TMA026/MMA430

Exercises and solutions

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Exercise session 1: A6, A11, A15.

Exercise A.6

Let Ω be the unit ball in $\mathbb{R}^d$, $d = 1, 2, 3$, i.e., $\Omega = \{x \in \mathbb{R}^d : |x| < 1\}$. For which values of $\lambda \in \mathbb{R}$ does the function $v(x) = |x|^\lambda$ belong to $L_2(\Omega)$ respectively $H^1(\Omega)$?

<u>Solution:</u> It holds that $v \in L_2(\Omega)$ if the function is bounded in $L_2(\Omega)$-norm, i.e. we want to check for what values of $\lambda \in \mathbb{R}$ it holds that

$$\|v\|_{L_2(\Omega)} := \left(\int_\Omega |v(x)|^2 \,\mathrm{d}x \right)^{1/2} < \infty.$$

For convenience, we look at the squared norm and note that by using spherical coordinates we have

$$\|v\|^2_{L_2(\Omega)} = \int_\Omega |v(x)|^2 \,\mathrm{d}x = \int_\Omega |x|^{2\lambda} \,\mathrm{d}x = \int_{S_1^{d-1}(0)} \mathrm{d}\sigma \int_0^1 r^{2\lambda} r^{d-1} \,\mathrm{d}r,$$

where $S_1^{d-1}(0)$ denotes the surface of the unit ball in $\mathbb{R}^d$. The surface integral will be evaluated to a constant C depending on the dimension d. Hence it suffices to analyze for what values of $\lambda \in \mathbb{R}$ it holds that

$$\int_0^1 r^{2\lambda} r^{d-1} \,\mathrm{d}r < \infty.$$

It is well known that this integral converges whenever the exponent is larger than -1, hence we seek values of λ such that

$$2\lambda + d - 1 > -1 \implies \lambda > -\frac{d}{2}.$$

We may thus conclude that

$$|x|^\lambda \in L_2(\Omega) \text{ if } \lambda > -\frac{d}{2},$$
$$|x|^\lambda \notin L_2(\Omega) \text{ if } \lambda \leq -\frac{d}{2}.$$

To check when $v \in H^1(\Omega)$, we check when it holds that

$$\|v\|_{H^1(\Omega)} := \left(\|v\|^2_{L_2(\Omega)} + |v|_1^2\right)^{1/2} < \infty.$$

Here, it suffices to look at the seminorm $|\cdot|_1$, since we already have the result for $\|v\|_{L_2(\Omega)}$. We check for what values of $\lambda \in \mathbb{R}$ it holds that

$$|v|_1^2 := \int_\Omega |\nabla v|^2 \,\mathrm{d}x = \int_\Omega |\nabla(|x|^\lambda)|^2 \,\mathrm{d}x < \infty.$$

At first, we note that for each partial derivative we have

$$\partial_{x_k}(|x|^\lambda) = \partial_{x_k}(x_1^2 + ... + x_d^2)^{\lambda/2} = \frac{\lambda}{2}(x_1^2 + ... + x_d^2)^{\lambda/2-1} \cdot 2x_k = \lambda |x|^{\lambda-2} x_k.$$

Consequently, we get

$$(\nabla(|x|^\lambda))^2 = \nabla(|x|^\lambda) \cdot \nabla(|x|^\lambda) = (\lambda |x|^{\lambda-2})^2 \underbrace{(x_1^2 + ... + x_d^2)}_{=|x|^2} = \lambda^2 |x|^{2\lambda-2}.$$

Hence, by once again applying spherical coordinates, we find for the integral

$$|v|_1^2 = \int_\Omega \lambda^2 |x|^{2\lambda-2} \, \mathrm{d}x = \lambda^2 \int_{S_1^{d-1}(0)} \mathrm{d}\sigma \int_0^1 r^{2\lambda-2} r^{d-1} \, \mathrm{d}r.$$

Similarly as above, the surface integral equals a constant C dependent of the dimension d, and the integral over the radius converges whenever

$$2\lambda - 2 + d - 1 > -1 \implies \lambda > \frac{2-d}{2} \implies \lambda > 1 - \frac{d}{2}.$$

The conclusion is

$$|x|^\lambda \in H^1(\Omega) \text{ if } \lambda > 1 - \frac{d}{2},$$
$$|x|^\lambda \notin H^1(\Omega) \text{ if } \lambda \leq 1 - \frac{d}{2}.$$

Exercise A.11

Let $\Omega = (0,1)$ and $f(x) = 1/x$. Show that $f \notin L_2(\Omega)$. Show that $f \in H^{-1}(\Omega)$ by defining the linear functional $f(v) = (f, v)$, for all $v \in H_0^1(\Omega)$, and proving the inequality

$$|(f,v)| \leq C\|v'\|, \quad \forall v \in H_0^1(\Omega). \tag{1}$$

Conclude that $H^{-1}(\Omega) \not\subset L_2(\Omega)$.

<u>Solution:</u> The fact that $f \notin L_2(\Omega)$ follows from its unboundedness in $L_2(\Omega)$-norm, which is seen by

$$\|f\|_{L_2(\Omega)}^2 = \int_\Omega |f(x)|^2 \,\mathrm{d}x = \int_0^1 \frac{1}{x^2} \,\mathrm{d}x = -\left[\frac{1}{x}\right]_0^1 = \infty.$$

Next, we let f be the Riesz representative to the linear functional f, and want to show that this linear functional lies in $H^{-1}(\Omega)$. We note that it suffices to show the inequality (1), since this implies that

$$\|f\|_{H^{-1}(\Omega)} = \sup_{v \in H_0^1 \setminus \{0\}} \frac{|(f,v)|}{|v|_1} \leq C < \infty.$$

To show this, we first apply integration by parts and the compact support of v on Ω to find

$$(f,v) = \int_0^1 \frac{1}{x} v(x) \,\mathrm{d}x = \underbrace{\big[\log(x)v(x)\big]_0^1}_{=0} - \int_0^1 \log(x) v'(x) \,\mathrm{d}x.$$

Next, using Cauchy-Schwarz inequality we note that

$$|(f,v)| = \left|\int_0^1 \log(x) v'(x) \,\mathrm{d}x\right| \leq \underbrace{\left(\int_0^1 |\log(x)|^2 \,\mathrm{d}x\right)^{1/2}}_{=\|\log(\cdot)\|_{L_2(\Omega)}} \underbrace{\left(\int_0^1 |v'(x)|^2 \,\mathrm{d}x\right)^{1/2}}_{=\|v'\|_{L_2(\Omega)}}.$$

We are done if we can show that $\|\log(\cdot)\|_{L_2(\Omega)} \leq C$. This is seen by

$$\begin{aligned}
\int_\varepsilon^1 \log^2(x) \,\mathrm{d}x &= \int_\varepsilon^1 \log(x) \cdot \log(x) \,\mathrm{d}x \\
&= \big[\log(x)(x\log(x) - x)\big]_\varepsilon^1 - \int_\varepsilon^1 \log(x) - 1 \,\mathrm{d}x \\
&= \varepsilon \log(\varepsilon) - \varepsilon \log^2(\varepsilon) - \big[x\log(x) - x - x\big]_\varepsilon^1 \\
&= \varepsilon \log(\varepsilon) - \varepsilon \log^2(\varepsilon) - (-2 - \varepsilon\log(\varepsilon) + 2\varepsilon) \\
&= 2\varepsilon \log(\varepsilon) - \varepsilon \log^2(\varepsilon) + 2 - 2\varepsilon.
\end{aligned}$$

The limit for this is found by applying l'Hôspital on the first two terms. For the first we note that

$$\lim_{\varepsilon \to 0^+} \frac{\log(\varepsilon)}{1/\varepsilon} = -\lim_{\varepsilon \to 0^+} \frac{1/\varepsilon}{1/\varepsilon^2} = -\lim_{\varepsilon \to 0^+} \varepsilon = 0.$$

For the second, we can re-use the limit above and find

$$\lim_{\varepsilon \to 0^+} \frac{\log^2(\varepsilon)}{1/\varepsilon} = -\lim_{\varepsilon \to 0^+} \frac{2\log(\varepsilon)\frac{1}{\varepsilon}}{\frac{1}{\varepsilon^2}} = -2\lim_{\varepsilon \to 0^+} \frac{\log(\varepsilon)}{1/\varepsilon} = 0.$$

Consequently, we get

$$\begin{aligned}
\|\log(\cdot)\|^2_{L_2(\Omega)} &= \lim_{\varepsilon \to 0^+} \int_\varepsilon^1 \log^2(x)\,\mathrm{d}x \\
&= \lim_{\varepsilon \to 0^+} 2\varepsilon\log(\varepsilon) - \varepsilon\log^2(\varepsilon) + 2 - 2\varepsilon \\
&= 2 =: C^2,
\end{aligned}$$

and thus we have shown that

$$|(f,v)| \leq \sqrt{2}\|v'\|_{L_2(\Omega)}.$$

Exercise A.15

Let $\Omega = (0, L) \times (0, L)$ be a square of side L. Prove the scaled trace inequality

$$\|v\|_{L_2(\Gamma)} \leq C\Big(L^{-1}\|v\|^2_{L_2(\Omega)} + L\|\nabla v\|^2_{L_2(\Omega)}\Big)^{1/2}, \quad \forall v \in \mathcal{C}^1(\overline{\Omega}).$$

Hint: Apply (A.26) with $\hat{\Omega} = (0, 1) \times (0, 1)$ and use the scaling identities in Problem A.14.

<u>Solution:</u> We follow the given hint. Denote $\hat{\Omega} = (0, 1) \times (0, 1)$, and define for $v \in \mathcal{C}^1(\overline{\Omega})$ the function $\hat{v} : \hat{\Omega} \to \mathbb{R}$ by

$$\hat{v}(\hat{x}) = v(L\hat{x}), \quad \hat{x} \in \hat{\Omega}.$$

This transformation is the one done in the problem formulation of Problem A.14. It follows that $\hat{v} \in \mathcal{C}^1(\overline{\hat{\Omega}})$, and by Theorem A.4 (the trace theorem), there is a $C > 0$ such that

$$\|\hat{v}\|_{L_2(\hat{\Gamma})} \leq C\|\hat{v}\|_{H^1(\hat{\Omega})}, \quad \forall \hat{v} \in \mathcal{C}^1(\overline{\hat{\Omega}}).$$

Moreover, by Problem A.14, we have the following identities (with $d = 2$)

$$\begin{aligned}
\|\hat{v}\|_{L_2(\hat{\Gamma})} &= L^{-1/2}\|v\|_{L_2(\Gamma)},\\
\|\hat{v}\|_{L_2(\hat{\Omega})} &= L^{-1}\|v\|_{L_2(\Omega)},\\
\|\hat{\nabla}\hat{v}\|_{L_2(\hat{\Omega})} &= \|\nabla v\|_{L_2(\Omega)}.
\end{aligned}$$

Hence, we find that

$$\begin{aligned}
L^{-1/2}\|v\|_{L_2(\Gamma)} &= \|\hat{v}\|_{L_2(\hat{\Gamma})}\\
&\leq C\|\hat{v}\|_{H^1(\hat{\Omega})}\\
&= C\big(\|\hat{v}\|^2_{L_2(\hat{\Omega})} + \|\hat{\nabla}\hat{v}\|^2_{L_2(\hat{\Omega})}\big)^{1/2}\\
&= C\big(L^{-2}\|v\|^2_{L_2(\Omega)} + \|\nabla v\|^2_{L_2(\Omega)}\big)^{1/2},
\end{aligned}$$

and by passing $L^{-1/2}$ to the right hand side we conclude the inequality

$$\begin{aligned}
\|v\|_{L_2(\Gamma)} &\leq CL^{1/2}\big(L^{-2}\|v\|^2_{L_2(\Omega)} + \|\nabla v\|^2_{L_2(\Omega)}\big)^{1/2}\\
&= C\big(L^{-1}\|v\|^2_{L_2(\Omega)} + L\|\nabla v\|^2_{L_2(\Omega)}\big)^{1/2}.
\end{aligned}$$