TMA026/MMA430
Exercises and solutions

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Exercise session 2: 3.8, 3.9, 3.11.

Exercise 3.8

Give a variational formulation of the problem

$$\begin{aligned} -\nabla \cdot (a\nabla u) + cu &= f, \quad \text{in } \Omega, \\ a\frac{\partial u}{\partial n} + h(u - g) &= k, \quad \text{on } \Gamma, \end{aligned}$$

where $f \in L_2(\Omega)$, $g, k \in L_2(\Gamma)$, and the coefficients a, c, h are smooth and such that $a(x) \geq a_0 > 0$ and $c(x) \geq 0$ for $x \in \Omega$, and $h(x) \geq h_0 > 0$ for $x \in \Gamma$. Prove the existence and uniqueness of a weak solution. Prove the stability estimate

$$\|u\|_{H^1(\Omega)} \leq C\big(\|f\|_{L_2(\Omega)} + \|k\|_{L_2(\Gamma)} + \|g\|_{L_2(\Gamma)}\big).$$

Hint: Use Problem 3.4.

<u>Solution:</u> To find the variational formulation, we multiplty by a test function $v \in H^1(\Omega)$ and integrate over the domain Ω to get

$$-\int_\Omega \nabla \cdot (a\nabla u)v \,\mathrm{d}x + \int_\Omega cuv \,\mathrm{d}x = \int_\Omega fv \,\mathrm{d}x.$$

Applying Green's formula, we can rewrite the first term on the left hand side as

$$\begin{aligned} -\int_\Omega \nabla \cdot (a\nabla u)v \,\mathrm{d}x &= \int_\Omega a\nabla u \cdot \nabla v \,\mathrm{d}x - \int_\Gamma a\frac{\partial u}{\partial n} v \,\mathrm{d}s \\ &= \int_\Omega a\nabla u \cdot \nabla v \,\mathrm{d}x - \int_\Gamma kv \,\mathrm{d}s + \int_\Gamma huv \,\mathrm{d}s - \int_\Gamma hgv \,\mathrm{d}s, \end{aligned}$$

where we used the Robin boundary condition in the last step. The variational formulation states: Find $u \in H^1(\Omega)$ such that

$$\underbrace{(a\nabla u, \nabla v) + (cu, v) + (hu, v)_{L_2(\Gamma)}}_{=:a(u,v)} = \underbrace{(f, v) + (k + hg, v)_{L_2(\Gamma)}}_{=:L(v)},$$

for all $v \in H^1(\Omega)$. To show the existence of a unique weak solution, we wish to apply the Lax-Milgram lemma (Theorem A.3), i.e. it suffices to show that the bilinear form $a(\cdot, \cdot)$ is bounded and coercive, and that the linear functional $L(\cdot)$ is bounded.

For the boundedness of $a(\cdot, \cdot)$, take two arbitrary elements $v, w \in H^1(\Omega)$ and note that

$$\begin{aligned} |a(v, w)| &\leq \|a\|_{\mathcal{C}}\|\nabla v\|\|\nabla w\| + \|c\|_{\mathcal{C}}\|v\|\|w\| + \|h\|_{\mathcal{C}(\Gamma)}\|w\|_{L_2(\Gamma)}\|v\|_{L_2(\Gamma)} \\ &\leq \big(\|a\|_{\mathcal{C}} + \|c\|_{\mathcal{C}}\big)\|v\|_{H^1}\|w\|_{H^1} + \|h\|_{\mathcal{C}(\Gamma)}\|v\|_{L_2(\Gamma)}\|w\|_{L_2(\Gamma)} \\ &\leq \big(\|a\|_{\mathcal{C}} + \|c\|_{\mathcal{C}} + C^2\|h\|_{\mathcal{C}(\Gamma)}\big)\|v\|_{H^1}\|w\|_{H^1}, \end{aligned}$$

where in the first inequality we have applied the triangle inequality and the Cauchy-Schwarz inequality, in the second step the fact that $\|\cdot\| \leq \|\cdot\|_{H^1}$ and

$|\cdot|_{H^1} \leq \|\cdot\|_{H^1}$, and in the final step we applied the trace inequality. We conclude that $a(\cdot,\cdot)$ is bounded.

For the coercivity of $a(\cdot,\cdot)$, let $v \in H^1(\Omega)$ and note that

$$
\begin{aligned}
a(v,v) &= \int_\Omega a|\nabla v|^2 + cv^2 \,\mathrm{d}x + \int_\Gamma hv^2 \,\mathrm{d}s \\
&\geq a_0\|\nabla v\|^2 + h_0\|v\|^2_{L_2(\Gamma)} \\
&= \frac{a_0}{2}\|\nabla v\|^2 + \frac{a_0}{2}\|\nabla v\|^2 + h_0\|v\|^2_{L_2(\Gamma)} \\
&\geq \frac{a_0}{2}\|\nabla v\|^2 + \min\left\{\frac{a_0}{2}, h_0\right\}(\|\nabla v\|^2 + \|v\|^2_{L_2(\Gamma)}) \\
&\geq \frac{a_0}{2}\|\nabla v\|^2 + \min\left\{\frac{a_0}{2}, h_0\right\}C^{-2}\|v\|^2 \\
&\geq \min\left\{\frac{a_0}{2}, \min\left\{\frac{a_0}{2}, h_0\right\}C^{-2}\right\}\|v\|^2_{H^1}.
\end{aligned}
$$

For this calculation, we first used the fact bounds given on a, c, and h to bound them from above (c vanishes since it is bounded below by 0). The next part sets up for the usage of Friedrich's inequality (Problem 3.4 as given in the hint) which is applied in the fifth row. We conclude that $a(\cdot,\cdot)$ is coercive.

At last, we show the boundedness of the linear functional $L(\cdot)$. Let $v \in H^1(\Omega)$, and we find that

$$
\begin{aligned}
|L(v)| &\leq \|f\|\|v\| + \|k + gh\|_{L_2(\Gamma)}\|v\|_{L_2(\Gamma)} \\
&\leq \|f\|\|v\| + \big(\|k\|_{L_2(\Gamma)} + \|h\|_{\mathcal{C}(\Gamma)}\|g\|_{L_2(\Gamma)}\big)\|v\|_{L_2(\Gamma)} \\
&\leq \|f\|\|v\| + \big(\|k\|_{L_2(\Gamma)} + \|h\|_{\mathcal{C}(\Gamma)}\|g\|_{L_2(\Gamma)}\big)C\|v\|_{H^1} \\
&\leq \big(\|f\| + C(\|k\|_{L_2(\Gamma)} + \|h\|_{\mathcal{C}(\Gamma)}\|g\|_{L_2(\Gamma)})\big)\|v\|_{H^1},
\end{aligned}
$$

where we first use the triangle inequality and Cauchy-Schwarz, and in the third row the trace inequality. We conclude that $L \in H^{-1}(\Omega)$, with bounded norm

$$
\begin{aligned}
\|L\|_{H^{-1}} &\leq \|f\| + C\|k\|_{L_2(\Gamma)} + C\|h\|_{\mathcal{C}(\Gamma)}\|g\|_{L_2(\Gamma)} \\
&\leq \max\{1, C, C\|h\|_{\mathcal{C}(\Gamma)}\}\big(\|f\| + \|k\|_{L_2(\Gamma)} + \|g\|_{L_2(\Gamma)}\big).
\end{aligned}
$$

Now, the Lax-Milgram lemma partially yields the conclusion that there exists a unique solution $u \in H^1(\Omega)$ to

$$
a(u,v) = L(v), \quad \forall v \in H^1(\Omega),
$$

but it moreover gives the stability estimate

$$
\begin{aligned}
\|u\|_{H^1} &\leq \alpha^{-1}\|L\|_{H^{-1}} \\
&\leq \alpha^{-1}\max\{1, C, C\|h\|_{\mathcal{C}(\Gamma)}\}\big(\|f\| + \|k\|_{L_2(\Gamma)} + \|g\|_{L_2(\Gamma)}\big) \\
&= \tilde{C}\big(\|f\| + \|k\|_{L_2(\Gamma)} + \|g\|_{L_2(\Gamma)}\big),
\end{aligned}
$$

where α denotes the coercivity constant. This concludes the problem.

Exercise 3.9

Consider the Neumann problem

$$\begin{aligned} -\Delta u &= f, \quad \text{in } \Omega, \\ \frac{\partial u}{\partial n} &= 0, \quad \text{on } \Gamma. \end{aligned}$$

Assume that $f \in L_2(\Omega)$ and show that the condition

$$\int_\Omega f \,\mathrm{d}x = 0,$$

is necessary for the existence of a solution. Moreover, notice that if u satisfies the Neumann problem, then so does $u+c$ for any constant c. To obtain uniqueness, we add the extra condition

$$\int_\Omega u \,\mathrm{d}x = 0,$$

requiring the mean value of u to be zero. Give this problem a variational formulation using the space

$$V = \Big\{ v \in H^1(\Omega) : \int_\Omega v \,\mathrm{d}x = 0 \Big\}.$$

Prove that there is a unique weak solution. Hint: See Problem 3.5.

<u>Solution:</u> For the first part, we note that for a solution to exist it must hold that

$$\begin{aligned} \int_\Omega f \,\mathrm{d}x &= -\int_\Omega \Delta u \,\mathrm{d}x = -\int_\Omega \nabla \cdot \nabla u \,\mathrm{d}x = -\int_\Gamma \nabla u \cdot \mathbf{n} \,\mathrm{d}s \\ &= -\int_\Gamma \frac{\partial u}{\partial n} \,\mathrm{d}s = -\int_\Gamma 0 \,\mathrm{d}s = 0. \end{aligned}$$

Here we used Gauss divergence theorem in the third step. For the second part, we derive a variational formulation by standard procedure. Given the definition of V as in the problem formulation, multiply the equation by a test function $v \in V$ and integrate over the domain to get

$$-\int_\Omega \Delta u v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x.$$

Applying Green's formula to the left hand side gives

$$-\int_\Omega \Delta u v \,\mathrm{d}x = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x - \underbrace{\int_\Gamma \frac{\partial u}{\partial n} v \,\mathrm{d}s}_{=0} = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x,$$

due to the homogeneous Neumann condition. The variational formulation becomes: Find $u \in V$ such that

$$\underbrace{\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x}_{=:a(u,v)} = \underbrace{\int_\Omega f v \,\mathrm{d}x}_{=:L(v)}, \quad \forall v \in V.$$

We wish to apply Lax-Milgram to show the uniqueness of the solution. For the boundedness of $a(\cdot,\cdot)$ we immediately get for $v, w \in V$

$$|a(v,w)| \leq \|\nabla v\| \|\nabla w\| \leq \|v\|_{H^1} \|w\|_{H^1},$$

by Cauchy-Schwarz inequality. For the coercivity part, we want to use the hint, i.e. Problem 3.5, which gives the inequality

$$\|v\| \leq C\left(\|\nabla v\|^2 + \Big(\int_\Omega v \, \mathrm{d}x \Big)^2 \right)^{1/2}.$$

Note that given our space V, the integral will be evaluated to zero here, and we have a Poincaré type inequality that corresponds to function of zero mean. Using this, we get for $v \in V$

$$\begin{aligned} 2a(v,v) &= 2\|\nabla v\|^2 \\ &= \|\nabla v\|^2 + \|\nabla v\|^2 + \Big(\int_\Omega v \, \mathrm{d}x \Big)^2 \\ &\geq \|\nabla v\|^2 + C^{-1}\|v\|^2 \\ &\geq \min\{1, C^{-1}\}\|v\|_{H^1}^2, \end{aligned}$$

which gives coercivity with constant $\alpha = \frac{1}{2}\min\{C^{-1}, 1\}$. At last, the boundedness of $L(\cdot)$ follows immediately from Cauchy-Schwarz, as for $v \in V$ we get

$$|L(v)| \leq \|f\| \|v\| \leq \|f\| \|v\|_{H^1}.$$

The uniqueness now follows from Lax-Milgram.

Exercise 3.11

Assume that $\Omega \subset \mathbb{R}^2$ is a rectangle and that u is a smooth function with $u = 0$ on Γ. Prove that

$$|u|_2 = \|\Delta u\|.$$

Use this to prove (3.36) for $\mathcal{A} = -\Delta$.

Hint: Recall that

$$|u|_2^2 = \int_\Omega \Big(\big(\frac{\partial^2 u}{\partial x_1^2}\big)^2 + 2\big(\frac{\partial^2 u}{\partial x_1 \partial x_2}\big)^2 + \big(\frac{\partial^2 u}{\partial x_2^2}\big)^2 \Big)\,\mathrm{d}x$$

and integrate by parts in $\int_\Omega \big(\frac{\partial^2 u}{\partial x_1 \partial x_2}\big)^2\,\mathrm{d}x$. Then recall the definition $\|u\|_2 = (\|u\|^2 + |u|_1^2 + |u|_2^2)^{1/2}$ and prove that $\|u\| \leq C|u|_1$ and $|u|_1 \leq (\|u\|\ |u|_2)^{1/2}$.

Solution: At first, we mention that the proving (3.36) part with $\mathcal{A} = -\Delta$ is to prove that

$$\|u\|_2 \leq C\|\Delta u\|.$$

We define the rectangle we work with according to Figure 1. Given this domain, we have on each part of the boundary that

$$\begin{aligned} u = 0, \quad \frac{\partial u}{\partial x_1} = 0, \quad \frac{\partial^2 u}{\partial x_1^2} = 0, \quad n_1 = 0, \quad &\text{on } \Gamma_1, \\ u = 0, \quad \frac{\partial u}{\partial x_2} = 0, \quad n_2 = 0, \quad &\text{on } \Gamma_2. \end{aligned}$$

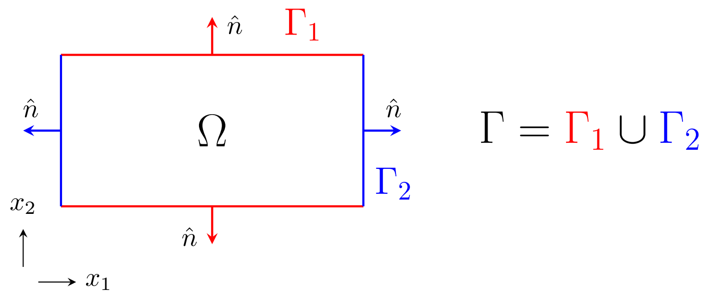

Figure 1: The rectangle with defined boundaries Γ_1 and Γ_2.

We now use the given hint, and begin by applying integration by parts on the integral

$$\begin{aligned} \int_\Omega \Big(\frac{\partial^2 u}{\partial x_1 \partial x_2}\Big)^2\,\mathrm{d}x &= \int_\Omega \frac{\partial}{\partial x_1}\frac{\partial u}{\partial x_2} \cdot \frac{\partial}{\partial x_2}\frac{\partial u}{\partial x_1}\,\mathrm{d}x \\ &= \int_\Gamma n_1 \frac{\partial u}{\partial x_2}\frac{\partial^2 u}{\partial x_1 \partial x_2}\,\mathrm{d}s - \int_\Omega \frac{\partial u}{\partial x_2} \cdot \frac{\partial}{\partial x_2}\frac{\partial^2 u}{\partial x_1^2}\,\mathrm{d}x \\ &= \int_\Gamma \Big(n_1 \frac{\partial u}{\partial x_2}\frac{\partial^2 u}{\partial x_1 \partial x_2} - n_2 \frac{\partial u}{\partial x_2}\frac{\partial^2 u}{\partial x_1^2}\Big)\,\mathrm{d}s + \int_\Omega \frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 u}{\partial x_2^2}\,\mathrm{d}x. \end{aligned}$$

Here, we first used integration by parts in the x_1-direction, and in the next step in the x_2-direction. Next, we check the boundary integral on Γ_1 and Γ_2 separately, and note that

$$\int_{\Gamma_1}\Big(\underbrace{n_1}_{=0}\frac{\partial u}{\partial x_2}\frac{\partial^2 u}{\partial x_1\partial x_2}-n_2\frac{\partial u}{\partial x_2}\underbrace{\frac{\partial^2 u}{\partial x_1^2}}_{=0}\Big)=0,$$
$$\int_{\Gamma_2}\Big(n_1\underbrace{\frac{\partial u}{\partial x_2}}_{=0}\frac{\partial^2 u}{\partial x_1\partial x_2}-\underbrace{n_2}_{=0}\frac{\partial u}{\partial x_2}\frac{\partial^2 u}{\partial x_1^2}\Big)=0,$$

and hence the whole integral over Γ evaluates to zero, and we thus have

$$\int_\Omega\Big(\frac{\partial^2 u}{\partial x_1\partial x_2}\Big)^2\,\mathrm{d}x=\int_\Omega\frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 u}{\partial x_2^2}\,\mathrm{d}x.$$

Consequently, we find that

$$\begin{aligned}|u|_2^2&=\int_\Omega\Big(\Big(\frac{\partial^2 u}{\partial x_1^2}\Big)^2+2\Big(\frac{\partial^2 u}{\partial x_1\partial x_2}\Big)^2+\Big(\frac{\partial^2 u}{\partial x_2^2}\Big)^2\Big)\,\mathrm{d}x\\&=\int_\Omega\Big(\Big(\frac{\partial^2 u}{\partial x_1^2}\Big)^2+2\frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 u}{\partial x_2^2}+\Big(\frac{\partial^2 u}{\partial x_2^2}\Big)^2\Big)\,\mathrm{d}x\\&=\int_\Omega\Big(\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}\Big)^2\,\mathrm{d}x\\&=\|\Delta u\|^2,\end{aligned}$$

and hence

$$|u|_2=\|\Delta u\|. \tag{1}$$

To prove (3.36), we follow the hint. First of all, the inequality

$$\|u\|\le C\|\nabla u\|=C|u|_1 \tag{2}$$

follows by the Poincaré inequality. The last inequality in the hint is found by applying Green's formula, the homogeneous boundary condition, and then Cauchy-Schwarz, i.e

$$\begin{aligned}|u|_1^2&=\int_\Omega\nabla u\cdot\nabla u\,\mathrm{d}x=\int_\Gamma\frac{\partial u}{\partial n}u\,\mathrm{d}s-\int_\Omega u\Delta u\,\mathrm{d}x=-\int_\Omega u\Delta u\,\mathrm{d}x\\&\le\Big|\int_\Omega u\Delta u\,\mathrm{d}x\Big|\le\|u\|\|\Delta u\|=\|u\||u|_2\le C|u|_1|u|_2\\&\le\frac{1}{2}|u|_1^2+\frac{C^2}{2}|u|_2^2.\end{aligned}$$

In the last three steps, we first applied the result (1), followed by the result (2), and at last the inequality $ab\le\frac{1}{2}a^2+\frac{1}{2}b^2$. Passing the first term to the left hand side yields

$$|u|_1^2\le C^2|u|_2^2. \tag{3}$$

With these results given, we now note that

$$\begin{aligned}\|u\|_2^2 = \|u\|^2 + |u|_1^2 + |u|_2^2 &\leq (1 + C^2)|u|_1^2 + |u|_2^2 \\ &\leq \underbrace{[(1 + C^2)C^2 + 1]}_{=:\tilde{C}} |u|_2^2 = \tilde{C}\|\Delta u\|^2,\end{aligned}$$

which concludes the exercise.