TMA026/MMA430
Exercises and solutions

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Exercise session 4: 6.1, 6.3.

Exercise 6.1

Consider the problem

$$
\begin{aligned}
&-(a\varphi')' + c\varphi = \lambda\varphi, \quad \text{in } \Omega = (0,1),\\
&\varphi(0) = \varphi(1) = 0,
\end{aligned}
$$

where $a(x)$ and $c(x)$ are smooth functions with bounds $a(x) \geq a_0 > 0$ and $c(x) \geq 0$.

(a) Show that if the functions $a(x)$ and $c(x)$ are increased, then all the corresponding eigenvalues increase.

(b) Find the eigenvalues when $a(x)$ and $c(x)$ are constant on Ω.

(c) Show that for given $a(x)$ and $c(x)$ there are constants k_1 and k_2 such that

$$0 < k_1 n^2 \leq \lambda_n \leq k_2 n^2.$$

Solution: (a) We multiply the equation by φ and integrate over the domain Ω to get

$$\int_0^1 -(a\varphi')'\varphi \,\mathrm{d}x + \int_0^1 c\varphi^2 \,\mathrm{d}x = \lambda \int_0^1 \varphi^2 \,\mathrm{d}x.$$

We can apply integration by parts on the first integral in combination with the homogeneous Dirichlet boundary condition to get

$$\int_0^1 -(a\varphi')'\varphi \,\mathrm{d}x = -\underbrace{\left[a\varphi'\varphi\right]_0^1}_{=0} + \int_0^1 a(\varphi')^2 \,\mathrm{d}x.$$

Replacing this in the first equation now gives

$$\underbrace{\int_0^1 a(\varphi')^2 \,\mathrm{d}x}_{=\|a^{1/2}\varphi'\|^2} + \underbrace{\int_0^1 c\varphi^2 \,\mathrm{d}x}_{=\|c^{1/2}\varphi\|^2} = \lambda \underbrace{\int_0^1 \varphi^2 \,\mathrm{d}x}_{\|\varphi\|^2}.$$

We can thus write λ as

$$\lambda = \frac{\|a^{1/2}\varphi'\|^2 + \|c^{1/2}\varphi\|^2}{\|\varphi\|^2}.$$

Here we see that if a or c increases, then λ increases as well.

(b) With a and c constant, the problem reduces to a standard ODE which we can solve by standard ODE solving methods. We can rewrite the ODE as

$$a\varphi'' + (c-\lambda)\varphi = 0.$$

The characteristic polynomial is

$$ar^2 + (c-\lambda) = 0 \implies r_{1,2} = \pm\sqrt{\frac{c-\lambda}{a}}.$$

To see how the solution becomes, we need to know whether $c - \lambda$ is positive or negative. From task (a), we have that (with a and c constant)

$$\begin{aligned}
\|\varphi\|^2\lambda = a\|\varphi'\|^2 + c\|\varphi\|^2 &\implies a\|\varphi'\|^2 + (c-\lambda)\|\varphi\|^2 = 0 \\
&\implies (c-\lambda)\|\varphi\|^2 \leq 0 \\
&\implies c - \lambda \leq 0.
\end{aligned}$$

Here, the second implication followed since $a\|\varphi'\|^2 \geq 0$, and the last implication from the fact that $\|\varphi\|^2 \geq 0$. The solutions to the characteristic polynomial is thus of the form

$$r_{1,2} = 0 + i\sqrt{\frac{\lambda - c}{a}},$$

which in turn yields the solution

$$\varphi(x) = Ae^{0x}\sin\left(\sqrt{\frac{\lambda - c}{a}}x\right) + Be^{0x}\cos\left(\sqrt{\frac{\lambda - c}{a}}x\right).$$

The first boundary condition $\varphi(0) = 0$ gives

$$\varphi(0) = B = 0.$$

The second condition $\varphi(1) = 0$ gives that

$$\sin\left(\sqrt{\frac{\lambda - c}{a}}\right) = 0 \implies \sqrt{\frac{\lambda - c}{a}} = n\pi.$$

Hence, solving this for λ, we find that for constant a and c each eigenvalue is given by the formula

$$\lambda_n = an^2\pi^2 + c.$$

(c) For the lower bound, we use the formula found for λ in task (a), and insert the lower bounds on $a(x)$ and $c(x)$, i.e.

$$\lambda = \frac{\|a^{1/2}\varphi'\|^2 + \|c^{1/2}\varphi\|^2}{\|\varphi\|^2} \geq a_0\frac{\|\varphi'\|^2}{\|\varphi\|^2}.$$

The lower bound is thus the eigenvalue corresponding to the eigenvalue problem with constant coefficients $a = a_0$ and $c = 0$. From task (b), we know that this is $= a_0\pi^2n^2$, so we let $k_1 = a_0\pi^2$, which gives

$$\lambda_n \geq k_1n^2.$$

For the upper bound, we use same expression but bound it by the max-norm of the coefficient functions, i.e.

$$\begin{aligned}
\lambda = \frac{\|a^{1/2}\varphi'\|^2 + \|c^{1/2}\varphi\|^2}{\|\varphi\|^2} &\leq \frac{\|a\|_\infty\|\varphi'\|^2 + \|c\|_\infty\|\varphi\|^2}{\|\varphi\|^2} \\
\leq \frac{\|a\|_\infty\|\varphi'\|^2 + \|c\|_\infty C_p\|\varphi'\|^2}{\|\varphi\|^2} &\leq \frac{(\|a\|_\infty + C_p\|c\|_\infty)\|\varphi'\|^2}{\|\varphi\|^2}.
\end{aligned}$$

Here we also applied Poincaré's inequality, which is alright since $\varphi \in H_0^1$. We know from earlier that $\|\varphi'\|^2/\|\varphi\|^2$ is the eigenvalue to corresponding problem with $a = 1$ and $c = 0$, so from (b) we know this is $\pi^2 n^2$. Hence the bound can be written as

$$\lambda \leq (\|a\|_\infty + C_p\|c\|_\infty)\pi^2 n^2 = k_2 n^2,$$

where the upper bound coefficient here is $k_2 = (\|a\|_\infty + C_p\|c\|_\infty)\pi^2$. Thus we have shown that

$$0 < k_1 n^2 \leq \lambda_n \leq k_2 n^2.$$

Exercise 6.3

(a) Use an argument similar to that of Theorem 6.4 to show that

$$v \in H^2 \cap H_0^1 \quad \text{if and only if} \quad \sum_{i=1}^{\infty} \lambda_i^2 (v, \varphi_i)^2 < \infty.$$

(b) Show that

$$\begin{aligned} (i) \quad & -\Delta v &= \sum_{i=1}^{\infty} \lambda_i (v, \varphi_i) \varphi_i, \\ (ii) \quad & \|\Delta v\|^2 &= \sum_{i=1}^{\infty} \lambda_i^2 (v, \varphi_i)^2, \end{aligned}$$

for $v \in H^2 \cap H_0^1$.

<u>Solution:</u> We are looking at the model problem

$$\begin{aligned} -\Delta \varphi_i &= \lambda_i \varphi_i, \quad && \text{in } \Omega, \\ \varphi_i &= 0, && \text{on } \Gamma. \end{aligned}$$

The eigenfunctions to this problem forms an ON-basis for $L_2(\Omega)$, and we can write a function $w \in L_2(\Omega)$ as

$$w = \sum_{i=1}^{\infty} (w, \varphi_i) \varphi_i. \tag{1}$$

We begin by solving task (b), and use the results to solve task (a).

(b) Since $v \in H^2$, we know that $-\Delta v \in L_2(\Omega)$. Hence, we can see that

$$-\Delta v = \sum_{i=1}^{\infty} (-\Delta v, \varphi_i) \varphi_i = \sum_{i=1}^{\infty} (\nabla v, \nabla \varphi_i) \varphi_i = \sum_{i=1}^{\infty} \lambda_i (v, \varphi_i) \varphi_i.$$

Here, we first wrote $-\Delta v$ using (1) and then used Green's formula (the boundary term vanish since $v \in H_0^1$). The final step follows from the weak form of the model problem. This shows (i). The (ii)-part follows by using the result (i) and the orthonormality of the eigenfunctions, i.e.

$$\begin{aligned} \|\Delta v\|^2 &= (-\Delta v, -\Delta v) \\ &= \Big(\sum_{i=1}^{\infty} \lambda_i (v, \varphi_i) \varphi_i, \sum_{j=1}^{\infty} \lambda_j (v, \varphi_j) \varphi_j \Big) \\ &= \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \lambda_i (v, \varphi_i) \lambda_j (v, \varphi_j) \underbrace{(\varphi_i, \varphi_j)}_{=\delta_{ij}} \\ &= \sum_{i=1}^{\infty} \lambda_i^2 (v, \varphi_i)^2. \end{aligned}$$

(a) Assume $v \in H^2 \cap H_0^1$. From task (b) we know that

$$\sum_{i=1}^{\infty} \lambda_i^2 (v, \varphi_i)^2 = \|\Delta v\|^2 < \infty,$$

since $v \in H^2$. This shows the first part of the task. Now we assume that

$$\sum_{i=1}^{\infty} \lambda_i^2 (v, \varphi_i)^2 < \infty,$$

and want to show that $v \in H^2 \cap H_0^1$. The assumption shows that $\lambda_i^2 (v, \varphi_i)^2 \to 0$ as $i \to \infty$ as this is necessary for the sum to converge. Consider $v \in L_2(\Omega)$ and define

$$v_N = \sum_{i=1}^{N} (v, \varphi_i) \varphi_i.$$

It holds that $v_N \to v$ in $L_2(\Omega)$. Moreover, note that $v_N \in H^2 \cap H_0^1$ since $\varphi \in H^2 \cap H_0^1$ and the space is closed under scalar multiplication. Moreover, we claim that $\{v_N\}$ is a Cauchy-sequence in H^2. This is seen by

$$\begin{aligned}
\|v_N - v_M\|_2^2 &\leq C\|\Delta(v_N - v_M)\|^2 && \text{(Elliptic regularity)} \\
&= C\Big\|\Delta \sum_{j=M+1}^{N} (v, \varphi_j)\varphi_j\Big\|^2 && \text{(Insert } v_{N,M}) \\
&= C\Big\| \sum_{j=M+1}^{N} (v, \varphi_j)\Delta\varphi_j\Big\|^2 && (\Delta \text{ linear}) \\
&= C\Big(\sum_{j=M+1}^{N} \lambda_j (v, \varphi_j)\varphi_j, \sum_{i=M+1}^{N} \lambda_i (v, \varphi_i)\varphi_i\Big) && (-\Delta\varphi_i = \lambda_i\varphi_i) \\
&= \sum_{j=M+1}^{N} \sum_{i=M+1}^{N} \lambda_j (v, \varphi_j)\lambda_i (v, \varphi_i) \underbrace{(\varphi_j, \varphi_i)}_{=\delta_{ij}} && ((\cdot,\cdot) \text{ linear}) \\
&= C \sum_{i=M+1}^{N} \lambda_i^2 (v, \varphi_i)^2 && (\{\varphi_i\}_i \text{ orthonormal})
\end{aligned}$$

and as seen in the beginning of the task, this expression $\to 0$ as $N, M \to \infty$. Since $\{v_N\}$ is a Cauchy-sequence in H^2 (which is complete), it holds that $v_N \to w \in H^2 \subset L_2$. But since $v_N \to v \in L_2$, it must hold that $w = v \in H^2$, so we have shown that $v \in H^2$. It remains to show that $v \in H_0^1$, which follows if the trace is zero. This follows since

$$\|\gamma v\|_{L_2(\Gamma)} = \|\gamma v - \gamma v_N\|_{L_2(\Gamma)} \leq C\|v - v_N\|_1 \leq C\|v - v_N\|_2 \to 0,$$

as $N \to 0$. Here we first used the fact that $v_N \in H_0^1$ so we could simply add the trace as $\gamma v_N = 0$. Then we used the trace inequality and moreover the fact that $\|\cdot\|_1 \leq \|\cdot\|_2$ followed by the fact that $v_N \to v$ in H^2. Hence $\gamma v = 0$ and moreover $v \in H_0^1$. We have thus shown that $v \in H^2 \cap H_0^1$ and we are done.