FOURIER ANALYSIS & METHODS 2020.02.24

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Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

1. Dirichlet problem in a quadrant

Consider the problem

$$u_{xx} + u_{yy} = 0, \quad x, y > 0, \quad u(x, 0) = f(x), \quad u(0, y) = g(y).$$

To deal with these inhomogeneities let us instead solve two nicer problems:

(1) $w_{xx} + w_{yy} = 0, \quad x, y > 0, \quad w(x, 0) = f(x), \quad w(0, y) = 0.$
(2) $v_{xx} + v_{yy} = 0, \quad x, y > 0, \quad v(x, 0) = 0, \quad v(0, y) = g(y).$

The full solution will then be obtained by setting

$$u(x, y) = w(x, y) + v(x, y).$$

Exercise 1. *Verify that if w and v solve the problems above, then indeed u solves the original problem.*

We would like to use Fourier methods, but the problems we have above $x, y > 0$. The Fourier transform is defined on the whole plane. So, we may wish to use an even or odd extension.

> Idea: To solve a problem like $w_{xx}+w_{yy} = 0, \quad x, y > 0, \quad w(x, 0) = f(x), \quad w(0, y) = 0$, look at the boundary condition. The solution should vanish at $x = 0$. Now think about sine and cosine. Which of these vanishes at $x = 0$? The sine. That is an odd function. So this gives us the clue to extend oddly.

We define therefore

$$f_o(x) := \begin{cases} f(x) & x > 0 \\ -f(-x) & x < 0 \end{cases}.$$

Now, we take the Fourier transform of the PDE in the x variable. We obtain:

$$-\xi^2 \hat{w}(\xi, y) + \partial_{yy}\hat{w}(\xi, y) = 0 \implies \hat{w}(\xi, y) = A(\xi)e^{-|\xi|y} + B(\xi)e^{|\xi|y}.$$

The functions A and B can depend on ξ but not on y. We would like to use Fourier methods which requires staying within $\mathcal{L}^2$. Hence we do not want the second solution because $y > 0$ so it is very much not in $\mathcal{L}^2$. Thus, we keep the first solution. The boundary condition at $y = 0$ acts like an initial condition, at least from x's perspective:

$$\hat{w}(\xi, 0) = A(\xi) = \hat{f}_o(\xi) \implies \hat{w}(\xi, y) = \hat{f}_o(\xi) e^{-|\xi| y}.$$

We look at table 2 of Folland to find a function whose Fourier transform is $e^{-|\xi|y}$. OBS! The transform is occurring in the x variable, from whose perspective y is a constant. Thus, the item on the table is a slight modification of 10, in particular the function

$$\frac{y}{\pi}(x^2+y^2)^{-1} \text{ has Fourier transform in the } x \text{ variable } e^{-y|\xi|}.$$

Thus, we have found

$$\hat{w}(\xi, y) = \hat{f}_o(\xi) \widehat{\frac{y}{\pi}(x^2+y^2)^{-1}}(\xi).$$

The Fourier transform sends convolutions to products, which tells us that

$$w(x,y) = \int_{-\infty}^{\infty} f_o(z) \frac{y}{\pi((x-z)^2+y^2)} dz = \int_{-\infty}^{0} f_o(z) \frac{y}{\pi((x-z)^2+y^2)} dz + \int_0^\infty f(z) \frac{y}{\pi((x-z)^2+y^2)} dz$$

We do a substitution in the first integral, with $t = -z$

$$= \int_{-\infty}^{0} f_o(z) \frac{y}{\pi((x-z)^2+y^2)} dz = -\int_{\infty}^{0} f_o(-t) \frac{y}{\pi((x+t)^2+y^2)} dt$$

$$= \int_{\infty}^{0} f(t) \frac{y}{\pi((x+t)^2+y^2)} dt = -\int_0^\infty f(t) \frac{y}{\pi((x+t)^2+y^2} dt.$$

Re-naming the variable of integration z, we get

$$w(x,y) = \int_0^\infty f(z) \left[\frac{y}{\pi((x-z)^2+y^2)} - \frac{y}{\pi((x+z)^2+y^2)} \right] dz.$$

The other problem is basically identical, we simply Fourier transform in the y variable. Thus the solution to the second problem is

$$v(x,y) = \int_0^\infty g(z) \left[\frac{x}{\pi((y-z)^2+x^2)} - \frac{x}{\pi((y+z)^2+x^2)} \right] dz.$$

We obtain the full solution by adding:

$$u(x,y) = w(x,y) + v(x,y).$$

2. The Laplace transform

We shall now enter Chapter 8, and learn about another useful transform, known as the Laplace transform.

Definition 1. *Assume that*

lap0 (2.1)
$$f(t) = 0 \quad \forall t < 0,$$

and that there exists $a, C > 0$ such that

lapa (2.2)
$$|f(t)| \le Ce^{at} \quad \forall t \ge 0.$$

Then for we define for $z \in \mathbb{C}$ with $\Re(z) > a$ the Laplace transform of f at the point z to be

$$\mathfrak{L}f(z) = \hat{f}(-iz) = \int_0^\infty f(t) e^{-zt} dt.$$

We may also use the notation

$$\widetilde{f}(z) = \mathfrak{L}f(z).$$

Let us verify that the Laplace transform is well defined. To do so, we estimate

$$|\mathfrak{L}f(z)| \leq \int_0^\infty |f(t)e^{-zt}|dt \leq \int_0^\infty Ce^{at}|e^{-zt}|dt = \int_0^\infty e^{at}e^{-\Re(z)t}dt$$

$$= \left.\frac{e^{t(a-\Re(z))}}{a-\Re(z)}\right|_0^\infty = \frac{1}{\Re(z)-a}.$$

Above we have used the fact that

$$\left|e^{\text{complex number}}\right| = e^{\text{real part}}.$$

Due to this beautiful convergence, $\mathfrak{L}f(z)$ is holomorphic in the half plane $\Re(z) > a$. This is because we may differentiate under the integral sign due to the absolute convergence of the integral. The assumption that $f(t) = 0$ for all negative t is not actually necessary, we could just make it so. For this purpose we define the *heavyside* function, commonly denoted by

$$\Theta(t) := \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}.$$

If we have some f defined on $\mathbb{R}$ which satisfies (lapa 2.2) but is not (lap0 2.1), we can apply the Laplace transform to Θf. Another thing which can happen is that we have a function which is only defined on $[0, \infty)$. In that case, we can just extend it to be identically zero on $(-\infty, 0)$.

Let's familiarize ourselves with the Laplace transform by demonstrating some of its fundamental properties.

Proposition 2 (Properties of $\mathfrak{L}$)**.** *Assume f and g satisfy (lapa 2.2) and (lap0 2.1), then*

(1) $\mathfrak{L}f(x+iy) \to 0$ *as* $|y| \to \infty$ *for all* $x > a$.
(2) $\mathfrak{L}f(x+iy) \to 0$ *as* $x \to \infty$ *for all* y.
(3) $\mathfrak{L}(\Theta(t-a)f(t-a))(z) = e^{-az}\mathfrak{L}f(z)$.
(4) $\mathfrak{L}(e^{ct}f(t))(z) = \mathfrak{L}f(z-c)$.
(5) $\mathfrak{L}(f(at)) = a^{-1}\mathfrak{L}f(a^{-1}z)$.
(6) **** If f is continuous and piecewise $\mathcal{C}^1$ on $[0,\infty)$, and f' satisfies (lapa 2.2) and (lap0 2.1), then*

$$\mathfrak{L}(f')(z) = z\mathfrak{L}f(z) - f(0).$$

(7) $\mathfrak{L}(\int_0^t f(s)ds)(z) = z^{-1}\mathfrak{L}f(z)$.
(8) $\mathfrak{L}(tf(t))(z) = -(\mathfrak{L}f)'(z)$.
(9) $\mathfrak{L}(f * g)(z) = \mathfrak{L}f(z)\mathfrak{L}g(z)$.
(10) *If $t^{-1}f(t)$ satisfies (lap0 2.1) and (lapa 2.2), then*

$$\mathfrak{L}(t^{-1}f(t))(z) = \int_z^\infty \mathfrak{L}f(w)dw.$$

The integral is any contour in the w-plane which starts at z along which $\Im w$ stays bounded and $\Re w \to \infty$.

Proof: There's a bunch of stars next to #6 because it's the reason the Laplace transform is useful for solving PDEs and ODEs. It's quite similar to how the Fourier transform takes in derivatives and spits out multiplication. Intuitively, this fact about $\mathfrak{L}$ should jive with the similar fact about $\mathcal{F}$ because well, the Laplace transform is just the Fourier transform taken at a complex point.

(1) The first statement

$$\mathfrak{L}f(z) = \int_0^\infty e^{-(x+iy)t} f(t)dt = \int_0^\infty e^{-xt} f(t) e^{-iyt} dt = \hat{g}(y),$$

for the function

$$g(t) = e^{-xt} f(t).$$

The Riemann-Lebesgue Lemma says that $\hat{g}(y) \to 0$ when $|y| \to \infty$.

(2) The second statement is more satisfying because we just compute and estimate directly. We did this estimate above already, where we got

$$|\mathfrak{L}f(z)| \le \frac{1}{\Re(z) - a} \to \infty \text{ when } \Re(z) = x \to \infty.$$

(3) The third statement is also a direct computation:

$$\mathfrak{L}(\Theta(t-a)f(t-a))(z) = \int_0^\infty \Theta(t-a)f(t-a)e^{-zt}dt = \int_{-a}^\infty \Theta(s)f(s)e^{-z(s+a)}ds.$$

Above we did the substitution $s = t - a$ so $ds = dt$. Since f and the Heavyside function are zero for negative s, and the Heavyside function is 1 for positive s, this is

$$e^{-za} \int_0^\infty f(s) e^{-zs} ds = e^{-za} \mathfrak{L}f(z).$$

(4) Similarly, we directly compute

$$\mathfrak{L}(e^{ct} f)(z) = \int_0^\infty e^{ct} e^{-zt} f(t) dt = \int_0^\infty e^{-(z-c)t} f(t) dt = \mathfrak{L}f(z-c).$$

(5) Again no surprise, we compute

$$\mathfrak{L}(f(at))(z) = \int_0^\infty e^{-zt} f(at) dt = \int_0^\infty e^{-zs/a} f(s) \frac{ds}{a} = a^{-1} \mathfrak{L}f(z/a).$$

Here we used the substitution $s = at$ so $a^{-1} ds = dt$.

(6) Now we are finally getting to the important one:

$$\mathfrak{L}(f')(z) = \int_0^\infty e^{-zt} f'(t) dt = e^{-zt} f(t) \Big|_0^\infty + \int_0^\infty z e^{-zt} f(t) dt.$$

We have used integration by parts above. By (lapa 2.2) and since $\Re(z) > a$, the limit as $t \to \infty$ is zero, and so we get

$$\mathfrak{L}(f')(z) = -f(0) + z\mathfrak{L}f(z).$$

Awesome.

(7) Next we define

$$F(t) = \int_0^t f(s) ds.$$

Then, we use the preceding fact:

$$\mathfrak{L}(F')(z) = z\mathfrak{L}F(z) - F(0) = z\mathfrak{L}F(z).$$

Since $F' = f$ we get

$$z^{-1}\mathfrak{L}(f)(z) = \mathfrak{L}(\int_0^t f(s) ds)(z).$$

(8) Next, we compute:

$$\mathfrak{L}(tf(t))(z) = \int_0^\infty te^{-zt} f(t)dt = \int_0^\infty \frac{d}{dz}\left(-e^{-zt}\right) f(t)dt$$
$$= \frac{d}{dz}\left(-\int_0^\infty e^{-zt} f(t)dt\right) = -(\mathfrak{L}f)'(z).$$

Yes, we have used the absolute convergence of the integral to swap limits. It's legit yo.

(9) Nearing the finish line, we compute

$$\mathfrak{L}(f * g)(z) = \mathcal{F}(f * g)(-iz) = \hat{f}(-iz)\hat{g}(-iz) = \mathfrak{L}f(z)\mathfrak{L}g(z).$$

(10) Finally, note that by (2.2), if $t^{-1}f(t)$ satisfies this, then at the point $t = 0$ apparently the function f vanishes, so that the function $t^{-1}f(t)$ is well defined. So, don't panic about this point!!! We next define the holomorphic function

$$F(z) = \int_z^\infty \widetilde{f}(w)dw.$$

Since $\widetilde{f}(w) \to 0$ when $\Re(w) \to \infty$ and $\Im(w)$ stays bounded, the fundamental theorem of calculus says that

$$F'(z) = -\widetilde{f}(z).$$

On the other hand,

$$\frac{d}{dz}\int_0^\infty t^{-1}f(t)e^{-zt}dt = \int_0^\infty -f(t)e^{-zt}dt = -\widetilde{f}(z).$$

Hence,

$$F(z) = \int_0^\infty t^{-1}f(t)e^{-zt}dt + c,$$

for some constant c. Since

$$\lim_{\Re z \to \infty} F(z) = 0 = \lim_{\Re(z) \to \infty} \int_0^\infty t^{-1}f(t)e^{-zt}dt \implies c = 0.$$

2.1. **Application to solving ODEs.** We see that

$$\mathfrak{L}(f')(z) = z\mathfrak{L}f(z) - f(0).$$

Let's do it again:

$$\mathfrak{L}(f'')(z) = z\mathfrak{L}(f')(z) - f'(0) = z\left(z\mathfrak{L}f(z) - f(0)\right) - f'(0) = z^2\mathfrak{L}f(z) - zf(0) - f'(0).$$

In general:

Proposition 3. *Assume that everything is defined, then*

$$\mathfrak{L}(f^{(k)})(z) = z^k\mathfrak{L}f(z) - \sum_{j=1}^{k} f^{(k-j)}(0)z^{j-1}.$$

Proof: Well clearly we should do a proof by induction! Check the base case first:

$$\mathfrak{L}(f')(z) = z\mathfrak{L}f(z) - f(0).$$

Here $k = 1$ and the sum has only one term with $j = k = 1$. It works. Now we assume the above formula holds and we show it for $k+1$. We compute

$$\mathfrak{L}(f^{(k+1)})(z) = \mathfrak{L}((f^{(k)})')(z) = z\mathfrak{L}(f^{(k)})(z) - f^{(k)}(0).$$

By induction this is

$$z\left(z^k\mathfrak{L}f(z) - \sum_{j=1}^{k} f^{(k-j)}(0)z^{j-1}\right) - f^{(k)}(0).$$

This is

$$z^{k+1}\mathfrak{L}f(z) - \sum_{j=1}^{k} f^{(k-j)}(0)z^{j} - f^{(k)}(0).$$

Let us change our sum: let $j+1 = l$. Then our sum is

$$\sum_{l=2}^{k+1} f^{k-(l-1)}(0)z^{l-1} = \sum_{l=2}^{k+1} f^{(k+1-l)}(0)z^{l-1}.$$

Observe that

$$f^{(k)}(0) = f^{k+1-1}(0)z^{1-1}.$$

Hence

$$-\sum_{j=1}^{k} f^{(k-j)}(0)z^{j} - f^{(k)}(0) = -\sum_{l=1}^{k+1} f^{(k+1-l)}(0)z^{l-1}.$$

So, we have computed

$$\mathfrak{L}(f^{(k+1)})(z) = z^{k+1}\mathfrak{L}f(z) - \sum_{l=1}^{k+1} f^{(k+1-l)}(0)z^{l-1}.$$

That is the formula for $k+1$, which is what we needed to obtain.

For this reason one can use $\mathfrak{L}$ to solve linear constant coefficient ODEs which can be *non-homogeneous!* Let us see how this works... A linear, constant coefficient ODE of order n looks like:

$$\sum_{k=0}^{n} c_k u^{(k)}(t) = f(t).$$

In order for the solution to be unique, there must be specified initial conditions on u, that is

$$u(0), u'(0), \ldots u^{(n-1)}(0).$$

We are not requiring $f(t)$ to be the zero function, so the ODE could be *inhomogeneous.* Notoriously difficult to solve right? NOT ANYMORE! We hit both sides of the ODE with $\mathfrak{L}$:

$$\sum_{k=0}^{n} c_k \mathfrak{L}(u^{(k)})(z) = \widetilde{f}(z).$$

Let's write out the left side using our proposition. First we have

$$c_0\widetilde{u}(z).$$

Then we have

$$c_1\left(z\widetilde{u}(z)-u(0)\right).$$

By our proposition, we computed that for $k \geq 1$,

$$\mathfrak{L}(c_k u^{(k)})(z) = c_k\left(z^k\widetilde{u}(z) - \sum_{j=1}^{k} u^{(k-j)}(0)z^{j-1}\right).$$

Therefore the left side of the ODE becomes

$$c_0\widetilde{u}(z) + \sum_{k=1}^{n} c_k\left(z^k\widetilde{u}(z) - \sum_{j=1}^{k} u^{(k-j)}(0)z^{j-1}\right)$$

$$= \sum_{k=0}^{n} c_k z^k\widetilde{u}(z) - \sum_{k=1}^{n} c_k \sum_{j=1}^{k} u^{(k-j)}(0)z^{j-1}.$$

We therefore define two polynomials

$$P(z) := \sum_{k=0}^{n} c_k z^k,$$

$$Q(z) := -\sum_{k=1}^{n} c_k \sum_{j=1}^{k} u^{(k-j)}(0)z^{j-1}.$$

Our ODE has been LAPLACE-TRANSFORMED into

$$P(z)\widetilde{u}(z) + Q(z) = \widetilde{f}(z).$$

We can solve this for $\widetilde{u}(z)$:

$$\widetilde{u}(z) = \frac{\widetilde{f}(z) - Q(z)}{P(z)}.$$

Hence to get our solution $u(t)$ we just need to invert the Laplace transform of the right side, that is our solution will be

$$u(t) = \mathfrak{L}^{-1}\left(\frac{\widetilde{f}(z) - Q(z)}{P(z)}\right).$$

2.2. **Exercises for the week.** These exercises will be demonstrated for you.

(1) (Eö 55)
(2) (7.4.4) Solve the heat equation $u_t = ku_{xx}$ on the half line $x > 0$ with boundary conditions $u(x, 0) = f(x)$ and initial condition $u(0, t) = 0$. Do the same for the inhomogeneous heat equation $u_t = ku_{xx} + G(x, t)$ with the same initial and boundary conditions.
(3) (7.4.6) Solve Laplace's equation $u_{xx} + u_{yy} = 0$ in the semi-infinite strip $x > 0$, $0 < y < 1$ with the boundary conditions $u_x(0, y) = 0$, $u_y(x, 0) = 0$, $u(x, 1) = e^{-x}$. Express the answer as a Fourier integral.
(4) (8.4.2) Find the temperature in a semi-infinite rod (the half-line $x > 0$) if its initial temperature is zero, and the end $x = 0$ is held at temperature 1 for $0 < t < 1$ and temperature 0 thereafter.
(5) (Eö 14)
(6) (Eö 45)

2.2.1. *Exercises for the week to be done oneself.*

(1) (7.3.1) Use the Fourier transform to find a solution of the ordinary differential equation $u'' - u + 2g(x) = 0$ where $g \in \mathcal{L}^1(\mathbb{R})$. The solution obtained in this way is the one that vanishes at $\pm\infty$.

(2) (7.4.7) Solve Laplace's equation $u_{xx} + u_{yy} = 0$ in the semi-infinite strip $x > 0$, $0 < y < 1$ with the boundary conditions $u(0, y) = 0$, $u(x, 0) = 0$, $u(x, 1) = e^{-x}$. Express the answer as a Fourier integral.

(3) (Eö 47)

(4) (8.4.1) Solve:

$$u_t = ku_{xx} - au, \quad x > 0, \quad u(x, 0) = 0, \quad u(0, t) = f(t).$$

(5) (8.4.3) Consider heat flow in a semi-infinite rod when heat is supplied to the end at a constant rate c:

$$u_t = ku_{xx} \text{ for } x > 0, \quad u(x, 0) = 0, \quad u_x(0, t) = -c.$$

With the aid of the computation:

$$\mathcal{L}\left(\frac{1}{\sqrt{\pi t}} e^{-a^2/(4t)}\right)(z) = \frac{e^{-a\sqrt{z}}}{\sqrt{z}},$$

show that

$$u(x, t) = c\sqrt{\frac{k}{\pi}} \int_0^t s^{-1/2} e^{-x^2/(4ks)} ds.$$

By substituting

$$\sigma = \frac{x}{\sqrt{4ks}}$$

and then integrating by parts, show that

$$u(x, t) = c\sqrt{\frac{4kt}{\pi}} e^{-x^2/(4kt)} - cx \operatorname{erfc}\left(\frac{x}{\sqrt{4kt}}\right).$$

(6) (Eö 12)

References

[1] Gerald B. Folland, *Fourier Analysis and Its Applications,* Pure and Applied Undergraduate Texts Volume 4, (1992).