Fourieranalys MVE030 och Fourier Metoder MVE290 juni.2019
Betygsgränser: 3: 40 poäng, 4: 53 poäng, 5: 67 poäng.
Maximalt antal poäng: 80.
Hjälpmedel: BETA.
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1. Lös problemet:

$$\begin{cases} u_x(0,t) = 0 & t > 0 \\ u_t(x,t) - u_{xx}(x,t) = 0 & t, x > 0 \\ u(x,0) = f(x) \in \mathcal{L}^2((0,\infty)) \cap \mathcal{C}^0((0,\infty)) & x > 0 \end{cases}$$

(10 p)

Well, this is a PDE in a half space. To figure out what we should do, let's investigate the boundary condition. The boundary condition is that

$$u_x(0,t) = 0.$$

This is rather nice. To achieve such a condition, as we have seen in examples and exercises, we should extend the initial data f evenly. Moreover, we also see that $f \in \mathcal{L}^2$, which indicates that Fourier transform methods have good odds of working. We know that the Fourier transform plays nicely with extending evenly and oddly, in the sense that the Fourier transform preserves these properties: Fourier transform an even function, the result is even; Fourier transform an odd function, the result is odd. On the other hand, the Fourier transform does *not* play nicely by say extending to be identically zero on the negative real line. If you extend this way, then apply the Fourier transform, the result will *not* necessarily be zero on the negative real line.

So, all these considerations tell us to extend f evenly or oddly, and due to the condition $u_x(0,t) = 0$, we shall extend evenly. (Just think about sine and cosine, the cosine is the even one, and it is the one whose derivative vanishes at zero).

Let

$$f_e(x) = f(x), \quad x > 0, \quad f_e(x) = f(-x), \quad x < 0.$$

Then let's apply the Fourier transform to the PDE in the x variable:

$$\hat{u}_t(\xi, t) - \widehat{u_{xx}}(\xi, t) = 0.$$

The properties of the Fourier transform (so generously given to us at the end of this exam) say that

$$\widehat{u_{xx}}(\xi, t) = (-i\xi)^2 \hat{u}(\xi, t),$$

so our equation becomes

$$\hat{u}_t(\xi, t) + \xi^2 \hat{u}(\xi, t) = 0 \implies \hat{u}(\xi, t) = a(\xi) e^{-\xi^2 t}.$$

(Above we have solved the ODE for the Fourier transform where the ODE variable is the variable t, and the variable ξ is an independent variable). The initial condition is that

$$\hat{u}(\xi, 0) = a(\xi) = \hat{f}_e(\xi).$$

So,

$$\hat{u}(\xi, t) = \hat{f}_e(\xi) e^{-\xi^2 t}.$$

Well, the Fourier transform sends a convolution to a product. We look at the table to find a function whose Fourier transform is $e^{-\xi^2 t}$. We know a function whose Fourier transform is $\hat{f}_e(\xi)$, simply f_e. So,

$$u(x, t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} f_e(y) e^{-(x-y)^2/(4t)} dy.$$

To put this in terms of the original function, and verify the boundary condition, we recall the definition of f_e as being an even extension, so

$$u(x, t) = \frac{1}{\sqrt{4\pi t}} \left(\int_{-\infty}^{0} f_e(y) e^{-(x-y)^2/(4t} dy + \int_0^\infty f(y) e^{-(x-y)^2/(4t)} dy \right).$$

We can turn the integral on the negative real axis into an integral on the positive real axis. To do this, let $z = -y$, then

$$\int_{-\infty}^{0} f_e(y) e^{-(x-y)^2/(4t} dy = \int_{\infty}^{0} f_e(-z) e^{-(x+z)^2/(4t)} (-dz) = \int_0^\infty f_e(-z) e^{-(x+z)^2/(4t)} dz.$$

Since

$$f_e(-z) = f_e(z) \quad z > 0,$$

this is

$$\int_0^\infty f(z)e^{-(x+z)^2/(4t)}dz.$$

Now, the name of the variable of integration is irrelevant, so we may as well re-name it back to y, and then we have

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_0^\infty f(y) \left(e^{-(x-y)^2/(4t)} + e^{-(x+y)^2/(4t)}\right) dy.$$

Since it might be helpful, here is basically how partial credit will be dished out. In case any of these items is somewhat messed up, but half-right, you'd get 1p instead of 2p.

(a) (2p) Choosing to use Fourier transform methods.

(b) (2p) Choosing to extend the initial condition evenly

(c) (2p) Correctly Fourier transforming the PDE.

(d) (2p) Solving the ODE for the Fourier transform of the solution correctly.

(e) (2p) Correctly inverting the Fourier transform to obtain the solution (going backwards correctly).

2. Lös problemet:

$$\begin{cases} u(0,t) = \cos(t) & t > 0 \\ u_t(x,t) - u_{xx}(x,t) = 0 & t, x > 0 \\ u(x,0) = 0 & x > 0 \end{cases}$$

(10 p)

This problem has different features. Specifically the boundary condition:

$$u(0,t) = \cos(t).$$

Moreover, the initial condition is zero. With the Fourier transform method, we are usually getting some convolution type stuff involving the initial data. If we were to obtain something like that here, it would just vanish since the initial data is zero. If we were to try Fourier transform methods in the t variable, it would fail miserably because $\cos(t)$ is not in $\mathcal{L}^2$.

So, this indicates that a different approach is required. In particular, all of these considerations suggest using the Laplace transform in the t variable. We Laplace transform the PDE in the t variable:

$$\widetilde{u_t}(x,z) - \widetilde{u}_{xx}(x,z) = 0.$$

We use the properties of the Laplace transform and the nice homogeneous initial condition to obtain:

$$z\widetilde{u}(x,z) - \widetilde{u}_{xx}(x,z) = 0.$$

We solve this ODE to obtain:

$$\widetilde{u}(x,z) = a(z)e^{-x\sqrt{z}} + b(z)e^{x\sqrt{z}}.$$

The properties of the Laplace transform imply (indeed it was a Theorem) that anything which is Laplace-transformable will $\to 0$ as the real part of z tends to infinity. For $x > 0$ (which it is since we work in the positive real line on this problem) the second term will not satisfy that unless b has some really great decay properties. However b doesn't depend on x so if $x \to \infty$ also, then b cannot save this term from growing exponentially. Thus, we try to solve the problem using only the other term. The boundary condition says:

$$\widetilde{u}(0,z) = \widetilde{\cos(t)}(z) = a(z) \implies \widetilde{u}(x,z) = \widetilde{\cos(t)}(z)e^{-x\sqrt{z}}.$$

Now, we could compute the Laplace transform of $\cos(t)$. The easiest way (for me at least) to do this is to write

$$\cos(t) = \frac{e^{it} + e^{-it}}{2}.$$

Then we compute (by definition of the Laplace transform)

$$\int_0^\infty \cos(t)e^{-tz}dt = \frac{e^{it-tz}}{2(i-z)} + \frac{e^{-it-tz}}{2(-i-z)}\Bigg|_{t=0}^{\infty} = -\frac{1}{2(i-z)} + \frac{1}{2(i+z)}$$

$$= \frac{-i-z+i-z}{2(i-z)(i+z)} = \frac{-z}{-1-z^2} = \frac{z}{1+z^2}.$$

So, this is just fine and dandy.

We know that the Laplace transform takes a convolution to a product. We know where the first term came from, so we look for a function whose Laplace transform is $e^{-x\sqrt{z}}$. We look at the lovely table. We see that to get $2a^{-1}\sqrt{\pi}e^{-a\sqrt{z}}$ as the Laplace transform we should start with $H(t)t^{-3/2}e^{-a^2/(4t)}$. So with our problem, we would want $a = x$, and to obtain $e^{-x\sqrt{z}}$ as the Laplace transform we should start with

$$\frac{x}{2\sqrt{\pi}t^{3/2}}H(t)e^{-x^2/(4t)}.$$

Hence

$$u(x,t) = \int_{\mathbb{R}} H(s)\cos(s)H(t-s)\frac{x}{2\sqrt{\pi}(t-s)^{3/2}}e^{-x^2/(4(t-s))}ds.$$

This is because the Laplace transform is in the t variable, so that's the variable for the convolution, and also because the Laplace transform needs the functions inside to be zero for negative values (hence the Heavyside factors). With these Heavyside factors in mind, we obtain

$$u(x,t) = \int_0^t \cos(s)\frac{x}{2\sqrt{\pi}(t-s)^{3/2}}e^{-x^2/(4(t-s))}ds.$$

Since it might be helpful, here is basically how partial credit will be dished out. In case any of these items is somewhat messed up, but half-right, you'd get 1p instead of 2p.

(a) (2p) Choosing to use Laplace transform methods.

(b) (2p) Correctly Laplace transforming the PDE.

(c) (2p) Solving the ODE for the Laplace transform of the solution correctly to get the general solution.

(d) (2p) Discarding the non-Laplace-transformable part of the solution and using the BC to determine the Laplace transform of the solution to the PDE. (Basically going from the general solution of the ODE to the particular solution correctly here).

(e) (2p) Correctly inverting the Laplace transform to obtain the solution (going backwards correctly).

3. Lös ekvationen:

$$u(t) + \int_{-\infty}^{\infty} e^{-|t-\tau|} u(\tau) d\tau = \frac{1}{1+t^2}.$$

(10p)

We have seen such equations in the exercises. The second term is a convolution, and the term on the right is one of the items on our list of Fourier transforms. So let us Fourier transform this entire equation:

$$\hat{u}(\xi) + \hat{u}(\xi)\frac{2}{\xi^2+1} = \pi e^{-|\xi|}.$$

This is because the Fourier transform of a convolution is the product of the Fourier transforms, and the Fourier transforms of all functions except for the unknown function are conveniently found in the table. So we solve this equation for $\hat{u}(\xi)$:

$$\hat{u}(\xi)\left(1 + \frac{2}{\xi^2+1}\right) = \pi e^{-|\xi|} \implies \hat{u}(\xi) = \pi e^{-|\xi|}\frac{\xi^2+1}{\xi^2+3}.$$

This looks a little alarming so let us re-write the right side as

$$\pi e^{-|\xi|}\left(1 - \frac{2}{\xi^2+3}\right) = \pi e^{-|\xi|} - \pi e^{-|\xi|}\frac{2}{\xi^2+3}.$$

Thus we have found that

$$\hat{u}(\xi) = \pi e^{-|\xi|} - \pi e^{-|\xi|}\frac{2}{\xi^2+3}.$$

To invert the Fourier transform, we use the fact that everything is linear. We know which function has Fourier transform equal to $\pi e^{-|\xi|}$, and we also know that the Fourier transform turns a convolution into a product. So, we just need a function whose Fourier transform is $\frac{2}{\xi^2+3}$. The function

$$e^{-\sqrt{3}|x|} \text{ has Fourier transform } \frac{2\sqrt{3}}{\xi^2+3}.$$

So, the function

$$\frac{e^{-\sqrt{3}|x|}}{\sqrt{3}} \text{ has Fourier transform } \frac{2}{\xi^2+3}.$$

Hence

$$u(x) = \frac{1}{1+x^2} - \int_{\mathbb{R}} \frac{1}{1+(x-y)^2} \frac{e^{-\sqrt{3}|y|}}{\sqrt{3}} dy.$$

Points:

(a) (2p) Choosing to use Fourier transform methods.

(b) (2p) Correctly Fourier transforming the equation.

(c) (3p) Correctly solving for the Fourier transform of u.

(d) (3p) Inverting the Fourier transform to obtain u.

4. Lös problemet:

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) = \cos(x) & 0 < t,\, 0 < x < 1 \\ u(x,0) = x^2 - 1 & x \in [0,1] \\ u_t(x,0) = 0 & \\ u_x(0,t) = 0 = u(1,t) & t > 0 \end{cases}$$

(10p)

Now we have entered the geometric realm of bounded intervals. Indeed $0 < x < 1$. The boundary conditions are fantastic. The initial conditions are fine. The only issue is that the PDE is not homogeneous. However, it is *time independent*. So we can attempt to deal with this by finding a steady state (that means time independent) solution. So we first seek a function ϕ which satisfies

$$-\phi''(x) = \cos(x).$$

We would also like to preserve the beautiful boundary conditions, so we politely request that

$$\phi'(0) = \phi(1) = 0.$$

Now the function $\cos(x)$ will certainly satisfy this ODE. Solutions to the homogeneous version of this ODE are linear functions. So a general solution is

$$\phi(x) = \cos(x) + ax + b,$$

for some constants a and b. To achieve the boundary condition at zero, we need $a = 0$. To achieve the boundary condition at 1 we need

$$0 = \cos(1) + b \implies b = -\cos(1).$$

So we define

$$\phi(x) = \cos(x) - \cos(1).$$

Now, we just need to solve a nicer problem:

$$\begin{cases} v_{tt}(x,t) - v_{xx}(x,t) = 0 & 0 < t,\, 0 < x < 1 \\ v(x,0) = x^2 - 1 - \phi(x) & x \in [0,1] \\ v_t(x,0) = 0 & x \in [0,1] \\ v_x(0,t) = 0 = v(1,t) & t > 0 \end{cases}.$$

Then, the full solution will be

$$u(x,t) = \phi(x) + v(x,t).$$

Note that our initial data is still beautiful, continuous, and certainly therefore in $\mathcal{L}^2(0,1)$. Moreover, the boundary conditions are fantastic (self adjoint in particular). So Fourier series methods ought to work here.

We approach the problem at hand now by separating variables writing

$$v = X(x)T(t).$$

We put this into the PDE:

$$T''(t)X(x) - X''(x)T(t) = 0.$$

We tidy it up so that all time dependent terms are on one side, and all space dependent terms are on the other side. So, to achieve this we first divide by XT and then re-arrange:

$$\frac{T''}{T} = \frac{X''}{X}.$$

Since the two sides depend on different variables, they must both be constant. So, we look for solutions to

$$\frac{T''}{T} = \text{ constant } = \frac{X''}{X}.$$

We start with the X side because its conditions are homogeneous and simple. In particular, we seek to solve

$$X'' = \lambda X, \quad X'(0) = X(1) = 0.$$

First case, $\lambda = 0$. Then X would be a linear function. The condition that $X'(0)$ would mean that the slope of the linear function is zero, so that it is a constant function. The condition that $X(1) = 0$ means that X is the constant zero function. So, no solutions when $\lambda = 0$. In the next case $\lambda > 0$. So, the solution to the equation could be written as either a linear combination of $e^{\pm\sqrt{\lambda}x}$ or as a linear combination of hyperbolic sine and cosine. Let us use the latter, because 0 is in our interval. Writing

$$a\cosh(\sqrt{\lambda}x) + b\sinh(\sqrt{\lambda}x)$$

the condition for the derivative to vanish at $x = 0$ requires that $b = 0$. The condition to vanish at $x = 1$ would require (if we want $b \neq 0$) that $\cosh(\sqrt{\lambda}) = 0$. The only real number at which the cosh vanishes is at zero. So we would need $\lambda = 0$. However that contradicts the case we are in. Therefore the case $\lambda > 0$ yields no non-zero solutions.

Finally, we have the case $\lambda < 0$. In this case the solutions are linear combinations of $\sin(\sqrt{|\lambda|}x)$ and $\cos(\sqrt{|\lambda|}x)$. The condition for the derivative to vanish at zero means that there cannot be a sine term. Moreover, the condition to vanish at $x = 1$ means that we need $\sqrt{|\lambda|}$ to be of the form $(2n + 1)\pi/2$. Consequently, all solutions we find in this way are, up to constant factors,

$$X_n(x) = \cos((2n + 1)\pi x/2), \quad \lambda_n = -\pi^2 \frac{(2n + 1)^2}{4}.$$

This informs us what the T function must be since

$$\frac{T_n''}{T_n} = \lambda_n \implies T_n(t) = \text{ a linear combination of } \sin((2n+1)\pi t/2) \text{ and } \cos((2n+1)\pi t/2).$$

In the last step, we put together all the X_nT_n pairs, by the superposition principle, because the PDE is homogeneous, thereby creating our super solution:

$$v(x, t) = \sum_{n \geq 0} X_n(x) \left(a_n \cos((2n + 1)\pi t/2) + b_n \sin((2n + 1)\pi t/2)\right).$$

We shall need the constant factors now to guarantee that the initial conditions are satisfied. First we have the condition at $t = 0$ for the function,

$$v(x,0) = \sum_{n\geq 0} a_n X_n(x) = x^2 - 1 - \phi(x) \implies a_n = \frac{\int_0^1 (x^2 - 1 - \phi)\overline{X_n}}{\int_0^1 |X_n|^2}.$$

The reason we can expand the function in a Fourier X_n series is that the SLP theory guarantees that the functions X_n form an orthogonal basis for $\mathcal{L}^2$ on the interval $[0,1]$. Moreover, the functions $x^2 - 1$ and ϕ are continuous on the closed interval, hence bounded on that interval, hence certainly elements of the Hilbert space $\mathcal{L}^2([0,1])$. So they can indeed be expanded in terms of the functions X_n.

Next we have the condition for the derivative at zero, so

$$v_t(x,0) = \sum_{n\geq 0} b_n((2n+1)\pi/2)X_n(x) = 0 \implies b_n = 0 \quad \forall n.$$

We have therefore specified all quantities in our solution.

Points:

(a) (1p) Choosing to find a steady state solution to deal with the inhomogeneity in the PDE.

(b) (2p) Correctly solving for the steady state solution to solve the inhomogeneous PDE and not screw up the nice BC.

(c) (1p) Setting up the next problem to solve correctly. (homog. PDE, modified IC, same BC, then observe full solution will be sum of these two).

(d) (2p) Choosing to use separation of variables.

(e) (2p) Obtaining the X_n part of the solution correctly.

(f) (2p) Obtaining the T_n part of the solution, in particular getting the a_n and the b_n coefficients correctly.

5. Beräkna:

$$\lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{e+n}.$$

(Tips: beräkna Fourier-serien av den 2π periodiska funktionen som är lika med $\cos(ex)$ i intervallet $(-\pi, \pi)$.)

(10p)

Let's follow the hint. The Fourier coefficients are

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(ex) e^{-inx} dx.$$

For me, it is easier to turn the cosine into

$$\cos(ex) = \frac{e^{iex} + e^{-iex}}{2}.$$

So I will do this and then compute

$$c_n = \frac{1}{4\pi} \int_{-\pi}^{\pi} e^{iex-inx} + e^{-iex-inx} dx$$

$$= \frac{1}{4\pi(ie-in)} \left(e^{\pi(ie-in)} - e^{-\pi(ie-in)} \right) + \frac{1}{4\pi(-ie-in)} \left(e^{\pi(-ie-in)} - e^{-\pi(-ie-in)} \right).$$

$$= \frac{1}{2\pi(e-n)} (-1)^n \sin(e\pi) + \frac{1}{2\pi(e+n)} (-1)^n \sin(e\pi)$$

$$= \frac{(-1)^n \sin(e\pi)}{2\pi} \left(\frac{e+n+(e-n)}{(e-n)(e+n)} \right)$$

$$= \frac{(-1)^n \sin(e\pi)}{2\pi} \frac{2e}{e^2-n^2}.$$

So the Fourier series for the function which is equal to $\cos(ex)$ in the interval $(-\pi, \pi)$ and is 2π periodic is

$$\sum_{n \in \mathbb{Z}} \frac{(-1)^n \sin(e\pi) e}{\pi(e^2-n^2)} e^{inx}.$$

Now we are going to have to think a bit. We want to use this somehow to compute the rather mysterious sum

$$\sum_{n=-N}^{N} \frac{1}{e+n}.$$

Note that we can pair up terms of the form $\pm n$ for all non-zero n. When we do this we get:

$$\frac{1}{e} + \sum_{n=1}^{N} \frac{1}{e+n} + \frac{1}{e-n} = \frac{1}{e} + \sum_{n=1}^{N} \frac{e-n+e+n}{e^2-n^2} = \frac{1}{e} + \sum_{n=1}^{N} \frac{2e}{e^2-n^2}.$$

Hope springs eternal! To get our Fourier series looking like this we want to get rid of the pesky alternation $(-1)^n$. To do that we choose to evaluate the series at $x = \pi$. What is the limit of the series? We must use the theorem on the pointwise convergence of Fourier series. When we do this, we get that the series at $x = \pi$ converges to the sum of the left and right limits of our function. It is 2π periodic. So

$$\lim_{x \to \pi, x < \pi} \text{ is } \cos(e\pi).$$

On the other hand

$$\lim_{x \to \pi, x > \pi} \text{ is } \lim_{x \to -\pi, x > -\pi} = \cos(-e\pi).$$

(maybe draw a picture to see this? It is because of the 2π periodicity.) So the average of these limits gives us that the Fourier series converges to

$$\frac{\cos(e\pi) + \cos(-e\pi)}{2} = \sum_{n \ in \mathbb{Z}} \frac{(-1)^n \sin(e\pi) e}{\pi(e^2-n^2)} e^{in\pi}.$$

Note that $e^{in\pi} = (-1)^n$. So this series simplifies to

$$\sum_{n \in \mathbb{Z}} \frac{e \sin(e\pi)}{\pi(e^2-n^2)}.$$

The terms are the same for $n = \pm 1, \pm 2, \ldots$, so the series simplifies to

$$\frac{e \sin(e\pi)}{\pi e^2} + \sum_{n \geq 1} \frac{2e \sin(e\pi)}{\pi(e^2-n^2)}.$$

Hence we have the equality

$$\frac{\cos(e\pi) + \cos(-e\pi)}{2} = \frac{e \sin(e\pi)}{\pi e^2} + \sum_{n \geq 1} \frac{2e \sin(e\pi)}{\pi(e^2-n^2)}.$$

The cosine is even, so we can simplify and re-arrange things a bit to obtain

$$\cos(e\pi) - \frac{\sin(e\pi)}{e\pi} = \frac{\sin(e\pi)}{\pi} \sum_{n\geq 1} \frac{2e}{e^2 - n^2}.$$

We therefore obtain:

$$\sum_{n\geq 1} \frac{2e}{e^2-n^2} = \left(\cos(e\pi) - \frac{\sin(e\pi)}{e\pi}\right) \frac{\pi}{\sin(e\pi)} = \pi\cot(e\pi) - \frac{1}{e}.$$

So, we therefore have that

$$\lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{e+n} = \lim_{N\to\infty} \frac{1}{e} + \sum_{n=1}^{N} \frac{2e}{e^2-n^2} = \frac{1}{e} + \sum_{n\geq 1} \frac{2e}{e^2-n^2}$$

$$= \frac{1}{e} + \pi\cot(e\pi) - \frac{1}{e} = \pi\cot(e\pi).$$

Points:

(a) (1p) Correct definition of Fourier coefficient c_n for the function $\cos(ex)$.

(b) (1p) Correctly computing these coefficients.

(c) (4p) Correctly applying the theorem on pointwise convergence of Fourier series to evaluate the series at $x = \pi$.

(d) (2p) Correctly manoeuvring the series in question from the statement of the problem to make it look like the Fourier series.

(e) (2p) Solving for the sum and getting it right.

6. Lös problemet:

$$\begin{cases} u_{xx} + u_{yy} = y, & 0 < x < 2, \quad 0 < y < 1 \\ u(x,0) = 0, & u(x,1) = 0 \\ u(0,y) = y - y^3, & u(2,y) = 0. \end{cases}$$

(10p)

We see an inhomogeneity in the PDE. Fortunately it only depends on the variable y rather than on both variables. So, we can look for a function which depends only on y and solves this. OBS! We do not want to screw up the boundary conditions. So, we seek a function f to solve:

$$f''(y) = y, \quad f(0) = f(1) = 0.$$

We integrate both sides of the eODE twice, obtaining

$$f(y) = \frac{y^3}{6} + ay + b.$$

To obtain the desired boundary conditions, we need $b = 0$ so that $f(0) = 0$, and then we need $a = -1/6$ so that $f(1) = 0$. Hence this part of our solution is

$$f(y) = \frac{y^3}{6} - \frac{y}{6}.$$

To continue we look for a function to solve

$$\begin{cases} v_{xx} + v_{yy} = 0, & 0 < x < 2, \quad 0 < y < 1 \\ v(x,0) = 0, & v(x,1) = 0 \\ v(0,y) = y - y^3 - f(y), & v(2,y) = -f(y). \end{cases}$$

Our full solution will then be

$$u(x,y) = v(x,y) + f(y).$$

So, now to look for v we use separation of variables in the PDE, writing (means to an end, that's all it is) $v = X(x)Y(y)$ so that

$$X''(x)Y(y) + Y''(y)X(x) = 0.$$

Tidying things up so that all x dependent terms are on the left, and all y dependent terms are on the right, we have

$$\frac{X''}{X} = -\frac{Y''}{Y}.$$

These two sides of the equation must then both be constant. To obtain the vanishing at $y = 0$ and $y = 1$ we will need to have

$$Y(0) = Y(1) = 0.$$

This is rather lovely. We have $-Y''/Y$ = constant. Let us call the constant λ. If $\lambda = 0$ then Y is a linear function. The only linear function that has $Y(0) = Y(1) = 0$ is the constant 0 function. In case $\lambda < 0$, then $-\lambda > 0$. Solutions to the ODE will be linear combinations of $e^{\pm\sqrt{-\lambda}y}$. It is a good **exercise** to show that no linear combination of these, (except 0), vanishes at 0 and 1. So we are left with the case $\lambda > 0$. In this case Y is a linear combination of $\sin(\sqrt{\lambda}y)$ and $\cos(\sqrt{\lambda}y)$. To get vanishing at $y = 0$, the coefficient of cosine must be zero. To get the vanishing at $y = 1$ we need $\sqrt{\lambda}$ to be an integer multiple of π. Hence we obtain, up to multiplication by non-zero coefficients, solutions

$$Y_n(y) = \sin(n\pi y), \quad \lambda_n = n^2\pi^2.$$

We return to the equation for its friend, X_n. The equation is

$$\frac{X_n''}{X_n} = n^2\pi^2.$$

This means that X_n is a linear combination of $e^{\pm n\pi x}$. Equivalently we can write X_n as a linear combination of hyperbolic sine and cosine. At this point, since there is both the hyperbolic sine and cosine, and we do not know anything more about their coefficients, we write

$$X_n(x) = a_n \cosh(n\pi x) + b_n \sinh(n\pi x).$$

Since our PDE is homogeneous, we can combine all these together using the superposition principle to obtain a super solution

$$v(x,y) = \sum_{n\geq 1} X_n(x)Y_n(y) = \sum_{n\geq 1} \sin(n\pi y)\left(a_n \cosh(n\pi x) + b_n \sinh(n\pi x)\right).$$

We need to determine the coefficients. To do this we use the boundary conditions for the x variable. We want

$$v(0,y) = \sum_{n\geq 1} \sin(n\pi y) a_n = y - y^3 - f(y).$$

So we want to expand the function $y - y^3 - f(y)$ in a Fourier sine series. We therefore set

$$a_n = \frac{\int_0^1 \sin(n\pi y)(y - y^3 - f(y))dy}{\int_0^1 |\sin(n\pi y)|^2 dy}.$$

Next, at $x = 2$ we wish to have

$$v(2, y) = \sum_{n \geq 1} \sin(n\pi y) \left(a_n \cosh(2n\pi) + b_n \sinh(2n\pi)\right) = -f(y).$$

So, we again expand in a Fourier sine series, making the request that

$$(a_n \cosh(2n\pi) + b_n \sinh(2n\pi)) = \frac{\int_0^1 \sin(n\pi y)(-f(y))dy}{\int_0^1 |\sin(n\pi y)|^2 dy}.$$

Solving the equation for b_n we obtain

$$b_n = \left(\frac{\int_0^1 \sin(n\pi y)(-f(y))dy}{\int_0^1 |\sin(n\pi y)|^2 dy} - a_n \cosh(2n\pi) \right) \frac{1}{\sinh(2n\pi)}.$$

We have already specified a_n above. Our solution v is now defined and the whole problem has solution $u = v + f$.

Points:

(a) (4p) Dealing with the "steady state solution" inhomogeneity in the PDE, that is finding a function of y alone (independent of x to solve:

$$f''(y) = y, \quad f(0) = f(1) = 0.$$

(Idea to do this steady state approach gets 2 points, solving the ODE $f'' = y$ gets 1 point and getting the boundary conditions right gets another 1 point).

(b) (1p) Setting up the new problem correctly (like remembering to subtract off the "steady state solution.")

(c) (1p) Idea to use separation of variables.

(d) (2p) Solving the Y part of the problem.

(e) (2p) Solving the X part of the problem.

7. (Bevisa Samplingsatsen) Låt $f \in L^2(\mathbb{R})$. Antar att det finns $L > 0$ så att om $|x| > L$, $\hat{f}(x) = 0$. Visa att gäller:

$$f(t) = \sum_{n \in \mathbb{Z}} f\left(\frac{n\pi}{L}\right) \frac{\sin(n\pi - tL)}{n\pi - tL}.$$

(10p)

Please see the theory proofs document for the proof!

Points: (note that you don't have to do everything in the same order listed below!)

(a) (2p) Idea to expand $\hat{f}(x)$ in a Fourier series on the interval $[-L, L]$.

(b) (2p) Correct calculation of the Fourier coefficients as

$$c_n = \frac{1}{2L} \int_{-L}^{L} e^{-in\pi x/L} \hat{f}(x) dx.$$

(c) (2p) Using the FIT to relate f and $\hat{f}$, that is

$$f(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ixt} \hat{f}(x) dx = \frac{1}{2\pi} \int_{-L}^{L} e^{ixt} \hat{f}(x) dx.$$

(d) (1p) Substituting the Fourier series into this expression, to get $f(t)$ is equal to an integral of the Fourier series of $\hat{f}$.

(e) (2p) Using the FIT to obtain:

$$c_n = \frac{1}{2L} \int_{\mathbb{R}} e^{ix(-n\pi/L)} \hat{f}(x) dx = \frac{2\pi}{2L} f\left(\frac{-n\pi}{L}\right).$$

(f) (1p) Correctly computing the integral of the Fourier series of $\hat{f}$ to obtain the statement in the theorem for $f(t) = \dots$

8. (Generating function for Bessel functions) Bevisa att $\forall x \in \mathbb{R}$ och $z \in \mathbb{C}$ med $z \neq 0$ Bessel funktionerna, J_n, uppfyller

$$\sum_{n=-\infty}^{\infty} J_n(x) z^n = e^{\frac{\pi}{2}\left(1-\frac{1}{z}\right)}.$$

(10 p)

Please see the theory proofs document for the proof!

Points:

(a) (2p) Idea to Taylor expand the exponential function on the right side.

(b) (2p) Idea to expand *each* of the two functions,

$$e^{xz/2} = \sum_{j \geq 0} \frac{\left(\frac{xz}{2}\right)^j}{j!},$$

and

$$e^{-x/(2z)} = \sum_{k \geq 0} \frac{\left(\frac{-x}{2z}\right)^k}{k!}$$

as their own Taylor series.

(c) (2p) One point each for getting these expansions right.

(d) (2p) Variable change to get a sum from $-\infty$ to ∞ rather than 0 to ∞. (1p for the idea and 1p for doing it right).

(e) (2p) Correct algebraic manipulations to make the series expansion for the Bessel functions to appear.

So now you can check for yourself to verify that these rules of grading were precisely followed on each exercise. It is rare, but possible, that a mistake could occur, so if you find anything which is inconsistent with this point scheme, please let us know and we shall correct it! ♡

Fourier transforms

In these formulas below $a > 0$ and $c \in \mathbb{R}$.

$f(x)$	$\hat{f}(\xi)$
$f(x-c)$	$e^{-ic\xi}\hat{f}(\xi)$
$e^{ixc}f(x)$	$\hat{f}(\xi-c)$
$f(ax)$	$a^{-1}\hat{f}(a^{-1}\xi)$
$f'(x)$	$i\xi\hat{f}(\xi)$
$xf(x)$	$i(\hat{f})'(\xi)$
$(f*g)(x)$	$\hat{f}(\xi)\hat{g}(\xi)$
$f(x)g(x)$	$(2\pi)^{-1}(\hat{f}*\hat{g})(\xi)$
$e^{-ax^2/2}$	$\sqrt{2\pi/a}e^{-\xi^2/(2a)}$
$(x^2+a^2)^{-1}$	$(\pi/a)e^{-a\|\xi\|}$
$e^{-a\|x\|}$	$2a(\xi^2+a^2)^{-1}$
$\chi_a(x) = \begin{cases} 1 & \|x\| < a \\ 0 & \|x\| > a \end{cases}$	$2\xi^{-1}\sin(a\xi)$
$x^{-1}\sin(ax)$	$\pi\chi_a(\xi) = \begin{cases} \pi & \|\xi\| < a \\ 0 & \|\xi\| > a \end{cases}$

$$H(t) := \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$$

Laplace transforms

In these formulas below, $a > 0$ and $c \in \mathbb{C}$.

$H(t)f(t)$	$\widetilde{f}(z)$
$H(t-a)f(t-a)$	$e^{-az}\widetilde{f}(z)$
$H(t)e^{ct}f(t)$	$\widetilde{f}(z-c)$
$H(t)f(at)$	$a^{-1}\widetilde{f}(a^{-1}z)$
$H(t)f'(t)$	$z\widetilde{f}(z) - f(0)$
$H(t)\int_0^t f(s)ds$	$z^{-1}\widetilde{f}(z)$
$H(t)(f*g)(t)$	$\widetilde{f}(z)\widetilde{g}(z)$
$H(t)t^{-1/2}e^{-a^2/(4t)}$	$\sqrt{\pi/z}e^{-a\sqrt{z}}$
$H(t)t^{-3/2}e^{-a^2/(4t)}$	$2a^{-1}\sqrt{\pi}e^{-a\sqrt{z}}$
$H(t)J_0(\sqrt{t})$	$z^{-1}e^{-1/(4z)}$
$H(t)\sin(ct)$	$c/(z^2+c^2)$
$H(t)\cos(ct)$	$z/(z^2+c^2)$
$H(t)e^{-a^2t^2}$	$(\sqrt{\pi}/(2a))e^{z^2/(4a^2)}\operatorname{erfc}(z/(2a))$
$H(t)\sin(\sqrt{at})$	$\sqrt{\pi a/(4z^3)}e^{-a/(4z)}$

Lycka till! May the force be with you! ♡ Julie Rowlett.