Fourieranalys MVE030 och Fourier Metoder MVE290 20.mars.2020
Betygsgränser: 3: 40 poäng, 4: 53 poäng, 5: 67 poäng.
Maximalt antal poäng: 80.
Examinator: Julie Rowlett.
Telefonvakt: Julie 0317723419. OBS! Om ni är osäker på något fråga! (If you are unsure about anything whatsoever, please ask!)

Study aids: All study aids are allowed, but you are kindly requested not to communicated with anyone else in any form during the writing of this exam, except Julie in case of questions. Please write your name and personal number on each page of your exam to make sure that no part of your exam is lost.

The exam is a blended format and is presented here in both Swedish and English. You may write in English, Swedish (German and French are also fine if you want to have even more fun). You are free to switch between these languages as you wish. You may submit your exam in any readable format as well as using a combination of formats (hand-written for some parts, typed for other parts), just do what works best for you and make sure it's readable! Lycka till, and may the mathematical force be with you ♡

Hjälpmedel: Alla hjälpmedel är tillåtna, men du ombeds att ej komunnicera med någon annan på något vis under skrivningen av denna tentamen, förutom Julie då frågor förekommer. Vänligen skriv ditt namn och personnumer på varje sida så att ingen del av dina svar kommer bort.

Tentan är i ett blandat format och finns på svenska och engelska. Du får skriva på engelska och svenska (tyska och franska är också tillåtet om du vill ha ännu mer kul). Du får byta mellan dessa språk. Du får lämna in examen i vilket läsbart format som helst eller en kombination av format. Gör vad som fungerar bäst för dig, sålänge som det är läsbart! Lycka till, and may the mathematical force be with you ♡

Contents

OBS! For some problems, you may need to select more than one choice to receive the points for the problem. (i.e. to receive the points one may need to select *both* a and b or both a, b, and c, etc).

1. English version

(1) (10 p total) We are faced with the following problem.

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = \sin(t)\cos(x) & 0 < t,\ -\pi < x < \pi \\ u(x,0) = |x| - \pi & x \in [-\pi, \pi] \\ u(-\pi, t) = u(\pi, t) & t \geq 0 \end{cases}$$

(a) (2p) What should we do first?
   (i) Apply the Fourier transform.
   (ii) Apply the Laplace transform.
   (iii) Solve the homogeneous PDE.
   (iv) Find a steady state solution.
   (v) None of these.

(b) (2p) Are the boundary conditions self adjoint?
   (i) Yes.
   (ii) No.

(c) (2p) Which technique can NOT be used to correctly solve this problem?
   (i) Separation of variables.
   (ii) Fourier series.
   (iii) A regular Sturm-Liouville problem.
   (iv) Fourier transform.

(d) (2p) How do we deal with the inhomogeneity in the PDE?
   (i) Express it as a Fourier series in t.
   (ii) Express it as a Fourier series in x.
   (iii) Apply the Laplace transform in t.
   (iv) Apply the Fourier transform in x.
   (v) None of these

(e) (2p) What form will the solution take?
   (i) A Fourier series.
   (ii) A Fourier transform.
   (iii) A convolution.
   (iv) A Laplace transform.
   (v) A distribution.
   (vi) None of these.

(2) (10 p total)

(a) (2p) What is the difference between a partial differential equation and an ordinary differential equation?

(b) (2p) Consider the following problem:

$$\begin{cases} u_{xx}(x,y) + u_{yy}(x,y) = 0 & x > 0, \quad y > 0 \\ u(0,y) = f(y) & \in \mathcal{L}^2(0,\infty) \\ u(x,0) = g(x) & \in \mathcal{L}^2(0,\infty) \end{cases}$$

Which technique or techniques could be used to solve this problem?

(i) The Fourier transform.
(ii) The Fourier sine trasform.
(iii) The Fourier cosine transform.
(iv) The Laplace transform.
(v) A Fourier series.
(vi) A regular Sturm-Liouville problem.
(vii) None of these.

(c) (2p) Consider the following problem:

$$\begin{cases} u(0,t) = e^t & t > 0 \\ u_t(x,t) - u_{xx}(x,t) = 0 & t, x > 0 \\ u(x,0) = 0 & x > 0 \end{cases}$$

Which technique or techniques could be used to solve this problem?

(i) A steady state solution.
(ii) The Fourier transform.
(iii) The Fourier sine trasform.
(iv) The Fourier cosine transform.
(v) The Laplace transform.
(vi) A Fourier series.
(vii) A regular Sturm-Liouville problem.
(viii) Separation of variables.
(ix) None of these.

(d) (2p) Consider the following problem:

$$\begin{cases} \sqrt{1+t}\,u_{xx} = u_t & 0 < x < 1, \quad t > 0 \\ u(0,t) = 1, & u(1,t) = 0 \\ u(x,0) = 1 - x. & \end{cases}$$

Which technique or techniques could be used to solve this problem?

(i) A steady state solution.
(ii) Separation of variables.
(iii) The Fourier transform.
(iv) The Fourier sine trasform.
(v) The Fourier cosine transform.
(vi) The Laplace transform.
(vii) A Fourier series.
(viii) A regular Sturm-Liouville problem.
(ix) None of these.

(e) (2p) For the problem in the preceding question, are the boundary conditions self-adjoint?

(i) Yes.
(ii) No.

(3) (10 p total)

(a) (2p) What is the Fourier series of the function $\phi(x) = 1 \forall x \in \mathbb{R}$?

(b) (4p) The Fourier series of the function $f(x) = x$ for $x \in (-\pi, \pi)$ is

$$2 \sum_{n \geq 1} \frac{(-1)^{n+1} \sin(nx)}{n}.$$

Differentiating the Fourier series we obtain

$$2 \sum_{n \geq 1} (-1)^{n+1} \cos(nx).$$

Is this the same as the Fourier series for the function which is equal to $f'(x)$ for $x \in (-\pi, \pi)$?

If yes, explain why it is.

If no, explain why it is not.

(c) (4p) Let

$$c_n := \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{x-inx} dx.$$

Compute

$$\sum_{n \in \mathbb{Z}} c_n e^{in\pi}.$$

(4) (10 p total)

(a) (5p) We wish to compute

$$\sum_{n\geq 1} \frac{1}{n^4}.$$

Find a function whose Fourier series you could use to compute this series, and explain how to use it to compute the series.

(b) (5p) Find a function $\varphi(x)$ whose Fourier coefficients satisfy

$$c_n := \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(x)e^{-inx}dx \neq 0 \quad \forall n \in \mathbb{Z},$$

and

$$\lim_{|n|\to\infty} n^k c_n = 0, \quad \forall k \in \mathbb{N}.$$

(5) (10p total)

(a) (5p) What is the polynomial $p(x)$ of at most degree 17 that minimizes the following integral

$$\int_{-4}^{4} |e^{\cos(x)} - p(x)|^2 dx?$$

(b) (5p) In what types of geometric settings do Bessel functions arise in solving PDEs like the heat equation and the wave equation?

(6) (10 p total)

(a) (2p) Can you solve a regular Sturm-Liouville problem correctly and obtain $\sqrt{-1}$ as an eigenvalue?

(i) Yes.

(ii) No.

(b) (2p) You have found all the eigenfunctions f_n and corresponding eigenvalues λ_n to the regular Sturm-Liouville problem

$$L(f) + \lambda f = 0, \qquad \text{on the interval } (a, b),$$

subject to the boundary conditions

$$B_i(f) = 0, \quad i = 1, 2.$$

What happens to λ_n when $n \to \infty$?

(c) (2p) Assume that u and v are solutions to the aforementioned regular SLP and have eigenvalues 2 and 4, respectively. Compute

$$\int_a^b u(x)\overline{v(x)}dx.$$

(d) (4p) Use $\{f_n\}_{n\geq 1}$ and $\{\lambda_n\}_{n\geq 1}$ to obtain the solution $u(x,t)$ to the following problem

$$\partial_t u - L(u) = 0, \quad t > 0, \quad x \in (a, b),$$
$$B_i(u) = 0, \quad i = 1, 2,$$
$$u(x, 0) = \varphi(x) \text{ is a bounded, continuous function on } [a, b].$$

1.1. **Theory.**

(1) (15p) I've attempted to prove the BBC, but I keep getting stuck. Can you help me finish the proof?

Theorem 1.1. *Let $g \in L^1(\mathbb{R})$. Define*

$$\alpha = \int_{-\infty}^{0} g(x)dx, \quad \beta = \int_{0}^{\infty} g(x)dx.$$

Assume that f is piecewise continuous on $\mathbb{R}$ and its left and right sided limits exist for all points of $\mathbb{R}$. Assume that either f is bounded on $\mathbb{R}$ or that g vanishes outside of a bounded interval. Let, for $\varepsilon > 0$,

$$g_\epsilon(x) := \frac{g(x/\epsilon)}{\epsilon}.$$

Then

$$\lim_{\epsilon \to 0^+} f * g_\epsilon(x) = \alpha f(x+) + \beta f(x-) \quad \forall x \in \mathbb{R}.$$

Proof: First, since this should hold for all $x \in \mathbb{R}$, let us fix the point x.

(a) **What is the meaning of $f(x+)$ and $f(x-)$? Explain what are these things? (1p)**

(b) **Why is the statement in the theorem equivalent to proving the following statement (1p)?**

$$\lim_{\epsilon \to 0} \int_{\mathbb{R}} f(x-y)g_\varepsilon(y)dy - \alpha f(x+) - \beta f(x-) = 0.$$

So, now that you've explained why the statement in the theorem is equivalent to proving the statement above, this can be achieved by proving that

$$\lim_{\varepsilon \to 0} \int_{-\infty}^{0} f(x-y)g_\varepsilon(y)dy - \int_{-\infty}^{0} f(x+)g(y)dy = 0$$

and also

$$\lim_{\varepsilon \to 0} \int_{0}^{\infty} f(x-y)g_\varepsilon(y)dy - \int_{0}^{\infty} f(x-)g(y)dy = 0.$$

(c) **Why is it sufficient to prove that only *one* of the above limits is zero? (1p)**

(d) **Having chosen the first of these two limits, why will it complete the proof to prove that for a given $\delta > 0$, choosing $\varepsilon > 0$ sufficiently small, we can guarantee that the following equation is true? (2p)**

$$\left| \int_{-\infty}^{0} f(x-y)g_\varepsilon(y)dy - \int_{-\infty}^{0} f(x+)g(y)dy \right| < \text{ a constant multiple of } \delta.$$

We would therefore like to show that by choosing ε sufficiently small, we can make

$$\int_{-\infty}^{0} f(x-y)g_\varepsilon(y)dy - \int_{-\infty}^{0} f(x+)g(y)dy$$

as small as we like. In particular, let $\delta > 0$ be arbitrarily small. We wish to show that by choosing ε sufficiently small, we can make

$$\left| \int_{-\infty}^{0} f(x-y)g_\varepsilon(y)dy - \int_{-\infty}^{0} f(x+)g(y)dy \right| < \text{ a constant multiple of } \delta.$$

(e) **Why will it complete the proof to show that the inequality above is true? How does that prove the limit below? (2p)**

$$\lim_{\varepsilon \to 0} \int_{-\infty}^{0} f(x-y)g_\varepsilon(y)dy - \int_{-\infty}^{0} f(x+)g(y)dy = 0?$$

(f) **How do we obtain the equation below? It looks like magic to me, please explain?!? (1p)**

$$\int_{-\infty}^{0} f(x-y)g_\varepsilon(y)dy - \int_{-\infty}^{0} f(x+)g(y)dy = \int_{-\infty}^{0} g_\varepsilon(y)\left(f(x-y) - f(x+)\right)dy.$$

(g) **Why can we find $y_0 < 0$ to make the inequality below true? (1p)**

$$|f(x-y) - f(x+)| < \delta \forall y \in [y_0, 0).$$

(h) **How do we use that above to obtain the inequality below? (1p)**

$$\left|\int_{y_0}^{0} (f(x-y) - f(x+))g_\varepsilon(y)dy\right| \leq \delta ||g||_{L^1(\mathbb{R})}.$$

(i) **What is $||g||_{L^1(\mathbb{R})}$? (1p)**

Next, there are two cases. In the first case, f is bounded, which means that there exists $M > 0$ such that $|f(x)| \leq M$ holds for all $x \in \mathbb{R}$.

(j) **How do we use this to obtain the inequality below? (1p)**

$$\left|\int_{-\infty}^{y_0} (f(x-y) - f(x+))g_\varepsilon(y)dy\right| \leq 2M \int_{-\infty}^{y_0/\varepsilon} |g(y)|dy.$$

(k) **How can we use ε to obtain the inequality below? (1p)**

$$2M \int_{-\infty}^{y_0/\varepsilon} |g(y)|dy < \delta?$$

In this case, we therefore have the estimate

$$\left|\int_{-\infty}^{0} f(x-y)g_\varepsilon(y)dy - \int_{-\infty}^{0} f(x+)g(y)dy\right|$$

$$\leq \left|\int_{-\infty}^{y_0} (f(x-y) - f(x+))g_\varepsilon(y)dy\right| + \left|\int_{y_0}^{0} (f(x-y) - f(x+))g_\varepsilon(y)dy\right| \leq \delta + \delta||g||_{L^1(\mathbb{R})}.$$

(l) **Why is the proof complete in this case now? (1p)**

(m) **In the second case in the theorem, when g vanishes outside a bounded interval, how can we use ε to obtain the equation below 'for ε small enough?' (2p)**

$$\int_{-\infty}^{y_0} (f(x-y) - f(x+))g_\varepsilon(y)dy = 0.$$

(2) (5p) Explain your favorite proof from all of the theory-item-proofs in this course. Why is that proof your favorite? What do you like about it?