2020.03.02 B

EÖ 28:

$$u_t - u_{xx} = t \sin x \qquad 0<x<1$$

$$t>0$$

$$u(0,t) = u(1,t) = 0$$

$$u(x,0) = \sin(2\pi x)$$

Q Hur ska man börja?

♡ $w_t - w_{xx} = 0$

$$w(0,t) = w(1,t) = 0$$

$$w(x,0) = \sin(2\pi x)$$

Q Och nu?

Variabel-Separation. $w(x,t) = X(x)T(t)$.

$$\frac{T'}{T} - \frac{X''}{X} = 0 \Rightarrow \frac{T'}{T} = \frac{X''}{X} = \lambda.$$

Q T eller X?

=

$X(0) = X(1) = 0$. $X'' = \lambda X$.

$\boxed{Q}$ Lösningarna?

$X_n(x) = \sin(n\pi x)$ $n \geq 1$, $n \in \mathbb{N}$. $\lambda_n = -n^2\pi^2$

$\boxed{Q}$ $T_n(t) = ?$

$$T_n(t) = e^{-n^2\pi^2 t}.$$

$\boxed{Q}$ $w(x,t) = ?$

$$w(x,t) = \sum_{n\geq 1} a_n \sin(n\pi x)\, e^{-n^2\pi^2 t}$$

$$a_n = \frac{\langle \sin(2\pi x), \sin(n\pi x) \rangle}{\|\sin(n\pi x)\|^2} = \begin{cases} 1 & n=2 \\ 0 & \text{else} \end{cases}$$

$$\Rightarrow \quad w(x,t) = \sin(2\pi x)\, e^{-4\pi^2 t}.$$

$\boxed{Q}$ Och nu?

♡♡ : $v_t - v_{xx} = t\sin(x)$

$v(0,t) = v(1,t) = 0$

$v(x,0) = 0$

$$v(x,t) = \sum_{n\geq 1} c_n(t)\, X_n(x).$$

$\boxed{Q}$ Vad gör vi med v?

$$\text{PDE} \Rightarrow \sum_{n\geq 1} \left(c_n'(t) - n^2\pi^2 c_n(t)\right) X_n(x) = t \sin(x)$$

$\boxed{Q}$ Vad gör vi med t sin(x)?

$$t\sin(x) = \sum_{n\geq 1} t\, b_n X_n(x), \quad b_n = \frac{\int_0^1 \sin(x)\, \overline{\sin(n\pi x)}\, dx}{\int_0^1 |\sin(n\pi x)|^2\, dx}$$

$\boxed{Q}$ Och nu?

$$c_n'(t) - n^2\pi^2 c_n(t) = b_n t. \qquad c_n(0) = 0$$

$$\text{Homog. ODE} \Rightarrow c_n(t) = A_n e^{n^2\pi^2 t}$$

$$\text{Inhomog. ODE} \Rightarrow \alpha_n t + \beta_n \xrightarrow{d/dt} \alpha_n - n^2\pi^2(\alpha_n t + \beta_n) = b_n t$$

$$\Rightarrow -n^2\pi^2 \alpha_n = b_n,$$

$$\alpha_n - n^2\pi^2 \beta_n = 0$$

$$\Rightarrow \frac{-b_n}{n^2\pi^2} t + \frac{-b_n}{(n^2\pi^2)^2}$$

$$\Rightarrow A_n e^{n^2\pi^2 t} - \frac{b_n}{n^2\pi^2}\left(t + \frac{1}{n^2\pi^2}\right) = c_n(t)$$

$$c_n(0) = 0 \Rightarrow A = \frac{b_n}{(n^2\pi^2)^2}$$

$v(x,t) = \sum_{n\geq 1} c_n(t) X_n(x)$ och $u(x,t) = w(x,t) + v(x,t)$.

$\boxed{\cancel{5.5.1}}$ $\boxed{\text{Eö } 30}$

$$\begin{cases} \Delta u = 0 & 0 < \theta < \frac{\pi}{4} \\ & 1 < r < 2 \\ u(1,\theta) = 0 & u(r,0) = 0 \\ u_r(2,\theta) = 0 & u(r,\pi/4) = r-1 \end{cases}$$

$\boxed{Q}$ Hur börjar vi?

$\Delta = \partial_r^2 + r^{-1}\partial_r + r^{-2}\partial_\theta^2$. $u = R(r)\,\Theta(\theta)$.

$R''\Theta + r^{-1}R'\Theta + r^{-2}R\Theta'' = 0$ $\quad * r^2,\ \div R\Theta$

$\rightarrow \dfrac{r^2 R''}{R} + \dfrac{rR'}{R} + \dfrac{\Theta''}{\Theta} = 0$

$\rightarrow \dfrac{r^2 R''}{R} + \dfrac{rR'}{R} = -\dfrac{\Theta''}{\Theta} = \text{constant}$

$\boxed{Q}$ Vilken skulle vi hitta först?

$\dfrac{r^2R'' + rR'}{R} = \lambda \rightarrow r^2R'' + rR' - \lambda R = 0$

$\boxed{Q}$ ↑↑ Vad heter eqn.?

Euler. $R(r) = r^{\nu}$. $\longrightarrow$ $\nu(\nu-1) + \nu - \lambda = 0$

$\nu^2 = \lambda$ $\cancel{\Rightarrow R(r) = ar^{\sqrt{\lambda}} + br^{-\sqrt{\lambda}}}$.

$\underline{\lambda = 0}$ $r^2 R'' + rR' = 0 \Rightarrow rR'' + R' = 0$

$\Rightarrow r\dfrac{R''}{R'} = -1 \Rightarrow \dfrac{R''}{R'} = -\dfrac{1}{r} \Rightarrow \log(R')' = -\dfrac{1}{r} \Rightarrow$

$\log(R') = -\log(r) \Rightarrow R' = \dfrac{1}{r} \Rightarrow R(r) = \log r.$

$R(1) = 0$, $R'(2) \neq 0$. Inga lösningar.

$\underline{\lambda > 0}$ $R(r) = ar^{\nu} + br^{-\nu}$. $R(1) = a + b \Rightarrow b = -a$.

$R'(r) = a(\nu r^{\nu-1} + \nu r^{-\nu-1}) \Big|_{r=2} \Rightarrow a = 0$ eller

$\nu(2^{\nu-1} + 2^{-\nu-1}) = 0$ $\lightning$

$\underline{\lambda < 0}$ $R(r) = ar^{i\nu} + br^{-i\nu} \longrightarrow \longrightarrow R(r) = a\cos(\nu \log r) + b\sin(\nu \log r)$

$\lambda = -\nu^2$,

$R(1) = a \Rightarrow a = 0.$

$R'(2) = \dfrac{b\nu}{2}\cos(\nu \log 2) \Rightarrow \cos(\nu \log 2) = 0$

$\Rightarrow \nu_n = \dfrac{(2n+1)\pi}{2\log 2}$. $\lambda_n = -\dfrac{(2n+1)^2\pi^2}{4\log 2^2}$.

$R_n(r) = \sin(\nu_n \log r)$.

六

$$-\frac{\Theta_n''}{\Theta_n} = \frac{(2n+1)^2\pi^2}{4(\log 2)^2} \Rightarrow \Theta_n(\theta) = a_n e^{\nu_n\theta} + b_n e^{-\nu_n\theta}$$

<u>OR</u> $a_n \cosh(\nu_n\theta) + b_n \sinh(\nu_n\theta)$

$$u(r,\theta) = \sum_{n\geq 0} R_n(r)\,\Theta_n(\theta). \qquad u(r,0) = 0 \Rightarrow$$

$\Rightarrow a_n = 0 \ \forall n.$

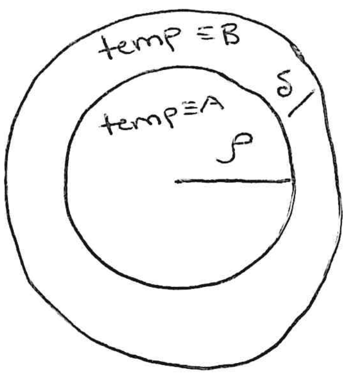
$$u(r,\pi/4) = \sum_{n\geq 0} R_n(r)\, b_n \sinh(\nu_n \pi/4) = r-1$$

$$\Rightarrow b_n = \frac{\langle r-1, R_n\rangle}{\sinh(\nu_n\pi/4)\,\|R_n\|^2}$$

$$= \frac{\int_1^2 (r-1)\sin(\nu_n \log r)\, r\,dr}{\sinh(\nu_n\pi/4)\int_1^2 \sin(\nu_n \log r)^2\, r\,dr}$$

5.5.2

$t^{\text{ime}} = 0$



" ends insulated "

⇒ no heat lost

⇒

indep. of ⟷ and θ.

$\boxed{5.5.2}$

$t > 0$, temp $\equiv B$ at boundary.

$\boxed{\text{Q? Hur hantera det?}}$

SS. $\equiv B$. ⇒ Problem
$$\begin{cases} w(\rho+\delta, t) = 0 \\ \Box w = 0 \\ w(r,0) = \begin{cases} A-B & : 0<r<\rho \\ 0 & : \rho<r<\rho+\delta \end{cases} \end{cases}$$

Lösning blir $u(r,t) = B + w(r,t)$.

$\boxed{\text{Q: Hitta } w\text{?}}$ $\qquad w = R(r)\,T(t)$.

PDE $T'R - R''T - r^{-1}R'T = 0$ →

$$\frac{T'}{T} = \frac{R''}{R} + \frac{r^{-1}R'}{R} = \lambda \quad \text{konstant.}$$

$\boxed{\text{Q: } T \text{ eller } R\text{?}}$ $\qquad R$. $\quad \lambda = 0 \Rightarrow R(r) = \log r$

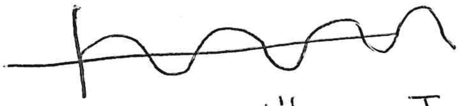
funkar inte.

$r^2R'' + rR' - r^2\lambda R$. $\quad \lambda \le 0 \Rightarrow J_0(\sqrt{|\lambda|}\, r)$

$\lambda > 0 \Rightarrow I_0, K_0$ ↯.

Behöver $J_0(\sqrt{|\lambda|}\,(\rho+\delta)) = 0 \Rightarrow$

$$\sqrt{|\lambda_n|} = \frac{z_n}{\rho+\delta}$$



$\{z_n\}_{n\geq 1} \in (0,\infty)$ där J_0 försvinner.

$$\Rightarrow \lambda_n = -\frac{(z_n)^2}{(\rho+\delta)^2}, \quad R_n(r) = J_0\left(\frac{z_n r}{\rho+\delta}\right).$$

$$\Rightarrow T_n(t) = e^{-(z_n^2/(\rho+\delta)^2)\,t} \cdot a_n$$

$$w(r,t) = \sum_{n\geq 1} R_n(r)\,T_n(t). \qquad \text{IC} \Rightarrow$$

$$a_n = \frac{\int_0^{\rho}(A-B)\,\overline{R_n(r)}\,r\,dr}{\int_0^{\rho+\delta}|R_n(r)|^2\,r\,dr} = \frac{(A-B)\int_0^{\rho} R_n(r)\,r\,dr}{\int_0^{\rho+\delta}|R_n(r)|^2\,r\,dr}.$$

$$u(r,t) = B + w(r,t).$$