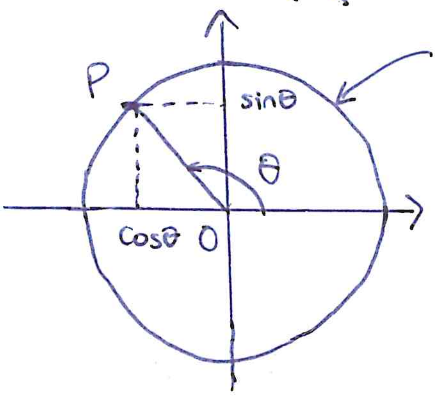
Appendix D Trigonometriska funktioner

Vinklar: π rad $= 180°$ (halv cirkel $180°$)



enhetcirkel $x^2+y^2=1$.

$P \in$ enhetcirkeln $\Rightarrow$ koordinater till P är $(\cos\theta, \sin\theta)$ där θ är vinkeln mellan den positiva x-axeln och OP.

Dvs $\boxed{x=\cos\theta}$, $\boxed{y=\sin\theta}$

Trigonometriska ettan: $\boxed{\cos^2\theta+\sin^2\theta=1.}$

(eftersom $P \in$ cirkeln $\Rightarrow x^2+y^2=1 \Rightarrow \cos^2\theta+\sin^2\theta=1$).

Andra trigonometriska funktioner

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ $(\cos\theta \neq 0)$ $\qquad$ $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$ $(\sin\theta \neq 0)$

$\sec\theta = \dfrac{1}{\cos\theta}$ $(\cos\theta \neq 0)$ $\qquad$ $\csc\theta = \dfrac{1}{\sin\theta}$ $(\sin\theta \neq 0)$

Användbara identiteter:

(i) $\cos(-x) = \cos x$

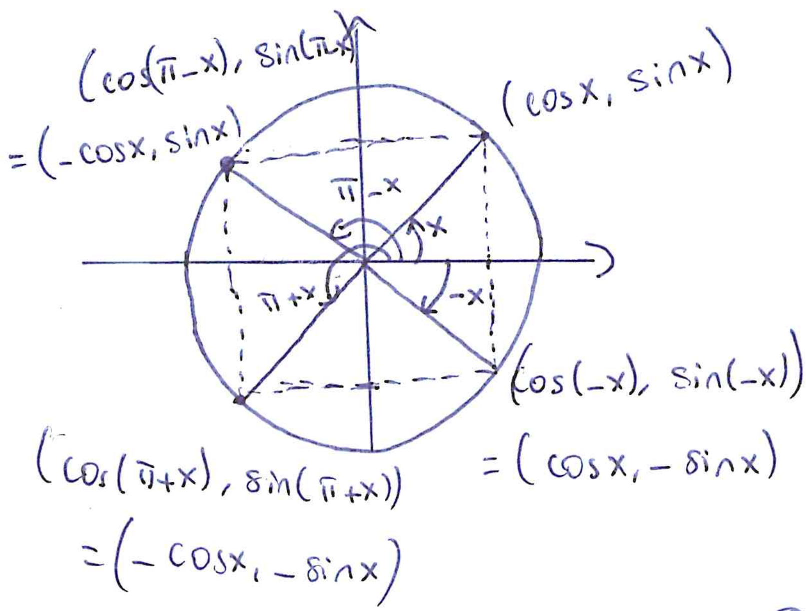
(ii) $\sin(-x) = -\sin x$

(iii) $\cos(\pi-x) = -\cos x$

(iv) $\sin(\pi-x) = \sin x$

(v) $\sin(\pi+x) = -\sin x$

(vi) $\cos(\pi+x) = -\cos x$



Periodiska funktioner: $\cos(x+2n\pi) = \cos x$ $\quad \sin(x+2n\pi) = \sin x$

$\tan(x+n\pi) = \tan x$ $\quad \cot(x+n\pi) = \cot x$

Additions formler:

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$
$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$
$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$
$$\sin(x-y) = \sin x \cos y - \cos x \sin y$$

Dubbelvinkel formler: $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

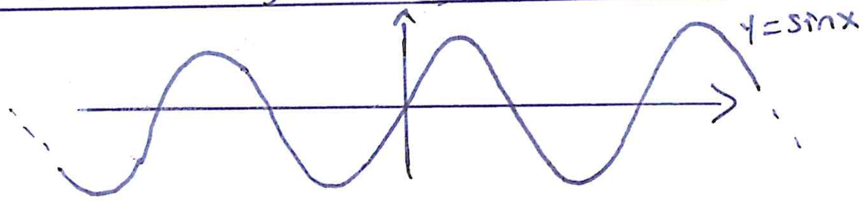
$\sin 2x = 2 \sin x \cos x.$

Andra formler: $\cos\left(\frac{\pi}{2} - x\right) = \sin x$ $\quad \sin\left(\frac{\pi}{2} - x\right) = \cos x.$

tabell om några sinus / cosinus värden finns på s.4

3.3 Gränsvärden och derivator till trigonometriska funktioner

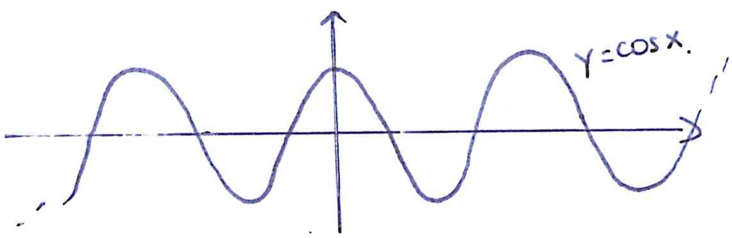
$\lim\limits_{x \to a} \sin x = \sin a$



$\lim\limits_{x \to a} \cos x = \cos a$

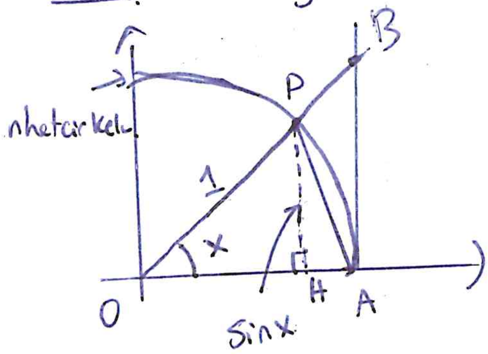
$\lim\limits_{x \to \pm\infty} \sin x$ och $\lim\limits_{x \to \pm\infty} \cos x$

existerar inte.



Sats: $\boxed{\lim\limits_{x \to 0} \frac{\sin x}{x} = 1}$

Bevis. Antag att $0 < x < \frac{\pi}{2}$



Arean av OAP < arean av cirkelsektorn OAP

< arean av OAB.

Arean av OAP $= \frac{PH \cdot \overset{=1}{OA}}{2} = \frac{\sin x}{2}.$

arean av cirkelsektorn $= \frac{x}{2}$ $\left(\begin{array}{l} \text{hela cirkeln} \\ 2\pi \to \pi r^2 \\ x \to \frac{x}{2} r^2 \text{ och } r=1 \end{array} \right)$

arean av $OAB = \frac{OA \cdot AB}{2}$ men $\tan x = \frac{AB}{OA} = \frac{AB}{1} = AB$

$\Rightarrow$ arean av $OAB = \frac{\tan x}{2}$

Alltså, $\frac{\sin x}{2} < \frac{x}{2} < \frac{\tan x}{2}$. $\Leftrightarrow$ $\sin x < x < \tan x = \frac{\sin x}{\cos x}$

$0 < x < \frac{\pi}{2} \Rightarrow \sin x > 0 \Rightarrow$ (dela med $\sin x$). $1 < \frac{x}{\sin x} < \frac{1}{\cos x}$.

Invertera $\Leftrightarrow$ $\cos x < \frac{\sin x}{x} < 1$

$$\Rightarrow \lim_{x \to 0^+} \cos x \leq \lim_{x \to 0^+} \frac{\sin x}{x} \leq 1 \quad \text{(Instängningsregel)}$$

$$\Rightarrow \quad 1 \leq \lim_{x \to 0^+} \frac{\sin x}{x} \leq 1 \quad \Rightarrow \lim_{x \to 0^+} \frac{\sin x}{x} = 1$$

Om $x \to 0^- \Leftrightarrow -x \to 0^+ \Rightarrow \lim_{-x \to 0^+} \frac{\sin(-x)}{-x} = 1$

$$\Rightarrow \lim_{x \to 0^-} \frac{-\sin x}{-x} = \lim_{x \to 0^-} \frac{\sin x}{x} = 1$$

Alltså $\lim_{x \to 0} \frac{\sin x}{x} = 1$

Ex. Beräkna $\lim_{x \to 0} \frac{\sin(2x)}{\sin(3x)}$

Lösn. $\lim_{x \to 0} \frac{\sin(2x)}{\sin(3x)} = \lim_{x \to 0} \underbrace{\frac{\sin(2x)}{2x}}_{\to 1} \cdot 2\cancel{x} \cdot \underbrace{\frac{3x}{\sin(3x)}}_{\to 1} \cdot \frac{1}{3\cancel{x}} = \frac{2}{3}$.

Sats $\boxed{\lim_{h \to 0} \frac{\cos h - 1}{h} = 0.}$

Bevis $\frac{\cos h - 1}{h} = \frac{\cos(2 \cdot \frac{h}{2}) - 1}{h} = \frac{1 - 2\sin^2 \frac{h}{2} - 1}{h} = -\underbrace{\frac{\sin \frac{h}{2}}{\frac{h}{2}}}_{\to 1} \cdot \underbrace{\sin \frac{h}{2}}_{\to 0}$

$\xrightarrow[h \to 0]{} 0$.

Sats (i) $\frac{d}{dx}\sin x = \cos x$ (ii) $\frac{d}{dx}\cos x = -\sin x.$

Bevis (i) $\frac{d}{dx}\sin x = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} = \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}$

$$= \lim_{h\to 0}\left(\sin x\underbrace{\left(\frac{\cos h-1}{h}\right)}_{\to 0} + \cos x\underbrace{\frac{\sin h}{h}}_{\to 1}\right) = \sin x\cdot 0+\cos x\cdot 1 = \cos x.$$

(ii) $\cos x = \sin\left(\frac{\pi}{2}-x\right) \Rightarrow \frac{d}{dx}\cos x = \frac{d}{dx}\sin\left(\frac{\pi}{2}-x\right) \overset{\text{Kedje regel}}{=} \cos\left(\frac{\pi}{2}-x\right)\cdot(-1)$

$= -\sin x.$

Sats $\frac{d}{dx}\tan x = 1+\tan^2 x = \frac{1}{\cos^2 x} = \sec^2 x.$

Bevis $\tan x = \frac{\sin x}{\cos x} \Rightarrow (\tan x)' = \frac{\cos x\cdot\cos x+\sin x\cdot\sin x}{\cos^2 x}$

$= \frac{\cos^2 x+\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$

eller $\frac{\cos^2 x+\sin^2 x}{\cos^2 x} = 1+\frac{\sin^2 x}{\cos^2 x} = 1+\left(\frac{\sin x}{\cos x}\right)^2 = 1+\tan^2 x.$

Tabell för sinus / cosinus.

x	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin x$	0	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}=\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	0
$\cos x$	1	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	0	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{2}}=-\frac{\sqrt{2}}{2}$	$-\frac{1}{2}$	-1