FOURIER ANALYSIS & METHODS 2020.02.26

JULIE ROWLETT

Abstract. Caveat Emptor! These are just informal lecture notes. Errors are inevitable! Read at your own risk! Also, this is by no means a substitute for the textbook, which is warmly recommended: *Fourier Analysis and Its Applications,* by Gerald B. Folland. He was the first math teacher I had at university, and he is awesome. A brilliant writer. So, why am I even doing this? Good question...

1. Geometric settings in which we can solve PDEs

We are on the home stretch! So far, the geometric settings we can handle are:

(1) finite intervals and rectangles, using Fourier series and SLP techniques;
(2) the entire real line, using Fourier transform;
(3) with nice boundary conditions, a half line using even or odd extensions;
(4) with a time-dependent boundary condition, a half line using Laplace transform;
(5) combining techniques to deal with half-spaces and quadrants.

2. Fun with drums and Bessel functions

Why do drums sound the way they do? This is actually a question that even today we do not completely understand. You'll soon understand why...

We shall solve the initial value problem for a vibrating drum. We begin by mathematicizing the drumhead as a circular membrane. Since it is a drumhead, the boundary is attached to the rest of the drum, so the boundary does not vibrate, it remains fixed. We think of the drumhead as being instantaneously still at the moment when we hit it. Consequently, the height on the drum at a point $z = (x, y)$ and time t satisfies:

$$u_{tt} - u_{xx} - u_{yy} = 0, \quad x^2 + y^2 \le L^2, \quad \begin{cases} u(x,y,t) = 0 & (x,y) \text{ on the boundary} \\ u_t(x,y,0) = 0 \\ u(x,y,0) = f(x,y) \end{cases}.$$

To solve this problem, we see that it is pretty decent and homogeneous, and it is also occurring in a bounded region of the plane. So we see if we can use separation of variables. For this we first separate the time and space variables. So our equation is

$$T''(t)S(x,y) - S_{xx}(x,y)T - S_{yy}(x,y)T = 0.$$

We divide everything by TS, move things around, and get

$$\frac{T''}{T} = \frac{S_{xx} + S_{yy}}{S}.$$

Since each side depends on a different variable, we have the equation

$$\frac{S_{xx}+S_{yy}}{S}=\lambda=\frac{T''}{T}.$$

Which side to solve first? We have the nice homogeneous boundary condition for the space variables, so we should solve for the space variables first. Consequently we seek a solution to:

$$S_{xx}+S_{yy}=\lambda S.$$

Expressing the boundary using x and y it is:

$$x^2+y^2=L^2.$$

This is not something of the form "variable equals value." It is more complicated. The reason is because the natural coordinate system for a disk is *not* the square Cartesian coordinates. The natural coordinate system is the polar coordinate system.

Exercise 1. *Show that the differential operator*

$$\partial_{xx}+\partial_{yy}$$

in polar coordinates (r,θ) *becomes*

$$\partial_{rr}+r^{-1}\partial_r+r^{-2}\partial_{\theta\theta}.$$

Hint: use the chain rule!

In terms of polar coordinates the boundary is at $r=L$. This is the type of expression we usually have for a boundary. The function S should vanish at $r=L$. Moreover, we are on a disk. So, the function S at θ and $\theta+2k\pi$ should be the same for all $k\in\mathbb{Z}$. Let us separate variables, writing $S=R(r)\Theta(\theta)$. Then our equation becomes

$$R''\Theta+r^{-1}R'\Theta+r^{-2}\Theta''=\lambda R\Theta,\quad R(L)=0,\quad \Theta(\theta+2k\pi)=\Theta(\theta).$$

Let's get the different variables cordoned off to different sides of the equation. So, we first divide by $R\Theta$:

$$\frac{R''}{R}+r^{-1}\frac{R'}{R}+r^{-2}\frac{\Theta''}{\Theta}=\lambda.$$

Multiply everything by r^2 to liberate the term with Θ from any r dependence:

$$r^2\frac{R''}{R}+r\frac{R'}{R}+\frac{\Theta''}{\Theta}=r^2\lambda \iff r^2\frac{R''}{R}+r\frac{R'}{R}-r^2\lambda=-\frac{\Theta''}{\Theta}.$$

Each side depends on a different variable, so they are both constant. Since we have the lovely periodicity condition for Θ, and its equation is more simple, let us look for its solution first. We have

$$-\frac{\Theta''}{\Theta}=\text{ constant }=\mu,\quad \Theta(\theta+2k\pi)=\Theta(\theta).$$

So, we are looking for a 2π periodic function which has Θ'' equal to a constant times Θ. The only functions which have this are sines and cosines! Equivalently, we may use complex exponentials. So, we may choose to use

$$\{\sin(nx),\cos(nx)\}_{n\in\mathbb{N}_0},\text{ or }\{e^{inx}\}_{n\in\mathbb{Z}}.$$

Either of these will do the job. The numbers

$$\mu=\mu_n=-n^2.$$

So, now let us take the value of μ_n and use it to find the partner function R_n. It satisfies

$$r^2\frac{R_n''}{R_n} + r\frac{R_n'}{R_n} - r^2\lambda = -\frac{\Theta_n''}{\Theta_n} = - - n^2 = n^2.$$

Re-arranging the equation, we get

eq:almostb

$$r^2 R_n'' + rR_n' - r^2\lambda - n^2 R_n = 0. \tag{2.1}$$

This is quite close to Bessel's equation.

Definition 1. *The differential equation*

$$x^2u''(x) + xu'(x) + (x^2 - \alpha^2)u(x) = 0, \quad \alpha \in \mathbb{C}$$

is Bessel's equation. The differential equation

$$u^2u''(x) + xu'(x) - (x^2 + \alpha^2)u(x) = 0,$$

is the modified Bessel equation.

So, let's try to relate our equation (eq:almostb 2.1). The main differences are: λ factor attached to r^2 term and different signs. Let us consider first the case in which $\lambda < 0$. Then $-\lambda > 0$. So, let us write

$$R_n(r) = F_n(x), \quad x = r\sqrt{|\lambda|} \implies R_n'(r) = F_n'(x)\sqrt{|\lambda|}.$$

So we also have

$$rR_n'(r) = \frac{x}{\sqrt{|\lambda|}}R_n'(r) = \frac{x}{\sqrt{|\lambda|}}F_n'(x)\sqrt{|\lambda|} = xF_n'(x).$$

Similarly we get

$$r^2R_n''(r) = x^2F_n''(x).$$

Moreover, since $\lambda < 0$,

$$-r^2\lambda = x^2.$$

So for the function F_n the differential equation (eq:almostb 2.1) is

$$x^2F_n''(x) + xF_n'(x) + x^2F_n(x) - n^2F_n(x).$$

This is

$$x^2F_n''(x) + xF_n'(x) + (x^2 - n^2)F_n(x) = 0.$$

This is Bessel's equation! The solution in this case is given by the function

$$F_n(x) = J_n(x) \implies R_n(r) = J_n(r\sqrt{|\lambda|}).$$

What should $\sqrt{|\lambda|}$ be? This comes from the boundary condition. We need

$$R_n(L) = 0 \implies J_n(L\sqrt{|\lambda|}) = 0 \implies L\sqrt{|\lambda|} \text{ is a number where } J_n \text{ vanishes.}$$

Theorem 2. *The Bessel function J_n has infinitely many zeros along the real axis. We may therefore write $\{z_{n,m}\}_{m\geq 1}$ to indicate the m^{th} positive zero of the Bessel function J_n.*

Consequently, we require

$$L\sqrt{|\lambda|} = z_{n,m} \quad \text{for some } m \geq 1.$$

This shows that (recalling $\lambda < 0$ in this case)

$$\lambda = \lambda_{n,m} = -\frac{z_{n,m}^2}{L^2}.$$

Exercise 2. *Consider the case $\lambda > 0$. Do a similar change of variables to* (eq:almostb 2.1) *to show that in this case we obtain the modified Bessel equation:*

$$x^2 F_n''(x) + xF_n'(x) - (x^2 + n^2)F_n(x) = 0.$$

Check the literature to see that the solutions are the modified Bessel functions, I_n and K_n. Verify in the literature that the functions $K_n(x) \to \infty$ when $x \to 0$. So, these do not yield physically viable solutions to the wave equation because there is no reason for our drum to go off to infinity at the center point. Verify that the functions $I_n(x)$ do not have any positive real zeros, so there is no way to obtain the boundary condition $R_n(L) = 0$. Hence, these too can be discarded.

So, with the exercise, we are able to conclude that only the case $\lambda < 0$ yields physically viable solutions. Equipped with this knowledge, we may return to our equation for the time dependent function.

$$\frac{T''_{n,m}}{T_{n,m}} = \lambda_{n,m} = -\frac{z_{n,m}^2}{L^2} \implies T_{n,m}(t) = a_{n,m}\cos(z_{n,m}t/L) + b_{n,m}\sin(z_{n,m}t/L).$$

The coefficients shall be determined by our initial conditions. Using superposition to create a super solution we have

$$u(t,r,\theta) = \sum_{n,m\geq 1} \left(a_{n,m}\cos(z_{n,m}t/L) + b_{n,m}\sin(z_{n,m}t/L)\right) J_n(z_{m,n}r/L)(\cos(n\theta)+\sin(n\theta)).$$

The time derivative should vanish when $t = 0$, which means that the coefficients

$$b_{n,m} = 0 \quad \forall n, m.$$

The other condition is

$$u(0,r,\theta) = \sum_{n,m\geq 1} a_{n,m} J_n(z_{m,n}r/L)(\cos(n\theta) + \sin(n\theta)) = f(r,\theta).$$

So, we would like to have a sort of Fourier expansion in terms of these Bessel functions and sines and cosines. We will have a theorem which says that indeed this is true. Thus

$$a_{n,m} = \frac{\langle f, J_n(z_{m,n}r/L)(\cos(n\theta) + \sin(n\theta))\rangle}{||J_n(z_{m,n}r/L)(\cos(n\theta) + \sin(n\theta))||^2}.$$

Here since we are doing things on a disk and using polar coordinates, our scalar products are:

$$\langle f, J_n(z_{m,n}r/L)(\cos(n\theta)+\sin(n\theta))\rangle = \int_0^L \int_0^{2\pi} f(r,\theta)\overline{J_n(z_{m,n}r/L)(\cos(n\theta) + \sin(n\theta))} r dr d\theta,$$

and

$$||J_n(z_{m,n}r/L)(\cos(n\theta)+\sin(n\theta))||^2 = \int_0^L \int_0^{2\pi} |J_n(z_{m,n}r/L)(\cos(n\theta)+\sin(n\theta))|^2 r dr d\theta.$$

2.1. **What are Bessel functions?** So, what exactly are these Bessel functions? We shall see that they are a bit like the redneck cousins of the sine and cosine functions. Let us write Bessel's equation in this way:

$$x^2 f'' + xf' + (x^2 - \nu^2)f = 0.$$

Assume that f has a series expansion (we will later see that this assumption luckily works out - if it didn't - we'd just have to keep trying other methods). Then we write

$$f(x) = \sum_{j \geq 0} a_j x^{j+b}.$$

Stick it into the ODE:

$$x^2 \sum_{j \geq 0} a_j (j+b)(j+b-1) x^{j+b-2} + x \sum_{j \geq 0} a_j (j+b) x^{j+b-1} + (x^2 - \nu^2) \sum_{j \geq 0} a_j x^{j+b} = 0.$$

Pull the factors of x inside the sum:

$$\sum_{j \geq 0} a_j (j+b)(j+b-1) x^{j+b} + \sum_{j \geq 0} a_j (j+b) x^{j+b} + \sum_{j \geq 0} a_j x^{j+b+2} - \nu^2 a_j x^{j+b} = 0.$$

Begin with $j = 0$. To make the sum vanish, it will certainly suffice to make all the individual terms in the sum vanish. So we would like to have

$$a_0 \left(b(b-1) + b - \nu^2 \right) x^b = 0.$$

This will be true if

$$a_0 = 0 \text{ or } b^2 - \nu^2 = 0 \implies b = \pm \nu.$$

Next look at $j = 1$. We need

$$a_1 \left((1+b)(1+b-1) + (1+b) - \nu^2 \right) x^{b+1} = 0.$$

Let's simplify what's in the parentheses, so we need

$$a_1 \left((1+b)^2 - \nu^2 \right) = 0.$$

So, here are our options:

(1) Let $b = \nu$, set $a_1 = 0$, and be free to choose a_0 OR
(2) Let $(1+b) = \nu$, set $a_0 = 0$, and be free to choose a_1.

If we think about it, the second option is rather like doing the first one for $\nu - 1$ instead of ν. So, the two options are basically equivalent, but the first one is a bit more simple, so that is what we choose to do. We set $b = \nu$, $a_1 = 0$, and we shall choose $a_0 \neq 0$ later.

What happens with the higher terms? Once $j \geq 2$ the term with $a_j x^{j+b+2}$ gets involved. Let's group the terms in the series in a nice way:

$$\sum_{j \geq 0} x^{j+b} a_j \left((j+b)(j+b-1) + (j+b) - \nu^2 \right) + a_j x^{j+b+2} = 0.$$

This is

$$\sum_{j \geq 0} x^{j+b} a_j \left((j+b)^2 - \nu^2 \right) + a_j x^{j+b+2} = 0.$$

We figured out how to make the terms with the powers x^b and x^{b+1} vanish. For the higher powers, the coefficient of

$$x^{j+b+2} \text{ is } a_{j+2} \left((j+2+b)^2 - \nu^2 \right) + a_j.$$

Therefore, we need

$$a_{j+2} \left((j+2+b)^2 - \nu^2 \right) = -a_j \implies a_{j+2} = -\frac{a_j}{(j+2+b)^2 - \nu^2)}.$$

Recalling that we picked $b = \nu$, this means

$$a_{j+2} = -\frac{a_j}{(j+2+\nu)^2 - \nu^2},$$

so we are not dividing by zero which is a good thing. Equivalently, for $j \geq 2$, we have
$$a_j = -\frac{a_{j-2}}{(j+\nu)^2 - \nu^2} = -\frac{a_{j-2}}{j^2 + 2\nu j} = -\frac{a_{j-2}}{j(j+2\nu)}.$$

We therefore see that since we picked $a_1 = 0$, all of the odd terms are zero. On the other hand, for the even terms, we can figure out what these are using induction. I claim that
$$a_{2k} = \frac{(-1)^k a_0}{2^{2k} k! (1+\nu)(2+\nu)\dots(k+\nu)}.$$
To begin we check the base case which has $k = 1$:
$$a_2 = -\frac{a_0}{2(2+2\nu)} = -\frac{a_0}{4(1+\nu)} = \frac{(-1)^1 a_0}{2^{2(1)} 1! (1+\nu)}.$$
So the formula is correct. We next assume that it holds for k and verify using what we computed above that it works for $k+1$. We have for $j = 2k+2$,
$$a_{2k+2} = -\frac{a_{2k}}{(2k+2)(2k+2+2\nu)}.$$
We insert the expression for a_{2k} by the induction assumption that the formula holds for k:
$$a_{2k+2} = -\frac{(-1)^k a_0}{(2k+2)(2k+2+2\nu) 2^{2k} k! (1+\nu)(2+\nu)\dots(k+\nu)}.$$
We note that
$$(2k+2)(2k+2+2\nu) = 4(k+1)(k+1+\nu) = 2^2(k+1)(k+1+\nu).$$
So
$$a_{2k+2} = -\frac{(-1)^k a_0}{2^{2(k+1)}(k+1)k!(1+\nu)(2+\nu)\dots(k+\nu)(k+1+\nu)}.$$
Finally we note that
$$(k+1)k! = (k+1)!.$$
So,
$$a_{2k+2} = -\frac{(-1)^k a_0}{2^{2(k+1)}(k+1)!(1+\nu)(2+\nu)\dots(k+\nu)(k+1+\nu)}.$$
This is the formula for $k+1$, so it is indeed correct. Before we proceed, we recall one of the many special functions,
$$\Gamma(s) := \int_0^\infty t^{s-1} e^{-t} dt, \quad s \in \mathbb{C}, \quad \Re(s) > 1.$$

Exercise 3. *Use integration by parts to show that*
$$s\Gamma(s) = \Gamma(s+1).$$
Next, show that $\Gamma(1) = 1$*. Use induction to show that* $\Gamma(n+1) = n!$ *for* $n \geq 1$*.*

Since $\Gamma(1) = 1$*, this is the reason we define*
$$0! := 1.$$
Moreover, viewing Γ *as an extension of the factorial function to real numbers, we can compute silly expressions like*
$$\pi! = \Gamma(\pi+1), \quad e! = \Gamma(e+1), \quad i! = \Gamma(i+1).$$

Use the so-called functional equation $s\Gamma(s) = \Gamma(s+1)$ to show that Γ extends to a meromorphic function whose only poles occur at the points 0 and the negative integers.

So, motivated by the form of the coefficients, the tradition is to choose

$$a_0 = \frac{1}{2^\nu \Gamma(\nu+1)}.$$

Therefore coefficient

$$a_{2k} = \frac{(-1)^k}{2^{2k+\nu} k!(1+\nu)(2+\nu)\dots(k+\nu)\Gamma(\nu+1)} = \frac{(-1)^k}{2^{2k+\nu} k!\Gamma(k+\nu+1)}.$$

This is because

$$(\nu+1)\Gamma(\nu+1) = \Gamma(\nu+2).$$

Next

$$(\nu+2)\Gamma(\nu+2) = \Gamma(\nu+3).$$

We continue all the way to

$$(\nu+k)\Gamma(\nu+k) = \Gamma(\nu+k+1).$$

We have therefore arrived at the definition of the Bessel function of order ν,

$$J_\nu(x) := \sum_{k \geq 0} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k+\nu}}{k!\Gamma(k+\nu+1)}.$$

For the special case $\nu = n \in \mathbb{N}$, the Bessel function is defined for good reason via

$$J_{-n}(x) = (-1)^n J_n(x).$$

The Weber Bessel function is defined for $\nu \notin \mathbb{N}$ to be

$$Y_\nu(x) = \frac{\cos(\nu\pi)J_\nu(x) - J_{-\nu}(x)}{\sin(\nu\pi)}.$$

The second linearly independent solution to Bessel's equation is then defined for $n \in \mathbb{N}$ to be

$$Y_n(x) := \lim_{\nu \to n} Y_\nu(x),$$

and this is well defined. If you are curious about Bessel functions, there are books by Olver, Watson, and Lebedev to name a few. What is most important about Y_n is that it blows up when $x \to 0$. That's okay. Since $J_n(x) \to 0$ as $x \to 0$, for $n \geq 1$, this shows that Y_n and J_n are certainly linearly independent! Hence they indeed form a basis of solutions to the Bessel equation.

2.1.1. *Solutions to: exercises for the week to be done oneself.*

(1) (7.3.1) Use the Fourier transform to find a solution of the ordinary differential equation

$$u'' - u + 2g(x) = 0, \quad g \in \mathcal{L}^1(\mathbb{R}).$$

Answer: $u(x) = g * e^{-|x|} = e^{-x} \int_{-\infty}^{x} e^y g(y)dy + e^x \int_x^\infty e^{-y} g(y)dy$.

(2) (7.4.7) We are tasked with solving the following problem:

$$u_{xx} + u_{yy} = 0, \quad x > 0, 0 < y < 1, \quad u(0,y) = 0, \quad u(x,0) = 0, \quad u(x,1) = e^{-x}.$$

Answer: $u(x,y) = \frac{2}{\pi} \int_0^\infty \frac{\xi \sin(\xi x)\sinh(\xi y)}{(1+\xi^2)\sinh(\xi)} d\xi$.

(3) (Eö 47) We wish to find a solution to

$$u_{xx} + u_{yy} = 0, \quad x \in \mathbb{R}, \quad 0 < y < a,$$

with

$$u(x,0) = 0, \quad u(x,a) = f(x).$$

Answer: The solution is

$$\frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(\xi) \frac{\sinh(\xi y)}{\sinh(\xi a)} e^{ix\xi} dx.$$

To obtain the inequality, one can use Plancharel's theorem which says that

$$\int_{\mathbb{R}} |u(x,y)|^2 dx = \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{u}(\xi,y)|^2 d\xi = \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{f}(\xi)|^2 \left| \frac{\sinh(\xi y)}{\sinh(\xi a)} \right|^2 d\xi \le \frac{1}{2\pi} \int_{\mathbb{R}} |\hat{f}(\xi)|^2 d\xi = \int_{\mathbb{R}} |f(x)|^2 dx.$$

We used that $y \le a$ to obtain that the ratio of hyperbolic sines is ≤ 1, and in the last step we used Plancharel again.

(4) (8.4.1) $u(x,t) = \frac{x}{\sqrt{4\pi k}} \int_0^t f(t-s) e^{-as} s^{-3/2} e^{-s^2/(4ks)} ds.$

(5) (8.4.3) Consider heat flow in a semi-infinite rod when heat is supplied to the end at a constant rate c:

$$u_t = k u_{xx} \text{ for } x > 0, \quad u(x,0) = 0, \quad u_x(0,t) = -c.$$

With the aid of the computation:

$$\mathcal{L}\left(\frac{1}{\sqrt{\pi t}} e^{-a^2/(4t)} \right)(z) = \frac{e^{-a\sqrt{z}}}{\sqrt{z}},$$

show that

$$u(x,t) = c\sqrt{\frac{k}{\pi}} \int_0^t s^{-1/2} e^{-x^2/(4ks)} ds.$$

Answer: We hit the PDE with the Laplace transform in the t variable. We get

$$\mathfrak{L}(u_t)(x,z) = k\mathfrak{L}(u_{xx})(x,z).$$

By the properties of the Laplace transform, and the IC,

$$\mathfrak{L}(u_t)(x,z) = z\mathfrak{L}(u)(x,z) - u(x,0) = z\mathfrak{L}(u)(x,z).$$

So we have the equation:

$$\frac{z}{k}\mathfrak{L}u(x,z) = \mathfrak{L}u(x,z)_{xx}.$$

This is an ODE now for the Laplace transform of our solution. The solution is of the form:

$$\mathfrak{L}u(x,z) = A(z)e^{-x\sqrt{z/k}} + B(z)e^{x\sqrt{z/k}}.$$

We want this to be bounded for large z so we strike the second solution. The boundary condition we have is that $u_x(0,t) = -c$, so when we transform this, we want

$$\mathfrak{L}u_x(0,z) = -\mathfrak{L}(c)(z).$$

We can Laplace transform the constant function:

$$\int_0^\infty c e^{-tz} dt = \frac{c}{z}.$$

On the other hand, taking the derivative of $A(z)e^{-\sqrt{z/k}x}$ with respect to x and then setting $x = 0$ we get:

$$-\sqrt{\frac{z}{k}}A(z) \implies -\sqrt{\frac{z}{k}}A(z) = -\frac{c}{z}.$$

So, we want

$$A(z) = \frac{c\sqrt{k}}{z^{3/2}}.$$

Thus our Laplace transformed solution is:

$$\mathfrak{L}u(x,z) = \frac{c\sqrt{k}}{z^{3/2}}e^{-x\sqrt{z/k}} = c\sqrt{k}\frac{1}{z}\left(\frac{e^{-x\sqrt{z/k}}}{\sqrt{z}}\right).$$

From here on out we can follow Folland's hint and use Table 3 which says that the Laplace transform of

$$\mathfrak{L}(\int_0^t f(s)ds)(z) = z^{-1}\mathfrak{L}(f)(z).$$

So, we have

$$\mathfrak{L}\left(\int_0^t \frac{1}{\sqrt{\pi s}}e^{-a^2/(4s)}ds\right)(z) = \frac{e^{-a\sqrt{z}}}{z\sqrt{z}}.$$

To get the correct right side, we choose

$$a = \frac{x}{\sqrt{k}}.$$

To get the constant factor of $c\sqrt{k}$ as well, we multiply both sides of the equation by $c\sqrt{k}$. So, we have

$$\mathfrak{L}\left(c\sqrt{k}\int_0^t \frac{1}{\sqrt{\pi s}}e^{-x^2/(4\sqrt{k}s)}ds\right)(z) = c\sqrt{k}e^{-x\sqrt{z/k}}.$$

Hence, the solution to the problem before it was hit with the Laplace transform is

$$c\sqrt{k}\int_0^t \frac{1}{\sqrt{\pi s}}e^{-x^2/(4\sqrt{k}s)}ds.$$

(6) (Eö 12) We define

$$f(t) = \int_0^1 \sqrt{w}e^{w^2}\cos(wt)dw.$$

We are supposed to then somehow compute

$$\int_{\mathbb{R}} |f'(t)|^2 dt.$$

Hint: This definition of f looks remarkably like a Fourier transform of something... The right side is an $\mathcal{L}^2$ norm, so we have the Parseval (is that the right name?) formula which says that

$$\int_{\mathbb{R}} |f'(t)|^2 dt = \frac{1}{2\pi}\int_{\mathbb{R}} |\widehat{f'(t)}|^2 dt.$$

Then we look to Table 2 of Folland which says that

$$\widehat{f'(\xi)} = i\xi\hat{f}(\xi).$$

So we just need to compute

$$\frac{1}{2\pi}\int_{\mathbb{R}} \xi^2 |\hat{f}(\xi)|^2 d\xi.$$

To solve this, the function f requires further inspection... it is very close to being a Fourier transform. Let us make it so. Begin by extending evenly (the presence of cosine hints at this...)

$$f(t) = \frac{1}{2}\int_{\mathbb{R}} \chi_{[-1,1]}(w)\sqrt{|w|}e^{w^2}\cos(wt)dw = \frac{1}{2}\int_{\mathbb{R}} \chi_{[-1,1]}(w)\sqrt{|w|}e^{w^2}e^{-iwt}dw.$$

The reason for the last step is that the function (without the cosine) is even. So if we throw in $e^{-iwt} = \cos(-wt) + i\sin(-wt) = \cos(wt) - i\sin(wt)$ the integral with the sine will be zero since sine is odd and the rest of the integrand is zero. So we recognize

$$f(t) = \mathcal{F}\left(\frac{1}{2}\chi_{[-1,1]}(w)\sqrt{|w|}e^{w^2}\right)(t).$$

By the FIT

$$\frac{1}{2}\chi_{[-1,1]}(w)\sqrt{|w|}e^{w^2} = \frac{1}{2\pi}\int_{\mathbb{R}} f(t)e^{iwt}dt = \frac{1}{2\pi}\hat{f}(-w) = \frac{1}{2\pi}\hat{f}(w).$$

This is because f is even and so it's Fourier transform is also even. So, we see that

$$\pi\chi_{[-1,1]}(w)\sqrt{|w|}e^{w^2} = \hat{f}(w).$$

Hence, we just need to compute

$$\frac{1}{2}\int_{\mathbb{R}} w^2\left(\chi_{[-1,1]}(w)\sqrt{|w|}e^{w^2}\right)^2 dw = \frac{1}{2}\int_{-1}^{1} |w|w^2 e^{2w^2}dw$$

$$= \int_0^1 w^3 e^{2w^2}dw.$$

Write the integrand as $(w^2)(we^{2w^2})$. Integrate by parts. It should end nicely.

References

[1] Gerald B. Folland, *Fourier Analysis and Its Applications,* Pure and Applied Undergraduate Texts Volume 4, (1992).