(z)$ is contained in a compact subset of $\mathbb{C}$ for all n.
On the other hand, if $|f^n| \leq M$ for all $n \in \mathbb{N}$ for all points in some $D_r(0)$, then let

$$\varphi_n(z) := \frac{1}{n} \sum_0^{n-1} \lambda^{-j} f^{j+1}(z), \quad n \geq 1$$

We clearly have

$$|\varphi_n(z)| \leq M \quad \forall z \in D_r(0).$$

Hence, by Montel's little theorem, it is normal, and contains a convergent subsequence. Note that

$$\varphi_n \circ f = \frac{1}{n} \sum_0^{n-1} \lambda^{-j} f^{j+2}(z) = \frac{1}{n} \sum_1^{n} \lambda^{-j+1} f^{j+1}(z) = \frac{1}{n} \lambda^{-n+1} f^{n+1}(z) + \frac{1}{n} \sum_1^{n-1} \lambda^{-j+1} f^{j+1}(z)$$

whereas

$$\lambda \varphi_n = \frac{1}{n} \sum_0^{n-1} \lambda^{-j+1} f^{j+1}(z) = \frac{1}{n} \lambda + \sum_1^{n-1} \lambda^{-j+1} f^{j+1}(z).$$

Hence

$$\varphi_n \circ f = \lambda \varphi_n + \mathcal{O}(1/n),$$

since f and f^n are uniformly bounded, as is $|\lambda^n| = 1$. Consequently, passing to a convergent subsequence of the φ_n (which we abuse notation and still call φ_n), we obtain in the limit that

$$\varphi \circ f = \lambda \varphi.$$

So defined, $\varphi_n(0) = 0$ for all n. Since $\lambda = f'(0)$ we have $\lambda^{-j}(f^j)'(0) = 1$, hence $\varphi_n'(0) = 1$ for all n. We therefore get that $\varphi'(0) = 1$. Consequently φ is biholomorphic in a neighborhood of 0.

The definition of a normal family should really be phrased on the Riemann sphere. The reason is that it is possible for a point to belong to the Fatou set, where in a neighborhood of that point, the iterates f^n converge locally uniformly to the function which is identically equal to infinity. Really, it sounds weird, but it is in fact precise as defined below.

Definition 15.6. Let R be a meromorphic function. The family $\{R^n\}$ is normal on an open set $U \subset \hat{\mathbb{C}}$ precisely when, the family is equicontinuous there with respect to the spherical metric on $\hat{\mathbb{C}}$. This means that for any compact $V \subset U$, for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$d_{\hat{\mathbb{C}}}(z, w) < \delta \implies d_{\hat{\mathbb{C}}}(R^n(z), R^n(w)) < \epsilon \quad \forall n \in \mathbb{N}, \quad \forall z, w \in V.$$

In this way, we can make sense of "locally uniform convergence to the constant function, ∞" on the Riemann sphere. For example, assume that R is a rational function which fixes infinity. Then let

$$f(z) := \frac{1}{R(1/z)}.$$

If 0 belongs to the Fatou set of f, then we define ∞ to belong to the Fatou set of R. Let U be the component of the Fatou set of f which contains 0, then for $\phi(z) = \frac{1}{z}$, $\phi(U)$ is the component of the Fatou set of R which contains ∞.

We therefore obtain, combining the previous two propositions, a necessary and sufficient condition for a neutral fixed point to belong to the Fatou set!

Corollary 15.7. *A neutral fixed point belongs to the Fatou set if and only if there is a solution to Schröder's equation.*

Proof: For the converse direction, if there is a solution to Schröder's equation, then we have proven that the iterates of f are uniformly bounded. Hence, by Montel's theorem, they are a normal family.

For the forward direction, assume that a neutral fixed point belongs to the Fatou set. Then we claim that the iterates are uniformly bounded in a neighborhood of the fixed point. Without loss of generality, let us henceforth assume that the fixed point is at $z_0 = 0$. We shall argue by contradiction. Assume that the iterates are *not* uniformly bounded in any $\overline{D_r(0)}$. This means that for every $k \in \mathbb{N}$ we can find $n \geq k$ and z_{n_k} with $|z_{n_k}| \leq \frac{1}{k}$ so that

$$|f^{n_k}(z_{n_k})| > k.$$

In particular, in any neighborhood $D_{1/k}(0)$ there are some iterates and corresponding points, which we are calling $f^{n_k}(z_{n_k})$, such that $f^{n_k}(z_{n_k})$ is super large.

Now, by assumption, the family of functions $\{f^n\}$ is normal on an open set which contains the fixed point, 0. So, for sufficiently large $k = K$, we have that

$$\overline{D_{1/K}(0)} \text{ is contained in the Fatou set.}$$

This means that the sequence

$$\{f^{n_k}\}_{k \geq K}$$

has a uniformly convergent subsequence on $D_{1/K}(0)$. Let us call this subsequence $\{f^{n_{k_j}}\}$. Note that on the one hand

$$f^{n_{k_j}}(0) = 0 \forall n_{k_j}.$$

However, we also have

$$|f^{n_{k_j}}(z_{n_{k_j}})| > k_j \to \infty.$$

In particular,
$$d_{\hat{\mathbb{C}}}(f^{n_{k_j}}(z_{n_{k_j}}), \infty) \to 0.$$
So, by the triangle inequality
$$d_{\hat{\mathbb{C}}}(f^{n_{k_j}}(z_{n_{k_j}}), f^{n_{k_j}}(0)) \geq d_{\hat{\mathbb{C}}}(0, \infty) - d_{\hat{\mathbb{C}}}(\infty, f^{n_{k_j}}(z_{n_{k_j}})).$$
For all j large, we can make
$$d_{\hat{\mathbb{C}}}(\infty, f^{n_{k_j}}(z_{n_{k_j}})) < \frac{1}{2} d_{\hat{\mathbb{C}}}(0, \infty).$$
Then for all j large we have
$$d_{\hat{\mathbb{C}}}(f^{n_{k_j}}(z_{n_{k_j}}), f^{n_{k_j}}(0)) \geq \frac{d_{\hat{\mathbb{C}}}(0, \infty)}{2}.$$
However, the definition of being normal requires that taking
$$\epsilon = \frac{d_{\hat{\mathbb{C}}}(0, \infty)}{3} > 0$$
there exists $\delta > 0$ such that
$$d_{\hat{\mathbb{C}}}(z, 0) < \delta \implies d_{\hat{\mathbb{C}}}(f^{n_{k_j}}(z), f^{n_{k_j}}(0)) < \epsilon.$$
Since $z_{n_{k_j}} \to 0$, as $j \to \infty$ we have for any $\delta > 0$ points with
$$d_{\hat{\mathbb{C}}}(z_{n_{k_j}}, 0) < \delta, \quad d_{\hat{\mathbb{C}}}(f^{n_{k_j}}(z_{n_{k_j}}), f^{n_{k_j}}(0)) \geq \frac{d_{\hat{\mathbb{C}}}(0, \infty)}{2} > \epsilon.$$
This is in direct violation of the definition of normal. $\lightning$ The proof is completed by this contradiction. 

So, now we know a necessary and sufficient condition for the irrationally neutral fixed points to belong to the Fatou set. Do they always belong to the Fatou set? Nope. It depends on the number theoretic properties of the angle, θ, in the exponent of the multiplier $e^{2\pi i\theta}$. We recall some famous (but too difficult for us to prove here) results.

Theorem 15.8 (Pfeiffer (1917))**.** *There is $\lambda = e^{2\pi i \phi}$ with ϕ irrational so that the Schröder equation has no solution for any polynomial f.*

Definition 15.9. ϕ is *Diophantine* (badly approximable by rational numbers) if there exists $c > 0, \mu < \infty$ so that
$$|\phi - \frac{p}{q}| \geq \frac{c}{q^{\mu}}$$
for all $p, q \in \mathbb{Z}, q \neq 0$. This is equivalent to
$$|\lambda^n - 1| \geq c n^{1-\mu}, \quad \forall n \neq 1.$$

Remark 12. Almost all real numbers are Diophantine - but not all! By the exercises, the Schröder equation has no solution for any rationally neutral fixed point. This shows that the connection between the number-theoretic nature of the angle of the multiplier at the fixed point is intimately related to the behavior of the iterates near the fixed point. It may very well be an open question to determine necessary and sufficient conditions on the angle at the fixed point (this is θ where the multiplier is $e^{2\pi i\theta}$) so that Schröder's equation admits a solution...

Theorem 15.10. *(Siegel, 1950s) If ϕ is Diophantine, $f(0) = 0$, $f'(0) = e^{2\pi i \phi}$, then there exists a solution h to Schröder's Equation.*

As a corollary we obtain

Corollary 15.11. *If f has a fixed point at z_0 with multiplier $e^{2\pi i \phi}$ such that ϕ is Diophantine, then z_0 is in the Fatou set of f.*

There is a necessary and sufficient condition for the multiplier, in case of irrationally neutral fixed points of quadratic polynomials, which determines whether or not such a point belongs to the Fatou set. It may well be an open problem to determine this in general...

15.2. **Classification of all fixed points as elements of either the Fatou or Julia set.** First we need a definition.

Definition 15.12. A simply connected component of the Fatou set such that f is conformally conjugate to an irrational rotation is a *Siegel disk*. Let such a component be denoted by Ω. This means that there exists a conformal map ϕ such that

$$\phi(f(z)) = \lambda\phi(z), \quad \lambda = e^{2\pi i\theta}\theta \in \mathbb{R} \setminus \mathbb{Q}.$$

Theorem 15.13. *Let z_0 be a fixed point of a holomorphic function f. Then*

(1) If z_0 is an attracting or super attracting fixed point, z_0 belongs to the Fatou set.
(2) If z_0 is a repelling fixed point, then it belongs to the Julia set.
(3) If z_0 is a rationally neutral fixed point, then it belongs to the Julia set.
(4) If z_0 is irrationally neutral, then it belongs to the Fatou set if and only if there is a solution to Schröder's equation. Otherwise it belongs to the Julia set.

Proof: In a neighborhood of an attracting fixed point, we have proven that the iterates $f^n(z) \to z_0$, where z_0 is the fixed point. Hence, all the iterates are uniformly convergent to the constant function z_0, and therefore the family $\{f^n\}$ is normal in this neighborhood of z_0. If z_0 is a super-attracting fixed point, the iterates $f^n(z) \to z_0$ in a neighborhood of the fixed point as well, so by the same argument, z_0 is in the Fatou set.
If z_0 is a repelling fixed point, then we have proven that (without loss of generality take $z_0 = 0$) f is conformally conjugate to

$$g(z) = \lambda z.$$

Note that for any $z \neq 0$,

$$g^n(z) = \lambda^n z \to \infty \text{ as } n \to \infty.$$

Now, I claim that the iterates of g cannot form a normal family in any neighborhood of 0. To see this, assume that some subsequence g^{n_k} converges on some neighborhood of 0. Then, since $g^{n_k}(0) = 0$ for all n_k, the limit at zero is zero. However, for any $z \neq 0$, $g^{n_k}(z) \to \infty$. So, the limit function would have to vanish at zero and be identically infinity on some $D_r(0) \setminus \{0\}$. This violates the definition of being normal. To see this, let

$$\epsilon := \frac{d_{\hat{\mathbb{C}}}(\infty, 0)}{3}.$$

Since the limit is ∞ for all $z \neq 0$, we have

$$d_{\hat{\mathbb{C}}}(g^{n_k}(z), g^{n_k}(0)) \geq d_{\hat{\mathbb{C}}}(0, \infty) - d_{\hat{\mathbb{C}}}(\infty, g^{n_k}(z)).$$

For large k, we can make the

$$d_{\hat{\mathbb{C}}}(\infty, g^{n_k}(z)) < \frac{1}{2} d_{\hat{\mathbb{C}}}(0, \infty)$$

thereby obtaining

$$d_{\hat{\mathbb{C}}}(g^{n_k}(z), g^{n_k}(0)) > \frac{1}{2} d_{\hat{\mathbb{C}}}(0, \infty).$$

Hence, there is no $\delta > 0$ such that

$$d_{\hat{\mathbb{C}}}(g^n(z), g^n(0)) < \epsilon \quad \forall d_{\hat{\mathbb{C}}}(z, 0) < \delta, \quad \forall n \in \mathbb{N}.$$

So, indeed, the family $\{g^n\}$ is not normal in any neighborhood of the repelling fixed point at zero. Consequently, neither is f.
The statement concerning rationally neutral fixed points is an exercise. Note that it implies that there is never a solution to Schröder's equation in the rationally neutral case (!)

If z_0 is an irrationally neutral fixed point, the statement has been demonstrated previously.

We know how to classify fixed points and whether they are in the Fatou or Julia set. If we are interested in iteration of meromorphic functions on $\hat{\mathbb{C}}$, we have proven that all such functions are rational functions. In this case we also know precisely how many fixed points such functions have.

Theorem 15.14. *A rational map of degree d has precisely $d+1$ fixed points, counting multiplicity, unless of course it is the identity.*

Proof: If the rational map is constant, that is of degree $d = 0$, then we have $R(z) = c$ for some $c \in \mathbb{C}$ for all $z \in \hat{C}$. Precisely one such point is equal to c, hence R has $d + 1 = 1$ fixed point. So, henceforth we assume that R is of degree $d \geq 1$ and therefore non-constant, and also that R is not the identity map.
To simplify our arguments, we will use a bit of conformal conjugation. Assume that there is a map $\phi : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ which is meromorphic and bijective. Then, consider

$$\tilde{R} := \phi \circ R \circ \phi^{-1}.$$

The R has a fixed point at z_0 if and only if $\tilde{R}$ has a fixed point at $\phi(z_0)$. Since ϕ is bijective, this shows that the number of fixed points of R is equal to the number of fixed points of $\tilde{R}$. We shall use this to justify the reduction to the case in which R does not map infinity to infinity. If infinity is a fixed point for R, then let us define

$$\phi(z) = \frac{1}{z} - c,$$

for a finite complex number, c. Then,

$$\phi^{-1}(z) = \frac{1}{z+c}.$$

Hence $\phi : \hat{C} \to \hat{C}$ is bijective and meromorphic. Moreover,

$$\tilde{R} = \phi^{-1} \circ R \circ \phi : \infty \to -c \to R(-c) \to \frac{1}{R(-c)+c}.$$

Thus, choose some $c \in \mathbb{C}$ such that $R(-c) \neq -c$, which is possible by the assumption that R is not the identity map. Then, $\tilde{R}(\infty) \neq \infty$. Since $\tilde{R}$ has fixed points in bijection with those of R, we may without loss of generality assume that R does not fix infinity. Proving the theorem in this case proves it for $\tilde{R}$, which implies the result for R since its fixed points are in bijection with those of R.
Let $\zeta \neq \infty$ be a fixed point of $R = P/Q$. We shall always assume that when we write

$$R(z) = \frac{P(z)}{Q(z)},$$

these two polynomials have no common factors. Thus, if $\zeta \neq \infty$ and

$$R(\zeta) = \zeta \implies Q(\zeta) \neq 0,$$

because P and Q have no common zeros. (If $Q(\zeta) = 0$, then since $P(\zeta) \neq 0$ this would make $R(\zeta) = \infty \neq \zeta$). Then we observe:

$$R(\zeta) = \zeta \iff R(\zeta) - \zeta = \frac{P(\zeta) - \zeta Q(\zeta)}{Q(\zeta)} = 0.$$

Since the denominator is nonzero, the degree of the zero of the function

$$R(z) - z$$

at the point ζ is the same as the degree of the zero of the function

$$P(z) - zQ(z)$$

at the point ζ. This is true for all $\zeta \neq \infty$. Hence the number of fixed points of R is equal to the number of zeros, counting multiplicity, of the polynomial

$$P(z) - zQ(z).$$

Since R does not map infinity to infinity, the degree of the numerator of R is less than or equal to the degree of the denominator of R, hence the degree of P is less than or equal to the degree of Q. Therefore, since the degree of R is

$$\max\{\deg(P), \deg(Q)\}$$

the degree of R, that is d, is equal to the degree of Q. Consequently the polynomial

$$P(z) - zQ(z) \text{ is of degree } d+1.$$

By the Fundamental Theorem of Algebra, this polynomials has $d+1$ zeros, counting multiplicity.

We will use the above result to prove the following awesome theorem on the Julia set!!!

Theorem 15.15. *The Julia set is not empty for rational functions with degree ≥ 2.*

We shall prove this next time.

15.3. **Homework.**

(1) * For a quadratic polynomial $P(z) = e^{2\pi i\theta}z + z^2$, where $\{p_n/q_n\}$ is the sequence of rational approximations to θ coming from its continued fraction expansion, show that a conjugation of P to $e^{2\pi i\theta}\zeta$ exists if and only if
$$\sum_{n=1}^{\infty} \frac{\log(q_{n+1})}{q_n} < \infty.$$
(2) Prove that such a conjugation exists for almost all θ, that is for all $\theta \in \mathbb{R}$ apart from a set of one-dimensional Hausdorff measure equal to zero. (Of course that is the same as all $\theta \in \mathbb{R}$ apart from a set of one-dimensional Lebesgue measure equal to zero).

15.4. **Hints.**

(1) There is a reason there are only two exercises, and the first exercise has a star. The sufficiency of the condition was proved by Brjuno in 1965. The necessity was proven by Yoccoz in 1988. If you find yourself stuck, just look up the proofs and work through the details to complete the exercise.
(2) As you may have noticed, much of this complex dynamics material is from the book by Lennart Carleson and Theodore W. Gamelin. You can find a proof of this in Section V.1 of that book, which is apparently due to Yoccoz.

16. The Fatou and Julia sets of rational functions

We begin by giving a useful characterization of being a normal family in the context of a compact, complete metric space. Note that $\hat{\mathbb{C}}$ is such a space!

Proposition 16.1 (Arzela-Ascoli)**.** *Assume that a family of functions are defined on a compact, complete metric space, (X, d), and that they are all maps from (X, d) to (X, d). Then, if the family is equicontinuous, every sequence from the family has a uniformly convergent subsequence, and the limit to which this sequence converges is a continuous function. Moreover, the converse holds: if every sequence of the family has a uniformly convergent subsequence, and the* *limit to which the sequence converges is a continuous function, then the family is equicontinuous. If either of these two equivalent conditions hold, the family is* normal.

Proof: If the family of functions are equicontinuous, and they map from a compact metric space into a compact metric space, then they are bounded in the sense that for all f in the family, $f(X) \subset X$ is compact. The Arzela-Ascoli theorem therefore guarantees the existence of a convergent subsequence. As the uniform limit of continuous functions, it is also continuous. For the converse, we argue by contradiction. Assume the family is not equicontinuous. Then at some point $z \in X$ there is a sequence of points $z_k \to z$ and functions in the family f_k such that for some fixed $\varepsilon > 0$,

$$d(f_k(z_k), f_k(z)) \geq \varepsilon.$$

Now, as a sequence of functions in the family $\{f_k\}$ has a convergent subsequence. Let us pass to this subsequence, but still call it $\{f_k\}$ because it has the same properties, it is just converging to some f which is by assumption continuous. Moreover the convergence is uniform. Since f is continuous there is $\delta > 0$ such that if

$$d(w,z) < \delta \implies d(f(w), f(z)) < \frac{\varepsilon}{3}.$$

Hence, by the triangle inequality

$$d(f_k(z_k), f_k(z)) \leq d(f_k(z_k), f(z_k)) + d(f(z_k), f(z)) + d(f(z), f_k(z)).$$

By the uniform convergence $f_k \to f$, for all k large we can make the first and last terms less than $\frac{\varepsilon}{3}$. Since the points $z_k \to z$, for all k large we can make $d(z_k, z) < \delta$ so that the middle term is also less than $\frac{\varepsilon}{3}$. We therefore get that

$$d(f_k(z_k), f, (z)) < \varepsilon \quad \text{for all } k \text{ large.}$$

$$\lightning$$

Theorem 16.2. *The Julia set is not empty for rational functions with degree* ≥ 2.

Proof: Let R be a rational function of degree ≥ 2.

Claim 3. R^n *has degree* d^n.

Proof of the claim: Write

$$R(z) = \frac{p(z)}{q(z)}, \quad p(z) = a\prod_1^n (z - r_k), \quad q(z) = b\prod_1^m (z - s_j).$$

Consider $R(R(z)) =$

$$\frac{a\prod_1^n (p/q - r_k)}{b\prod_1^m (p/q - s_j)} = \frac{aq(z)^{-n}\prod_1^n (p(z) - r_k q(z))}{bq(z)^{-m}\prod_1^m (p(z) - s_j q(z))} = \frac{a\prod_1^n (p(z) - r_k q(z))}{bq(z)^{n-m}\prod_1^m (p(z) - s_j q(z))}$$

Assume first that the degree of p is greater than or equal to the degree of q. Therefore the degree d of R is equal to n and $n \geq m$. The numerator in $R(R(z))$ has the factor $p(z)$ a total of n times, and is therefore of degree n^2. The denominator has the factor q^{n-m} which is of degree $m(n-m)$ times the product which has leading term p^m, which is of degree nm. So all together the denominator is of degree at most

$$m(n-m) + nm = 2nm - m^2.$$

The denominator could be of lower degree, which can only occur if p and q are of the same degree, and $a = bs_j$ for some s_j. However, this is not problematic for the proof, because since $n \geq m$

$$0 \leq (n-m)^2 = n^2 - 2mn + m^2 \implies 2mn - m^2 \leq n^2.$$

So, unless there is some other weird cancellation, $R(R(z))$ is of degree $n^2 = d^2$. Can there be any weird cancellation?
The numerator vanishes iff

$$p(z) = r_k q(z)$$

for some r_k and some z. The denominator vanishes iff

$$p(z) = s_j q(z)$$

for some s_j and some z, or if $n > m$ and q vanishes. I claim that the numerator and denominator can never both vanish. To see this, if we have both

$$p(z) = r_k q(z) \text{ and } p(z) = s_j q(z) \implies r_k q(z) = s_j q(z) \implies r_k = s_j \text{ or } q(z) = 0.$$

The first, $r_k = s_j$ never happens, and in the second case $q(z) = 0$ here together with $p(z) = r_k q(z)$ implies $p(z)$ also vanishes but they cannot both vanish at the same point! Hence, impossible. Similarly, if

$$p(z) = r_k q(z) \text{ and } q(z) = 0 \implies p(z) = 0 \lightning.$$

Hence, there is no cancelation of the numerator and denominator. So, the numerator and denominator of $R(R(z))$ have no common zeros and hence the degree of $R(R(z))$ is indeed $n^2 = d^2$.

Exercise 26. *Complete the proof of the claim (hint: induction) to prove that the degree of R^j is d^j. The case in which $m \geq n$ is very similar.*

We shall proceed with the proof by contradiction. We assume $\mathcal{J} = \emptyset$, so that the family of iterates of R is normal on $\hat{\mathbb{C}}$ which is compact, hence there exists a uniformly convergent subsequence. Let us call that subsequence R^{n_k} and their uniform limit f. Then this function f is continuous on $\hat{\mathbb{C}}$ with respect to the $d_{\hat{\mathbb{C}}}$ metric.
Let us dispatch with the case in which $f \equiv \infty$. In this case, we consider the rational function

$$\tilde{R}(z) := \frac{1}{R(1/z)}.$$

This function is conformally conjugate to R. Since $\phi(z) = 1/z$ is a bijection from $\hat{\mathbb{C}}$ to itself, and since R^{n_k} converges uniformly to ∞ on all of $\hat{\mathbb{C}}$, we get that $\tilde{R}$ converges uniformly to the constant function 0 on all of $\hat{\mathbb{C}}$. Thus, there exists $N \in \mathbb{N}$ such that for all $j \geq N$

$$d_{\hat{\mathbb{C}}}(R^{n_j}(z), 0) < \frac{1}{3} d_{\hat{\mathbb{C}}}(0, \infty) \quad \forall z \in \hat{\mathbb{C}}.$$

In particular, we get

$$d_{\hat{\mathbb{C}}}(R^{n_j}(z), \infty) \geq d_{\hat{\mathbb{C}}}(0, \infty) - d_{\hat{\mathbb{C}}}(0, R^{n_j}(z)) \geq \frac{1}{2} d_{\hat{\mathbb{C}}}(0, \infty) > 0.$$

So, in particular, the R^{n_j} have no poles in $\hat{\mathbb{C}}$, and moreover, they are entire and bounded. Thus the R^{n_j} are all constant for $j \geq N$. Since they converge to 0 this means that they are all equal to the constant zero function. This is a contradiction to the fact that R^{n_j} has degree d^{n_j}. So, we cannot have R^{n+j} converging uniformly on $\hat{\mathbb{C}}$ to the constant function, ∞.
What else can we determine about the limit function? First, note that with respect to $d_{\hat{\mathbb{C}}}$ the functions $\{R^n\}$ are all *continuous* on the entirety of $\hat{\mathbb{C}}$. Yep, even at their poles. Hence, the limit, being the uniform limit of continuous functions is also continuous on $\hat{\mathbb{C}}$. Let us call this limit function f. Whenever $f(z_0) \neq \infty$, by arguments from our previous results, we obtain that the family $\{R^n\}$ is uniformly bounded (in the usual sense, $|R^n| \leq M$ for some fixed $M > 0$ for all n) in some neighborhood of z_0. Consequently, we can apply the dominated convergence theorem to conclude that the integrals over closed curves in that neighborhood of f is zero, and we therefore obtain that f is holomorphic in such a neighborhood. Consequently, f is holomorphic whenever it is not infinity. Now, f can only have finitely many zeros of finite multiplicity because $f \equiv 0$ shall lead to a contradiction. If $f \equiv 0$, then we would get that the family $\{R^n\}$ is uniformly bounded on $\hat{\mathbb{C}}$, hence they are all constant, which is a contradiction. So, we know

that f has finitely many zeros of finite multiplicity. Therefore, $\frac{1}{f}$ is a meromorphic function on $\hat{\mathbb{C}}$. Hence, it is a rational function. Hence f is also a rational function. It therefore has some finite degree. Now, let us use the results concerning the fixed points of rational functions. The iterates R^{n_j} have $d^{n_j}+1$ fixed points (counting multiplicity). These are contained in a compact set, $\hat{\mathbb{C}}$. Hence they accumulate somewhere. Let us pass to a subsequence of R^{n_j} but still call it by the same name but such that we have fixed points for these R^{n_j} at z_{n_j} which tend to some $z_0 \in \hat{\mathbb{C}}$. By the uniform convergence to f we therefore have

$$f(z_0) = z_0.$$

Claim 4. *The limit function f is not the identity function. Nor is it an entire function.*

Let us prove this by contradiction. Assume that $f(z) = z$ for all $z \in \hat{\mathbb{C}}$. Fix $M > 0$. Then since f is holomorphic in $D_M(0)$, and it is bounded by M there, we get that all the iterates $\{R^n\}$ are bounded in this disk. Hence they have no poles in there. Letting $M \to \infty$, we get that $\{R^n\}$ can only have a pole at infinity for all n. Hence they are all polynomials. Since their degrees are greater than or equal to zero, they all have a super-attracting fixed point at ∞. Hence, some open neighborhood of ∞ belongs to the Fatou set, and all iterates converge to the constant function, ∞ in this neighborhood. Now that we assumed that the Julia set is empty, the limit to which our subsequence R^{n_j} converges must therefore also be the function which is constant and equal to infinity (or at least in an open neighborhood of ∞ we must have $f(z) \equiv \infty$). This violates the fact that the limit is meromorphic and thus has discrete zeros and poles. $\lightning$ Hence, the limit function, whatever it may be, is *not* the identity. This proves the first part of the claim. For the second part we may proceed similarly. Note that if f is entire, then f is uniformly bounded on any $D_M(0)$. Hence we obtain the same for all the iterates there. Hence, letting $M \to \infty$ we get that the only place the iterates can have a pole is at infinity. We therefore get the same contradiction in this case as well.
Now, let us assume that some z_{n_j} is a fixed point of a certain R^{n_j}. Then on the one hand

$$f \circ R^{n_j} \to f \circ f \text{ uniformly as } j \to \infty.$$

So, fixing the point z_{n_j} we get

$$f \circ R^{n_j}(z_{n_j}) \to f(f(z_{n_j})).$$

On the other hand, by definition of being a fixed point of that particular R^{n_j} we get *without letting $j \to \infty$ for this particular n_j,*

$$f(R^{n_j}(z_{n_j})) = f(z_{n_j}).$$

Hence

$$f(f(z_{n_j})) = f(z_{n_j}).$$

So, whenever R^{n_j} has a fixed point at z_{n_j}, f has a fixed point at $f(z_{n_j})$. Since R^{n_j} has $d^{n_j}+1$ fixed points, counting multiplicity, we get that f has a fixed point at $f(z_{n_j})$. However, the number of fixed points of f is finite, because f is not the identity. This means that the set $\{f(z_{n_j})\}$ is a finite set. Now, f is of finite degree, so it cannot send infinitely many points to the same point. Therefore, the set of fixed points of the R^{n_j} must be also finite, and the multiplicity at these fixed points must be tending to infinity. So, the set $\{z_{n_j}\}_{j\geq 1}$ of fixed points of $\{R^{n_j}\}_{j\geq 1}$ is finite and the degree of these fixed points tends to infinity. Near a fixed point, $p \neq \infty$, the power series of $R^{n_j}(z)$ is of the form

$$R^{n_j}(z) = p + \sum_{k\geq 1} a_k (z-p)^k,$$

where

$$a_k = \frac{(R^{n_j})^{(k)}(p)}{k!}.$$

The degree of the fixed point is the minimal k such that $a_k \neq 0$. Now, by the uniform convergence to f, we also get the uniform convergence of the derivatives to the corresponding derivatives of f. Therefore, since the degree of the fixed point $p \neq \infty$ is tending to infinity, this forces f to vanish at p to infinite order. By the identity theorem, then, $f \equiv 0 \lightning$. So, the only fixed point p can be infinity. However, then the same argument shows that f has a pole at infinity of infinite order, which violates the fact that f is a rational map (and hence of finite order). This is a $\lightning$ contradiction. 

16.1. **Self similar natural of the Julia set.**

Definition 16.3. Let R be meromorphic on $\hat{\mathbb{C}}$. (Thus it is a rational map). Then we say that E is *completely invariant* if E and E^c are invariant under R, in the sense that both

$$R(E) \subset E$$

and

$$R(E^c) \subset E^c.$$

This terminology may seem weird á priori, but let us see that it is sound.

Proposition 16.4. *Assume that the rational map is non-constant. Then E is completely invariant if and only if $R(E) = E$.*

Proof: Assume E is completely invariant. Note that the empty set is completely invariant as is $\hat{\mathbb{C}}$. We have proven that all rational maps which are non-constant are surjective from $\hat{\mathbb{C}}$ to $\hat{\mathbb{C}}$. So, we have

$$R(\hat{\mathbb{C}}) = R(E \cup E^c) = \hat{\mathbb{C}} = E \cup E^c.$$

On the other hand

$$\hat{\mathbb{C}} = R(E \cup E^c) = R(E) \cup R(E^c) \subset E \cup E^c.$$

Consequently, we must have

$$R(E) = E, \quad R(E^c) = E^c.$$

For the converse statement, if we assume $R(E) = E$, then we also get, via

$$R(\hat{\mathbb{C}}) = R(E) \cup R(E^c) = E \cup R(E^c) = \hat{\mathbb{C}} = E \cup E^c \implies R(E^c) = E^c.$$

Exercise 27. *What about when R is a constant map? Which sets are invariant? Also, observe that a set E is completely invariant if and only if its complement is completely invariant.*

With the notion of completely invariant, we can prove that the Julia set enjoys this property!

Theorem 16.5. *The Julia set $\mathcal{J}$ of a rational map is completely invariant. The Fatou set is also completely invariant.*

Proof: If either the Julia set or Fatou set is empty, we have proven that both $\emptyset$ and $\hat{\mathbb{C}}$ are completely invariant, so we are done. If the function R is constant, then the Fatou set is $\hat{\mathbb{C}}$, so we are done in this case. Let us assume that R is not constant. Let us also assume that the Fatou set is not empty. Let $D_r(z_0) \subset \mathcal{F}$. Then, let $\{R^{n_k}\}$ be a subsequence of $\{R^n\}$ which converges uniformly on $\overline{D_\rho(z_0)}$ for some $\rho < r$. Hence $\{R^{n_k - 1}\}$ converges uniformly on $R(\overline{D_\rho(z_0)})$. By continuity of R this set is bounded, and by the open mapping theorem, it is closed, hence it is compact. Therefore every uniformly convergent subsequence gives rise to a uniformly convergent subsequence on $R(\overline{D_\rho(z_0)})$. We can do this for any $\rho < r$. So, in particular, we get that $R(D\rho(z_0))$ is in the Fatou set for $\rho < r$. Since $R(z_0)$ is contained in this set, and since it is an open set by the open mapping theorem, we have

$$R(\mathcal{F}) \subset \mathcal{F}.$$

To obtain the converse, note that if R^{n_k} converge uniformly on $\overline{D_\rho(z_0)}$, then R^{n_k+1} converge uniformly on $R^{-1}(\overline{D_\rho(z_0)})$. Hence $R^{-1}(D_\rho(z_0))$, which is open by the continuity of R, and which contains $R^{-1}(z_0)$ (which could be several different points, but that is not important), is in the Fatou set. Thus

$$R^{-1}(\mathcal{F}) \subset \mathcal{F} \implies \mathcal{F} \subset R(\mathcal{F}).$$

Combining with the reverse containment, we have

$$R(\mathcal{F}) = \mathcal{F}.$$

Consequently, since R is non-constant in this case, we have that $\mathcal{F}$ is completely invariant, and so is its complement, $\mathcal{J}$. 

The next theorem shows that the Julia set is the same for all iterates of our rational function.

Theorem 16.6. $\forall N \geq 1$, $\mathcal{J}(R) = \mathcal{J}(R^N)$. *The same statement holds for the Fatou set, that is*

$$\mathcal{F}(R) = \mathcal{F}(R^N).$$

Proof: First, assume that R is constant. Then its Fatou set is $\hat{\mathbb{C}}$. Since $R^n = R$ is also constant for all n, the Fatou set of R^n is $\hat{\mathbb{C}}$ for all n. So, let us assume that R is not constant. Then it is surjective because R is a rational map. Let $D_r(z_0) \in \mathcal{F}(R)$. Then the family $\{R^n\}_{n\geq 1}$ is equicontinuous there. Since the family $\{(R^N)^n\}_{n\geq 1}$ is contained in $\{R^n\}_{n\geq 1}$ it is also equicontinuous in $D_r(z_0)$. Therefore this is also contained in $\mathcal{F}(R^N)$. Hence we obtain

$$\mathcal{F}(R) \subset \mathcal{F}(R^N).$$

For the reverse containment let $\{R^{n_k}\}$ be a subsequence of $\{R^n\}$. Then since it is a subsequence, $n_k \to \infty$ as $k \to \infty$. Hence, so, in particular, we can pick a subsequence of it which has

$$n_{k_{j+1}} - n_{k_j} = \text{ a positive integer multiple of } N.$$

Hence, this subsequence is of the form

$$\{R^{n_{k_j}}\} = \{R^{Nm_l+p}\}$$

for some sequence of integers $\{m_l\}$, for some fixed non-negative integer p. Assuming that the family $\{R^N\}$ is normal in a neighborhood of z, we get a uniformly convergent subsequence

$$\{R^{Nm_{l_q}}\}$$

consequently

$$R^{p+Nm_{l_q}} = R^p(R^{Nm_{l_q}})$$

also converges uniformly. This is a subsequence of $\{R^{n_{k_j}}\}$. So, it is in turn a uniformly convergent subsequence of $\{R^{n_k}\}$. Hence we see that whenever the family $\{R^{Nn}\}$ is normal, so is $\{R^n\}$. Therefore

$$\mathcal{F}(R^N) \subset \mathcal{F}(R).$$

Therefore these sets are equal, so that

$$\mathcal{F}(R) = \mathcal{F}(R^N), \quad \forall N \geq 1.$$

The same statement holds for the complements,

$$\mathcal{J}(R) = \mathcal{J}(R^N), \quad \forall N \geq 1.$$

We shall use but not prove the following lovely theorem.

Theorem 16.7 (Montel's Big Theorem)**.** *If a family of functions is meromorphic on domain G and the image under G of the family omits three points, then the family is normal. In particular, if $\exists z_1, z_2, z_3$ such that $f(G) \cap \{z_i\}_{i=1}^3 = \emptyset \quad \forall f \in \mathcal{F}$ then $\mathcal{F}$ is normal.*

16.2. **Homework.**

(1) Locate and read a proof of Montel's big theorem.
(2) Locate and read a proof of the Riemann-Hurwitz theorem.

17. Julia sets of rational maps

In addition to fixed points another type of distinctive point is a critical point.

Definition 17.1. A point $z \in \hat{\mathbb{C}}$ is a critical point if one of the three equivalent conditions below holds:

(1) R is not injective on any open neighborhood of z.
(2) $R'(z) = 0$.
(3) Let $f(w) := R(w) - R(z)$. Then f vanishes at z with multiplicity greater than one.

Definition 17.2. The multiplicity of $z \in \mathbb{C}$ is the degree of the zero of the function $R(w) - z$ for $w = z$ and is denoted by $\text{mult}(z)$.

We will use but not prove the following theorem.

Theorem 17.3 (Riemann-Hurwitz)**.** *Assume R is not constant and of degree d. Then*

$$\sum_{z \in \hat{\mathbb{C}}} \text{mult}(z) - 1 = 2(d-1).$$

The proof of this theorem relies upon some rather deep results in topology concerning the Euler characteristic of Riemann surfaces. Similar to the proof of the fact that the Julia set of any rational function is non-empty for all rational functions of degree at least two, which relied on the number of fixed points, we can use the Riemann Hurwitz theorem to prove that any coompletely invariant set for a rational map of degree at least two has at most two elements.

Theorem 17.4. *Any finite completely invariant set for R rational of degree at least two has at most 2 elements.*

Proof: Assume S is such a set. Then $R(S) = S$, and so R acts as a permutation on the elements of S. Assume S has n elements. Then R is uniquely identified with an element σ of the symmetric group S_n. This group has $n!$ elements hence the order of σ is finite. Let this order be k. This means that R^k acts as the identity element on S. We have already computed that the degree of R^k is d^k where d is the degree of R. Note that the multiplicity of the zero of $R^k(w) - z$ at $w = z$ is d^k. This is because the function $R^k(w) - z$ has precisely d^k zeros counting multiplicity by the Fundamental Theorem of Algebra. Perhaps that is not immediately apparent, but writing

$$R^k(w) = \frac{p(w)}{q(w)}, \quad R^k(w) = z \iff g(w) := p(w) - zq(w) = 0.$$

The function $g(w) : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a polynomial of degree equal to the degree of R^k, which is d^k. Hence this function has precisely d^k zeros counting multiplicity by the Fundamental Theorem of Algebra, and $g(w) = 0$ iff $R^k(w) = z$. So, if one of these zeros were to be some $w \neq z$, then

$R^{-k}(z) \ni w$ which shows that $w \in S$ because S is completely invariant. Then since R^k acts as the identity on S, this means that

$$R^k(w) = w \neq z = R^k(w).$$

This is a contradiction. So the only solutions to $R^k(w) - z = 0$ is $w = z$ and hence the multiplicity of z for R^k is d^k. This holds for each $z \in S$. So we have

$$\sum_{z \in S} \text{mult}(z) - 1 = n(d^k - 1) \leq \sum_{z \in \hat{\mathbb{C}}} \text{mult}(z) - 1 = 2(d^k - 1)$$

which shows that $n \leq 2$. 

Definition 17.5. The orbit of a point $z \in \hat{\mathbb{C}}$ is

$$\mathcal{O}(z) := \{R^n(z)\}_{n \in \mathbb{Z}}.$$

Note that this includes both the forwards and backwards orbits. If the orbit of a point is finite, then we say that point is exceptional. The set of all such points is denoted by $E(R)$.

Proposition 17.6. *The exceptional set of a rational map of degree at least two has 0, 1, or 2 points.*

Proof: If $z \in E(R)$, then by definition the orbit of z has finitely many elements. Since the orbit of z is the same as the orbit of $R(z)$ as well as the same as the orbit of $R^{-1}(z)$, the orbit is completely invariant. By the preceding theorem the orbit of z has 1 or 2 elements. It has at least one element because it contains $z = R^0(z)$. If the orbit of z contains only z, then it is a fixed point. If the orbit of z also contains w so that $R(z) = w \neq z$, then we know that either $R(R(z)) = R(w) = w$ or $R(R(z)) = z$. Hence either w is a fixed point of R or z is a fixed point of R^2. Consequently the total number of exceptional points is at most twice the number of fixed points of R plus the number of fixed points of R^2. This is finite because R has precisely $d+1$ fixed points, and R^2 has precisely d^2+1 fixed points. Since the orbit of any exceptional point is completely invariant, and the orbit of any point in the orbit of z is the same as the orbit of z, it follows that the orbit of each exceptional point is contained in $E(R)$. There are finitely many of these, they are each completely invariant, hence $E(R)$ is a finite, completely invariant set. By the preceding theorem it contains at most 2 points. 

Theorem 17.7. *The Julia set of any rational map of degree at least two is infinite, and the exceptional set is contained in the Fatou set.*

Proof: If the Julia set is finite, then because it is completely invariant, it contains at most 2 points. We know that the Julia set is not empty. So, first assume the Julia set contains one point. We can conjugate such that WLOG this point is ∞. Then since the Julia set is completely invariant,

$$R(\infty) \subset \mathcal{J} = \infty \implies R(\infty) = \infty,$$

and

$$R^{-1}(\infty) \subset \mathcal{J} = \infty \implies R^{-1}(\infty) = \infty.$$

Consequently, R has no poles in $\mathbb{C}$ and is an entire function. Since it has degree at least two, R is a polynomial. For any polynomial ∞ is a super-attracting fixed point, because 0 is a super-attracting fixed point for the function

$$\frac{1}{R(1/z)} = \phi^{-1} \circ R \circ \phi, \quad \phi(z) = \phi^{-1}(z) = 1/z,$$

and $\phi^{-1}(0) = \infty$. We have already seen that if two functions are conformally conjugate such as $\phi^{-1} \circ R \circ \phi = \widetilde{R}$, then R has a fixed point at ∞ if and only if $\widetilde{R}$ has a fixed point at $\phi^{-1}(\infty) = 0$. Moreover the multiplier at the fixed point is the same for R as for $\widetilde{R}$. Since the polynomial R is of degree $d \geq 2$, $1/R$ tends to 0 of order d as $z \to \infty$ hence $\widetilde{R}$ has a zero of order d at 0. By the Fundamental Theorem of Algebra, $\widetilde{R}$ has precisely d zeros counting multiplicity. Hence this function has only one zero of order d at zero so

$$\frac{1}{R(1/z)} = cz^d, \quad c \in \mathbb{C} \setminus \{0\} \implies R(z) = c^{-1} z^d.$$

Since 0 is a super-attracting fixed point for $\widetilde{R}$ it lies in the Fatou set for $\widetilde{R}$ and consequently $\phi^{-1}(0) = \infty$ also lies in the Fatou set of R. This is a contradiction because this point was assumed to be in the Julia set which is distinct from the Fatou set.

If the Julia set contains two points, we can again assume by conformal conjugation that these points are $\{0, \infty\}$. By the complete invariance of the Julia set we have a few possibilities. One possibility is that $R(0) = 0$ and $R(\infty) = \infty$. By the complete invariance of the Julia set, $R^{-1}(0) \in \{0, \infty\}$. If $R^{-1}(0) = \infty \implies R(\infty) = 0$ which is impossible. So, we also have that $R^{-1}(0) = 0$ and $R^{-1}(\infty) = \infty$. Consequently, R is a polynomial of degree $d \geq 2$. By the preceding argument ∞ is in the Fatou set, a contradiction.

The other possibility is that $R(0) = \infty$, and $R(\infty) = 0$, due to the complete invariance. In this case $R(z) = P(z)/Q(z)$ has a Laurent expansion about 0 of the form $c_j z^{-j} + \ldots$ with $c_j \neq 0$. Consequently when we consider long division of the polynomials P and Q it follows that the degree of Q is strictly larger than the degree of P. We can also see this because R vanishes at infinity, so the degree of Q must be larger than that of P. If there were any other point $p \in \mathbb{C}$ such that $R(p) = 0$, then again by the complete invariance of $\mathcal{J}$ such a point would necessarily be contained in $\mathcal{J}$ which it is not. Hence, the only zero of R is at infinity, and this zero must therefore be of degree d which is the degree of R. Consequently $R(z) = cz^{-d}$. Then

$$R^2(z) = R(R(z)) = c^{1-d} z^{d^2}$$

has a super-attracting fixed point at $z = 0$. It follows that 0 is in the Fatou set of R^2, and by one of our previous results, the Fatou set of R^N is the same as the Fatou set of R for any $N \in \mathbb{N}$. Hence 0 is in the Fatou set of R as well, which is a contradiction because 0 was assumed to be in the Julia set.

So, it is impossible for the Julia set to have 1 or 2 points, and this shows that it must have infinitely many points because it is not empty.

Next we consider the exceptional set. If it is just one point, by conformal conjugation we may assume that this point is ∞. Then the orbit of this point is ∞ and hence $R(\infty) = \infty = R^{-1}(\infty)$ and so R is a polynomial because it is an entire non-constant function with pole at infinity. As we have seen above ∞ is a super-attracting fixed point for any polynomial of degree at least two and hence lies in the Fatou set.

If the exceptional set contains two points, without loss of generality we assume these two points are 0 and ∞. Then we either have $R(0) = 0$, $R(\infty) = \infty$ which implies $R(z) = cz^d$, and both 0 and ∞ are in the Fatou set. By the above argument the other possibility is that $R(\infty) = 0$, $R(0) = \infty$. In this case we showed that $R(z) = cz^{-d}$, and again both 0 and ∞ lie in the Fatou set because this is true for R^2 (both 0 and ∞ are in the Fatou set of R^2 in this case).

So, in all cases the exceptional set lies in the Fatou set.

Theorem 17.8. *Any completely invariant closed set A satisfies one of the following: either $A \subset E(R) \subset \mathcal{F}$ or $A \supset \mathcal{J}$.*

Proof: Assume A is such a set, and let $U := \hat{\mathbb{C}} \setminus A$. Then U is open and completely invariant. Therefore the complement of U, being A is also completely invariant. If A is finite, then it has at most two points. It follows that since A is completely invariant, the orbit of each element of A lies in A and hence is finite, so $A \subset E(R)$. If A is infinite, consider $\{R^n\}$ on U. Since U is completely invariant, for each $z \in U$, $R^n(z) \subset U \subset \hat{\mathbb{C}} \setminus A$ and hence the family $\{R^n\}$ on U omits all points of A, of which there are more than three! So, the family R^n is normal on U, and hence $U \subset \mathcal{F}$. The reverse inclusion therefore holds for their complements, so $U^c = A \supset \mathcal{F}^c = \mathcal{J}$.

Theorem 17.9. *The Julia set is perfect for any non-constant rational map.*

Proof: Let $\mathcal{J}'$ denote the set of accumulation points of the Julia set. Then since $\mathcal{J}$ is closed it follows that $\mathcal{J}' \subset \mathcal{J}$. Note that since $\mathcal{J}$ is infinite and is contained in $\hat{\mathbb{C}}$ which is compact, the Julia set has accumulation points, so $\mathcal{J}' \neq \emptyset$. The idea is thus to show that $\mathcal{J}'$ is completely invariant because then we have proven that any completely invariant closed set is either in the Fatou set or it contains the Julia set. Since $\mathcal{J}'$ is in the Julia set, it cannot be in the Fatou set! First let's show that $\mathcal{J}'$ is closed. If z is an accumulation point of $\mathcal{J}'$, then any open neighborhood U of z contains an element of $\mathcal{J}'$, which is an element of $\mathcal{J}$ since $\mathcal{J}' \subset \mathcal{J}$. Therefore, this shows that z is an accumulation point of $\mathcal{J}$, hence $z \in \mathcal{J}'$. Hence, $\mathcal{J}'$ contains all its accumulation points and is therefore closed.
Next we show the complete invariance of $\mathcal{J}'$. Let $z \in \mathcal{J}'$. Then there is a sequence $\{z_n\} \subset \mathcal{J}$ which converges to z. The function R is continuous on $\hat{\mathbb{C}}$, and therefore $R(z_n) \to R(z)$. Since $R(z_n) \in \mathcal{J}$ for every n by the invariance of $\mathcal{J}$, we have a sequence in $\mathcal{J}$, namely $\{R(z_n)\}$ which converges to $R(z)$. Therefore $R(z)$ is an accumulation point of $\mathcal{J}$ and so $R(z') \in \mathcal{J}'$. Hence for any $z \in \mathcal{J}'$ we have $R(z) \in \mathcal{J}'$.We have thereby shown the inclusion

$$R(\mathcal{J}') \subset \mathcal{J}' \implies \mathcal{J}' \subset R^{-1}(\mathcal{J}').$$

Next let $z \in R^{-1}(\mathcal{J}')$, and $w = R(z) \in \mathcal{J}'$. Then since R is non-constant, it is an open map. Since $w = R(z) \in \mathcal{J}'$, for an open set U containing z, $R(U)$ is an open set containing w which is an accumulation point of $\mathcal{J}$, and so $R(U)$ has non-empty intersection with $\mathcal{J}$. Therefore since $R(\mathcal{J}) = \mathcal{J}$,

$$R^{-1}(R(U) \cap \mathcal{J}) = U \cap R^{-1}(\mathcal{J}) = U \cap \mathcal{J} \neq \emptyset.$$

So, for any open U containing z, $U \cap \mathcal{J} \neq \emptyset$. It follows that z is an accumulation point of $\mathcal{J}$ and so $z \in \mathcal{J}'$. This shows that

$$R^{-1}(\mathcal{J}') \subset \mathcal{J}' \implies \mathcal{J}' \subset R(\mathcal{J}') \subset \mathcal{J}'.$$

So

$$R(\mathcal{J}') = \mathcal{J}'$$

is completely invariant. Since it is a closed set, by the previous theorem it is either contained in $\mathcal{F}$ or it contains $\mathcal{J}$. Since $\mathcal{J}' \subset \mathcal{J}$ which is disjoint from $\mathcal{F}$, we cannot have $\mathcal{J}' \subset \mathcal{F}$, and so we must have

$$\mathcal{J}' \supset \mathcal{J} \supset \mathcal{J}' \implies \mathcal{J}' = \mathcal{J}.$$

Hence every point of $\mathcal{J}$ is an accumulation point of $\mathcal{J}$ which is the definition of being perfect.

17.1. **Homework.**

(1) Determine the Julia set of the function $R(z) = z^2$.
(2) Determine the Julia set of the function $R(z) = z^2 - 2$.
(3) Determine the Julia set of the Lattés function
$$\frac{(z^2+1)^2}{4z(z^2-1)}.$$
(4) Determine the Julia set of the function $1 - 2/z^2$.
(5) Show that the Julia set is the closure of the repelling periodic points. A repelling periodic point is a piont such that $R^n(z_0) = z_0$ for some n. Take the minimal such n. Then z_0 is a fixed point for R^n, and thus is termed attracting, repelling, rationally neutral or irrationally neutral according to the type of fixed point of R^n.
(6) Show that the Julia set of a Blaschke product $B(z)$ of degree $d \geq 2$ is either the unit circle or a Cantor set on the unit circle.

17.2. **Hints.**

(1) Consider $|z| < 1$ and $|z| > 1$.
(2) Recall a previous exercise about this function.
(3) Show that a dense subset of $\overline{\mathbb{C}}$ is iterated to the repelling fixed point at ∞.
(4) Show that this function has the same Julia set as the previous one.
(5) Show that the Julia set of R and R^n are the same for any n. Then use the fact that the Julia set contains all repelling fixed points.
(6) Show that the iterates of a Blaschke product are normal both inside and outside the unit disk. Thus the Julia set is a perfect subset of the unit disk. Consider what cases are possible and use the results we have proven.

18. Fractal nature of the Julia set and properties of the Mandelbrot Set

The Julia set of a rational map of degree at least two is either $\hat{\mathbb{C}}$ or has empty interior!

Theorem 18.1. *The Julia set of a rational map R of degree at least two is either $\hat{\mathbb{C}}$ or has empty interior.*

Proof: Let us decompose $\hat{\mathbb{C}}$ as a disjoint union
$$\hat{\mathbb{C}} = \partial\mathcal{J} \cup \mathring{\mathcal{J}} \cup \mathcal{F}.$$
Let us also assume that $z \in \mathring{\mathcal{J}}$, so the interior of $\mathcal{J}$ is not empty. Then there exists $r > 0$ such that $D_r(z) \subset \mathring{\mathcal{J}} \subset \mathcal{J}$. Applying R, by the Open Mapping Theorem, $R(D_r(z)) \ni R(z)$ is an open set. By the complete invariance of $\mathcal{J}$ this set lies in $\mathcal{J}$. Hence there is an open neighborhood of $R(z)$ in $\mathcal{J}$, so $R(z) \in \mathring{\mathcal{J}}$. This shows that
$$R(\mathring{\mathcal{J}}) \subset \mathring{\mathcal{J}}.$$
For the reverse inclusion we use continuity, because $R^{-1}(D_r(z))$ is an open set contained in $\mathcal{J}$ hence contained in $\mathring{\mathcal{J}}$ so
$$R^{-1}(\mathring{\mathcal{J}}) \subset \mathring{\mathcal{J}},$$
and we see that $\mathring{\mathcal{J}}$ is completely invariant. Since the Fatou set is also completely invariant, we have the following
$$R(\mathring{\mathcal{J}} \cup \mathcal{F}) = \mathring{\mathcal{J}} \cup \mathcal{F} \implies R(\partial\mathcal{J}) = \partial\mathcal{J},$$

so the boundary of $\mathcal{J}$ is also completely invariant. It is closed since its complement is by definition open. By a preceding result, since the intersection of the Julia set, which is closed and hence contains its boundary, with the Fatou set is empty, either the boundary of the Julia set contains the Julia set, or the boundary of the Julia set is empty. By assumption the Julia set has non-empty interior, so if it has non-empty boundary, then it cannot be contained in its boundary. It follows that the boundary of the Julia set is empty. This means that the Julia set is open as well as closed, and hence is the entire $\hat{\mathbb{C}}$. This shows that if the Julia set has non-empty interior, then it is $\hat{\mathbb{C}}$. On the other hand, if the Julia set is not $\hat{\mathbb{C}}$, by the contrapositive, it cannot have non-empty interior, so if the Julia set is not $\hat{\mathbb{C}}$, then it has empty interior. These are the only two mutually exclusive possibilities. 

The following proposition will allow us to prove the self-similarity property of $\mathcal{J}$. Basically, take any open set which has non-empty intersection with the Julia set. No matter how small that is, the inverse images of R of this open set will eventually cover all of $\mathcal{J}$.

Proposition 18.2. *Let R be a rational map of degree at least two, and U a non-empty open set such that $U \cap \mathcal{J} \neq \emptyset$. Then*

(1) We have
$$\bigcup_{n\geq 0} R^n(U) \supset \hat{\mathbb{C}} \setminus E(R) \supset \mathcal{J}.$$
(2) Moreover there exists $N \in \mathbb{N}$ such that
$$R^n(U) \supset \mathcal{J}$$
for all $n \geq N$.

Proof: Well, it makes sense to prove (1) first, because we will likely need it to prove (2) which is a stronger statement. Define
$$U_0 := \bigcup_{n\geq 0} R^n(U).$$
Define
$$V := \hat{\mathbb{C}} \setminus U_0.$$
If $V = \emptyset$ then we are done. If V has three or more points, we are led to a contradiction because this would mean that the family $\{R^n\}_{n\geq 1}$ on the set U is normal. Then we would have $U \subset \mathcal{F}$ which contradicts the fact that $U \cap \mathcal{J} \neq \emptyset$. So, V has at most 2 points. We wish to show that
$$V \subset E(R).$$
Then we get the reverse inclusion for the complements:
$$U_0 \supset \hat{\mathbb{C}} \setminus E(R).$$
So, for the sake of contradiction we assume there is some $z_0 \in V \setminus E(R)$. Then it must have an infinite orbit. We will show that a point has an infinite orbit iff the backwards orbit is infinite. Assume that the backwards orbit is finite,
$$\mathcal{O}^-(z_0) = K = \{z_0, \ldots, z_k\}.$$
Then consider R^{-1} on K. $R^{-1}(z_j)$ is a set of one or more points in K. If two points z_j and z_l have a common pre-image meaning the sets
$$R^{-1}(z_j) \cap R^{-1}(z_l) \neq \emptyset,$$
then applying R to a common point in this pre-image we get that $z_j = z_l$. Hence, for each $j = 0, \ldots k$,
$$R^{-1}(z_j) \subset K$$

is distinct. Each of these sets contains at least one point. Since K is a finite set, this means that each of these pre-images contains *exactly* one point, and so $R^{-1}: K \to K$ is a bijection. It can therefore be identified with a permutation, an element of the group S_{k+1}. This is a group of finite order, so there exists $n \in \mathbb{N}$ such that $(R^{-1})^n = R^{-n}$ acts as the identity on K. Now we consider the forward orbit. For each $z_j \in K$ we have

$$R^{-n}(z_j) = z_j \implies z_j = R^n(z_j)$$

for all $j = 0, 1, ..., n$. In particular $R^n(z_0) = z_0$. Hence

$$R^{n+k}(z_0) = R^k(z_0), \quad \forall k \in \mathbb{N}.$$

Consequently, the forward orbit $O^+(z_0)$ can have at most $n+1$ elements. This shows that if the backward orbit is finite, then the whole orbit is finite. Consequently, if the whole orbit is infinite, then the backwards orbit is infinite. Of course the reverse statement is also true: if the backwards orbit is infinite, then the whole orbit is infinite (because it contains the backward orbit!). So, we have shown the equivalence

$$\#\mathcal{O}^-(z) = \infty \iff \#\mathcal{O}(z) = \infty,$$

where in this statement z is arbitrary.
In our particular case of concern here, we have z_0 not in $E(R)$ hence it has infinite orbit, hence the backwards orbit is infinite. We will use this to achieve a contradiction. First, if some $R^{-m}(z_0) \in U_0$, for some $m \in \mathbb{N}$ then there is some $k \in \mathbb{N} \cup \{0\}$ such that

$$R^{-m}(z_0) \in R^k(U) \implies R^{-m}(z_0) = R^k(w), \quad w \in U.$$

Then applying R^m to both sides,

$$z_0 = R^{m+k}(w) \in R^{m+k}(U) \subset U_0.$$

This contradicts $z_0 \in V = \hat{\mathbb{C}} \backslash U_0$. So, this shows that we must have $R^{-m}(z_0) \ni U_0$ for all $m \in \mathbb{N}$. Since the backwards orbit of z_0 is infinite, there are infinitely many points $R^{-m}(z_0) \in \hat{\mathbb{C}} \setminus U_0$. By definition of U_0, the family of iterates R^n on U omits *all* these points, and there are not just three but infinitely many! By Montel's Theorem the family of iterates is therefore normal on U, so $U \subset \mathcal{F}$ which we have already seen is a contradiction since $U \cap \mathcal{J} \neq \emptyset$.
So, the assumption of a point $z_0 \in V \setminus E(R)$ leads in all cases to a contradiction, hence there can be no such problematic point! This shows that $V \subset E(R)$ and taking complements reverses the inclusion,

$$\hat{\mathbb{C}} \setminus V = U_0 \supset \hat{\mathbb{C}} \setminus E(R) \supset \mathcal{J}.$$

The second statement is rather ingenious. Since we know that the Julia set is infinite and perfect, the intersection $U \cap \mathcal{J}$ is not only not empty, but must contain infinitely many distinct points. Choose three distinct points. Since they are all in U which is open, let's call the points for instance z_1, z_2, z_3, and there exist $\epsilon_i > 0$ for $i \in I = \{1, 2, 3\}$ such that $D_{\epsilon_i}(z_i) \subset U$. Moreover we can choose

$$\epsilon = \frac{1}{2} \min\{\epsilon_1, \epsilon_2, \epsilon_3, |z_i - z_j| i \neq j \in I\}.$$

Then $D_\epsilon(z_i) := U_i$ are at a positive distance from each other, have non-empty intersection with $\mathcal{J}$, and are open sets contained in U.

Claim 5. *For each $i \in I$ there exists $j \in I$ and $n \in \mathbb{N}$ such that*

$$U_j \subset R^n(U_i)$$

Proof: By contradiction we assume not. Then there exists an $i \in I$ such that for each $j \in I$ and every $n \in \mathbb{N}$

$$U_j \not\subset R^n(U_i).$$

Hence

$$U_j \not\subset \cup_{n \geq 1} R^n(U_i), \quad j = 1, 2, 3.$$

Since these three sets are disjoint, there exist points in U_j which are not in $\cup_{n \geq 1} R^n(U_i)$, and which are distinct. Hence R^n on U_i omits these three points and is therefore normal. This is again a contradiction because it would imply $U_i \subset \mathcal{F}$ which it is not because $U_i \cap \mathcal{J} \neq \emptyset$.

Claim 6. *There exists $n \in \mathbb{N}$ and $i \in I$ such that*

$$U_i \subset R^n(U_i).$$

Proof: We have shown that there is some $j \in I$ such that

$$U_j \subset R^{n_1}(U_1).$$

If $j = 1$, then the claim is proven. Otherwise, without loss of generality (we can change their names) assume $U_j = U_2$. Then by the previous claim once more, we have some $k \in I$ and $n_2 \in \mathbb{N}$ such that

$$U_k \subset R^{n_2}(U_2).$$

If $k = 2$, the claim is proven. Otherwise, if $k = 1$, then

$$U_1 \subset R^{n_2}(U_2) \subset R^{n_2}(R^{n_1}(U_1)) = R^{n_2+n_1}(U_1),$$

and so in this case the claim is also proven. So, the remaining case is that $k = 3$. Then by the previous claim, there is $l \in I$ and $n_3 \in \mathbb{N}$ such that

$$U_l \subset R^{n_3}(U_3).$$

If $l = 3$, then the claim is proven. If $l = 2$, then

$$U_2 \subset R^{n_3}(U_3) \subset R^{n_3}(R^{n_2}(U_2)) = R^{n_3+n_2}(U_2),$$

and so the claim is proven. If $l = 2$, then

$$\begin{aligned} U_1 \subset R^{n_3}(U_3) \subset R^{n_3}(R^{n_2}(U_2)) &\subset R^{n_3}(R^{n_2}(R^{n_1}(U_1))) \\ &= R^{n_3+n_2+n_1}(U_1). \end{aligned}$$

So in this case the claim is also proven, and we have proven it in every possible case!

Now we can complete the proof of the proposition, which given the amount of work perhaps ought to be a theorem. For $U_i \subset R^n(U_i)$ as in the claim, let

$$S := R^n.$$

Then S is also a rational map of degree at least two. Since

$$U_i \subset S(U_i) \implies S(U_i) \subset S^2(U_i)$$

we have an increasing sequence

$$U_i \subset S(U_i) \subset \dots S^k(U_i) \subset S^{k+1}(U_i).$$

We have proven that the Julia set of R and any of its iterates R^n are identical. So the Julia set of R is the same as that of S, and we write both as $\mathcal{J}$. By definition of U_i,

$$U_i \cap \mathcal{J} \neq \emptyset,$$

and U_i is open, so by part (1) applied to U_i with respect to S,

$$\mathcal{J} \subset \cup_{n \geq 0} S^n(U_i).$$

On the right side we have an open cover by the open mapping theorem. The Julia set is a closed subset of $\mathbb{C}$ which is compact, hence $\mathcal{J}$ is also compact. Therefore any open cover admits a finite sub-cover and so there is $M \in \mathbb{N}$ such that

$$\mathcal{J} \subset \cup_{n=0}^{M} S^n(U_i) = S^M(U_i),$$

since $S^n(U_i) \subset S^M(U_i)$ for all $n \leq M$, $n \geq 0$. Note that $S^M = R^{nM}$. So, we have by complete invariance of $\mathcal{J}$ for any $s \in \mathbb{N}$

$$\mathcal{J} = R^s(\mathcal{J}) \subset R^s(R^{nM}(U_i)) = R^{nM+s}(U_i) \subset R^{nM+s}(U),$$

where the last statement follows since $U_i \subset U$. Hence for any $m \geq N := nM$ we have

$$\mathcal{J} \subset R^m(U).$$

We can now prove that the Julia set is self-similar!

Theorem 18.3. *The Julia set is self-similar in the sense that for any $z \in \mathcal{J}$,*

$$\mathcal{J} = \overline{\{R^{-n}(z)\}_{n\geq 1}}.$$

Proof: Let $z \in \mathcal{J}$. Then $z \notin E(R) \subset \mathcal{F}$, so the backwards orbit of z is infinite. Let $\epsilon > 0$ and $z_0 \in \mathcal{J}$. Consider $U := D_\epsilon(z_0)$. By the proposition there is $N \in \mathbb{N}$ such that

$$\mathcal{J} \subset R^N(U).$$

Moreover the Julia set is completely invariant which means that $R^{-n}(z) \in \mathcal{J} \subset R^N(U)$. So there exists $w \in U$ such that $R^{-n}(z) = R^N(w)$ and hence $w \in R^{-n-N}(z)$. By definition of $U \ni w$

$$|w - z_0| < \epsilon.$$

This shows that for each $z_0 \in \mathcal{J}$ and $\epsilon > 0$, there is an element of $\mathcal{O}^-(z) = \{R^{-n}(z)\}_{n\geq 1}$ which is at a distance less than ϵ from z_0. Hence $\mathcal{O}^-(z)$ is dense in $\mathcal{J}$. Therefore the closure of this set contains the closure of $\mathcal{J}$ which is equal to $\mathcal{J}$ because $\mathcal{J}$ is closed.

This last result as well as our previous result shows the connection between Julia sets and sets of non-integer Hausdorff dimension. Julia sets have an invariance property, a self-similarity property, and either have empty interior or are the whole space!

18.1. **The Mandelbrot set.** The Mandelbrot set focuses on the dynamics of quadratic polynomials

$$P_c(z) := z^2 + c.$$

As you will show in the exercises, the c-plane is like the moduli space of quadratic polynomials, because it is in bijection with the conjugacy classes of quadratic polynomials. So, when we look at the Mandelbrot set, we are looking at the dynamics of the conjugacy classes of quadratic polynomials. Recall the definition

$$\mathcal{M} := \{c \in \mathbb{C} : P_c^n(0) \text{ are bounded for all } n \in \mathbb{N}\}.$$

Let us prove a result which characterizes the Mandelbrot set.

Theorem 18.4. *The Mandelbrot set is a closed subset of the disk $\{|c| \leq 2\}$ which meets the real axis in the interval $[-2, 1/4]$. The Mandelbrot set consists of precisely those c such that $|P_c^n(0)| \leq 2$ for all n. The c in the Mandelbrot set are precisely those $c \in \mathbb{C}$ such that 0 does not belong to the basin of attraction of the superattracting fixed point at ∞. Moreover, the Mandelbrot set does not have any holes, in the sense that there are no open bounded components of $\mathbb{C} \setminus \mathcal{M}$.*

Proof: Assume that $|c| > 2$. We claim that for all $n \geq 2$ we have:

$$|P_c^n(0)| \geq |c|(|c|-1)^{2^{n-2}}.$$

Let us compute the base case, that is for $n = 2$:

$$|P_c^2(0)| = |c^2 + c| \geq |c|^2 - |c| = |c|(|c|-1) = |c|(|c|-1)^{2^{2-2}}.$$

So the base case is true, and now we assume it for some $n \geq 2$. Then we compute:

$$|P_c^{n+1}(0)| = |(P_c^n(0))^2 + c| \geq |P_c^n(0)|^2 - |c| \geq \left(|c|(|c|-1)^{2^{n-2}}\right)^2 - |c|$$
$$= |c|^2(|c|-1)^{2^{n-2}*2} - |c| = |c|^2(|c|-1)^{2^{n-1}} - |c|.$$

Since $|c| \geq 2$, for $n \geq 2$ note that

$$(|c|-1)^{2^{n-1}} \geq 1.$$

Consequently

$$|c|(|c|-1)^{2^{n-1}} \geq (|c|-1)^{2^{n-1}} + (|c|-1)^{2^{n-1}} \geq 1 + (|c|-1)^{2^{n-1}}.$$

Multiplying the far left and far right sides both by $|c|$ we get

$$|c|^2(|c|-1)^{2^{n-1}} \geq |c| + |c|(|c|-1)^{2^{n-1}}.$$

Therefore we have proven that

$$|P_c^{n+1}(0)| \geq |c|^2(|c|-1)^{2^{n-1}} - |c| \geq |c|(|c|-1)^{2^{n-1}}.$$

This completes the proof by induction, because it is the statement for $n+1$. So, whenever $|c| > 2$, and $n > 2$, the iterates of P_c are bounded below by

$$|c|(|c|-1)^{2^{n-1}} \to \infty \text{ as } n \to \infty \text{ since } |c| > 2 \implies |c| - 1 > 1.$$

So, we therefore see that the Mandelbrot set is contained in the disk $\overline{D_2(0)}$.
Moreover, if $|P_c^n(0)| \leq 2$ for all n, then this certainly implies that $c \in \mathcal{M}$. So, it is a sufficient condition. To see that it is also a necessary condition, we assume that $|P_c^m(0)| = 2 + \delta$ for some $\delta > 0$, and for some m. If $|c| = |P_c(0)| > 2$, then we have already proven that $c \notin \mathcal{M}$. So, now assume that

$$|P_c(0)| = |c| \leq 2,$$

and for some $m \geq 1$, we have

$$|P_c^{m+1}(0)| = |(P_c^m(0))^2 + c| \geq (2+\delta)^2 - |c| \geq (2+\delta)^2 - 2 \geq 2 + 4\delta.$$

We claim by induction that

$$|P_c^{m+k}(0)| \geq 2 + 4^k\delta.$$

The base case with $k = 1$ is proven. We assume it for some k and then need to show it for $k+1$. So we estimate

$$|P_c^{m+k+1}(0)| \geq |P_c^{m+k}(0)|^2 - |c| \geq (2 + 4^k\delta)^2 - 2 = 4 + 4(4^k)\delta + (4^k\delta)^2 - 2$$
$$= 2 + 4^{k+1}\delta + 4^{2k}\delta^2 \geq 2 + 4^{k+1}\delta.$$

This completes the proof by induction. Hence,

$$|P_c^{m+k}(0)| \geq 2 + 4^k\delta \to \infty \text{ as } k \to \infty,$$

which means that c cannot be by definition in $\mathcal{M}$. Hence, if $c \in \mathcal{M}$, the it is necessarily true that

$$|P_c^m(0)| \leq 2 \quad \forall m \geq 1.$$

To prove that $\mathcal{M}$ is closed, let us change our perspective a bit. I claim that for each n, the function

$$|P_c^n(0)|$$

is a continuous function of c. For $n=1$,

$$|P_c^1(0)| =: f_1(c) = |c|.$$

This is a continuous function. Proceeding by induction as usual then we assume that $f_n(c) := |P_c^n(0)|$ is a continuous function of c. Then

$$f_{n+1}(c) = |P_c^{n+1}(0)| = |f_n(c)^2 + c|.$$

The function $f_n(c)^2 + c$ is a continuous function of c since f_n is continuous. Moreover the absolute value function is also a continuous function. Therefore the composition is a continuous function. This completes the proof by induction. So, we can now write

$$\begin{aligned}\mathcal{M} &= \{c \in \mathbb{C} : f_n(c) \in \overline{D_2(0)} \quad \forall n \geq 1\} \\ &= \bigcap_{n\geq 1} f_n^{-1}(\overline{D_2(0)}).\end{aligned}$$

Since each f_n is a continuous function, the pre-image of the closed disk is a closed set. The intersection of any collection of closed sets is also a closed set. This shows that $\mathcal{M}$ is closed.
Next, we wish to prove that $\mathcal{M}$ does not have any holes in it. For this, let us define instead functions

$$\varphi_n(c) := P_c^n(0).$$

We claim that these are holomorphic (and indeed entire, polynomial) functions of c. The base case is certainly true. Then,

$$\varphi_{n+1}(c) = P_c^{n+1}(0) = (P_c^n(0))^2 + c = \varphi_n(c)^2 + c.$$

Since by induction, φ_n is a polynomial, we also have that $\varphi_n(c)^2$ is a polynomial. Consequently

$$\varphi_{n+1}(c)$$

is also a polynomial. Now, let's think about the open set, $\mathbb{C} \setminus \mathcal{M}$. For the sake of contradiction, assume that it has some bounded component, Ω. Then Ω is a bounded, open set, and its boundary is contained in $\mathcal{M}$. The functions φ_n are all holomorphic on Ω. Therefore, the maximum principle dictates that they achieve their maximum values on the boundary, that is for any $c \in \Omega$,

$$|\varphi_n(c)| \leq \sup_{z\in\partial\Omega} |\varphi_n(z)|.$$

Now, since $\partial\Omega \subset \mathcal{M}$, we have proven that

$$|\varphi_n(z)| \leq 2 \quad \forall z \in \mathcal{M}.$$

So then we get that

$$|\varphi_n(c)| \leq 2 \quad \forall c \in \Omega.$$

However, this immediately implies, by our characterization of the Mandelbrot set, that $\Omega \subset \mathcal{M}$. This is a contradiction. Hence, the Mandelbrot set has no holes.
So, let us proceed with the formulation of the Mandelbrot set in terms of the basin of attraction of the super attracting fixed point at infinity. It suffices to prove that

$$|P_c^n(0)| \leq 2 \forall n \iff 0 \notin A(\infty),$$

where $A(\infty)$ is the basin of attraction of the super attracting fixed point of P_c at ∞. Now $\implies$ is pretty clear, because if $0 \in A(\infty)$ then this requires $P_c^n(0) \to \infty$ which clearly cannot happen if the iterates are all bounded above by 2.
For the converse direction, we first show that $P_c(A(\infty)) = A(\infty)$. Assume that $z \in A(\infty)$. Then we have

$$P_c^n(z) \to \infty \implies P_c^n(P_c(z)) \to \infty \implies P_c(z) \in A(\infty) \implies P_c(A(\infty)) \subset A(\infty).$$

Next note that if $z \in A(\infty)$ then

$$P_c^{n-1}(z) \to \infty \implies P_c^n(P_c^{-1}(z)) \to \infty \implies P_c^{-1}(z) \in A(\infty) \implies P_c^{-1}(A(\infty)) \subset A(\infty)$$

so we get applying P_c to both sides

$$A(\infty) \subset P_c(A(\infty)).$$

Consequently, we have equality, that is

$$A(\infty) = P_c(A(\infty)).$$

We can repeat the argument to get that for all n

$$A(\infty) = P_c^n(A(\infty)), \quad P_c^{-n}(A(\infty)) = A(\infty).$$

If for some m we have $|P_c^m(0)| > 2$ we get that $P_c^m(0) \in A(\infty)$ and therefore $0 \in P_c^{-m}(A(\infty)) = A(\infty)$. So, this shows that if $0 \notin A(\infty)$ then we get

$$|P_c^m(0)| \le 2 \quad \forall m \ge 1 \implies c \in \mathcal{M}.$$

Finally, let us ponder the real numbers in the Mandelbrot set. If c is real, then the equation

$$P_c(x) - x = 0 \iff x^2 - x + c = 0 \iff x = \frac{1}{2} \pm \frac{\sqrt{1-4c}}{2},$$

has no real roots if $c > \frac{1}{4}$. The equation has one real root at $\frac{1}{2}$ if $c = \frac{1}{4}$, and two real roots if $c < \frac{1}{4}$. If $c > \frac{1}{4}$, then first note that $P_c^n(0)$ is real and increasing. This can be proven by induction:

$$P_c(0) = c, \quad P_c^2(0) = c^2 + c > c > \frac{1}{4}.$$

Similarly, for

$$P_c^{n+1}(0) = P_c^n(0)^2 + c,$$

we have that

$$P_c^n(0)^2 - P_c^n(0) + c > 0$$

because the equation

$$x^2 - x + c$$

has no real roots for $c > \frac{1}{4}$. By induction, $P_c^n(0) = x > \frac{1}{4}$. So, the sequence is real and increasing. If $P_c^n(0)$ had some finite limit point, call it x, then since the limit of real numbers is a real number when it exists, we would get

$$\lim_{n\to\infty} P_c^n(x) = \lim_{n\to\infty} P_c^{n+1}(x) = x \implies P_c(x) = x \lightning.$$

So, this shows that no real numbers greater than $\frac{1}{4}$ are in $\mathcal{M}$. Now, we already know that no real numbers less than 2 are in $\mathcal{M}$ because any number with modulus greater than 2 is not in $\mathcal{M}$. So, finally on the interval $[-2, 1/4]$, let

$$a = \frac{1}{2} + \frac{\sqrt{1-4c}}{2}$$

be the larger of the two real roots of $P_c(x) - x = 0$. Then, since $c \in [-2, 1/4]$, we have

$$a = \frac{1}{2} + \frac{\sqrt{1-4c}}{2} \ge \frac{1}{2} + \frac{\sqrt{1+8}}{2} = \frac{1}{2} + \frac{3}{2} = 2 \ge |c| = |P_c(0)|.$$

Then we claim by induction that $|P_c^n(0)| \le a$ for all n. We have proven the base case. Next,

$$|P_c^{n+1}(0)| = |(P_c^n(0))^2 + c| = \pm(P_c^n(0)^2 + c).$$

In case of $+$ we have

$$(P_c^n(0))^2 + c \le a^2 + c = a$$

since

$$a^2 - a + c = 0.$$

In case of $-$ we have

$$-(P_c^n(0))^2 - c \le -c \le a,$$

since we proved that

$$a \geq |c|.$$

So this proves that the iterates all have $|P_c^n(0)| \leq a$ for all n. By definition, $c \in \mathcal{M}$.
It is nice to recall, but we shall not prove, the following result which further characterizes the Mandelbrot set.

Theorem 18.5. *If $P_c^n(0) \to \infty$, then the Julia set of P_c is totally disconnected. Otherwise, $P_c^n(0)$ is bounded, and the Julia set is connected. Consequently, the Mandelbrot set consists of precisely those c such that the Julia set of the conjugacy class of P_c is connected.*

What is meant by totally disconnected?

Definition 18.6. A set S is totally disconnected in this context means that the connected components are single points.

The geometry of the Mandelbrot set and its various bits and pieces is therefore closely tied to the geometry of the Julia and Fatou sets of the conjugacy classes of quadratic polynomials. It could be interesting to investigate other Mandelbrot inspired type sets. For instance, what if we replace the conjugacy classes of quadratic polynomials with polynomials of a different degree? What happens? What if we instead look at such a thing but for conjugacy classes of rational functions of a certain degree, d? Do you have other ideas for interesting related topics to investigate? Have fun with it!

18.2. **Homework.**

(1) Show that any quadratic polynomial can be conjugated to a monic polynomial, $z^2 + \alpha z + \beta$.
(2) Show that any monic polynomial can be conjugated to move any given point to 0.
(3) Show that conjugating a fixed point to 0 you obtain $\lambda z + z^2$, where λ is the multiplier of the fixed point.
(4) Show that to uniquely determine the conjugacy class of the polynomial, you can move the critical point to 0, and then the polynomial is of the form $P_c(z) = z^2 + c$. In this way different c correspond to different conjugacy classes of quadratic polynomials.
(5) Show that the Hausdorff dimension of the Julia set of the polynomial $P_c(z)$ is 2 if c is on the boundary of the Mandelbrot set.
(6) * For small values of t, consider a family $\{f_t\}$ of maps of the form $z \mapsto e^{-2\pi i t} z + z^2$. Show that there exists a sequence $\{t_n\}$ such that the Hausdorff dimension d_n of the Julia set of f_{t_n} satisfies

$$\limsup_{n \to \infty} d_n = 2.$$

18.3. **Hints.** This hint is for exercise # 6. This is actually a pretty recent result contained in a research article from Heinemann & Stratmann published in 2001. If you get stuck on this exercise, find their paper and work through the proof. Interestingly, the proof connects IFS fractals to Julia sets of quadratic polynomials, thereby tying together the main topics of this course. It seemed like a nice way to wrap things up.

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