TMA026/MMA430
Exercises and solutions

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Exercise session 5: 8.6, 8.10, 8.16.

Exercise 8.6

Let u be the solution of

$$(u_t, \varphi) + a(u, \varphi) = (f, \varphi), \quad \forall \varphi \in H_0^1,\ t \in \mathbb{R}_+$$

with $u(\cdot, 0) = 0$. Show that

$$\int_0^t (\|u_t(s)\|^2 + \|\Delta u(s)\|^2)\,\mathrm{d}s \leq C \int_0^t \|f(s)\|^2\,\mathrm{d}s$$

for $t \geq 0$.

Solution: We assume sufficient regularity on u. Let $\varphi = u_t \in H_0^1$ in the weak form and we get

$$\|u_t\|^2 + a(u, u_t) = (f, u_t) \leq \|f\|\|u_t\| \leq \frac{1}{2}\|f\|^2 + \frac{1}{2}\|u_t\|^2,$$

where we first applied Cauchy-Schwarz followed by the inequality $ab \leq (a^2 + b^2)/2$. Move the last term to the left hand side and we have that

$$\frac{1}{2}\|u_t\|^2 + a(u, u_t) \leq \frac{1}{2}\|f\|^2.$$

For the a-bilinear form we note that

$$a(u, u_t) = \int_\Omega \nabla u \cdot \nabla u_t\,\mathrm{d}x = \int_\Omega \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(|\nabla u|^2)\,\mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u|_1^2.$$

We insert this and integrate from 0 to t to get

$$\int_0^t \frac{1}{2}\|u_t(s)\|^2\,\mathrm{d}s + \int_0^t \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(|u|_1^2)\,\mathrm{d}s \leq \int_0^t \frac{1}{2}\|f(s)\|^2\,\mathrm{d}s.$$

Cancel the 1/2 and evaluate the second integral in the left hand side and we arrive at

$$\int_0^t \|u_t(s)\|^2\,\mathrm{d}s + \underbrace{|u(t)|_1^2}_{\geq 0} \leq \underbrace{|u(0)|_1^2}_{=0} + \int_0^t \|f(s)\|^2\,\mathrm{d}s.$$

This simplifies to

$$\int_0^t \|u_t(s)\|^2\,\mathrm{d}s \leq \int_0^t \|f(s)\|^2\,\mathrm{d}s. \tag{1}$$

Now take $\varphi = -\Delta u \in H_0^1$. This gives

$$\underbrace{(u_t, -\Delta u)}_{(I)} + \underbrace{a(u, -\Delta u)}_{(II)} = (f, -\Delta u) \leq \frac{1}{2}\|f\|^2 + \frac{1}{2}\|\Delta u\|^2.$$

We apply integration by parts on the first term to get

$$(I) = \int_\Omega -(\nabla \cdot \nabla u) u_t \,\mathrm{d}x = \int_\Omega \nabla u \cdot \nabla u_t \,\mathrm{d}x + \underbrace{\int_\Gamma n \cdot \nabla u u_t \,\mathrm{d}s}_{=0} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u|_1^2,$$

where the boundary term vanishes since $u_t \in H_0^1$. Applying integration by parts on the second term gives

$$(II) = \int_\Omega \nabla u \cdot \nabla(-\Delta u) \,\mathrm{d}x = \int_\Omega (-\Delta u)^2 \,\mathrm{d}x + \int_\Gamma n \nabla u (\underbrace{-\Delta u}_{\in H_0^1}) \,\mathrm{d}s = \|\Delta u\|^2.$$

Insert the expressions for (I) and (II) and integrate from 0 to t to get

$$\int_0^t \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u(s)|_1^2 \,\mathrm{d}s + \int_0^t \frac{1}{2}\|\Delta u(s)\|^2 \,\mathrm{d}s \leq \int_0^t \frac{1}{2}\|f(s)\|^2 \,\mathrm{d}s.$$

Moreover, for the first integral we note that

$$\int_0^t \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|u(s)|_1^2 \,\mathrm{d}s = \underbrace{|u(t)|_1^2}_{\geq 0} - \underbrace{|u(0)|_1^2}_{=0} \geq 0,$$

so we can simply remove it from the above inequality and hence end up with

$$\int_0^t \|\Delta u(s)\|^2 \,\mathrm{d}s \leq \int_0^t \|f(s)\|^2 \,\mathrm{d}s. \tag{2}$$

We can now combine the results (1) and (2) and get the desired result with constant $C = 2$.

Exercise 8.10

Show that if u satisfies

$$\begin{aligned} u_t - \Delta u &= 0, \ \text{ in } \Omega \times \mathbb{R}_+, \\ u &= 0, \ \text{ on } \Gamma \times \mathbb{R}_+, \\ u(\cdot, 0) &= v, \ \text{ in } \Omega, \end{aligned}$$

then there is a constant C such that

$$\|u(t)\|_2^2 + \int_0^t |u_t(s)|_1^2 \,\mathrm{d}s \le C\|v\|_2^2, \quad \forall v \in H^2 \cap H_0^1, \ t \ge 0.$$

<u>Solution:</u> Assume that u has sufficient regularity and compact support. We take the gradient of the original equation to get

$$\nabla u_t - \nabla(\Delta u) = 0.$$

Multiply this by ∇u_t and integrate over Ω to get

$$\underbrace{\int_\Omega |\nabla u_t|^2 \,\mathrm{d}x}_{=|u_t|_1^2} - \underbrace{\int_\Omega \nabla(\Delta u) \cdot \nabla u_t \,\mathrm{d}x}_{(I)} = 0. \tag{3}$$

For the (I)-integral we apply integration by parts and find that

$$(I) = \int_\Omega \Delta u \Delta u_t \,\mathrm{d}x - \underbrace{\int_\Gamma n \cdot \nabla u_t \Delta u}_{=0} = \int_\Omega \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\Delta u|^2 \,\mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u(s)\|^2,$$

where the boundary integral vanishes due to compact support. Insert this into (3) and integrate from 0 to t to get

$$\int_0^t |u_t(s)|_1^2 \,\mathrm{d}s + \frac{1}{2}\int_0^t \frac{\mathrm{d}}{\mathrm{d}t}\|\Delta u(s)\|^2 \,\mathrm{d}s = 0.$$

Evaluate the second integral and recall that $u(\cdot, 0) = v(\cdot)$, and we have that

$$\int_0^t |u_t(s)|_1^2 \,\mathrm{d}s + \frac{1}{2}\|\Delta u\|^2 = \frac{1}{2}\|\Delta v\|^2 \le \frac{1}{2}\|v\|_2^2.$$

The last inequality holds since

$$\|\Delta v\|^2 = \Big\| \sum_{i=1}^d \frac{\partial^2 v}{\partial x_i^2} \Big\|^2 \le \sum_{i=1}^d \Big\| \frac{\partial^2 v}{\partial x_i^2} \Big\|^2 \le \|v\|_2^2.$$

Moreover, since $u \in H^2 \cap H_0^1$, we have the elliptic regularity estimate

$$\|u(t)\|_2^2 \le C_{\mathrm{ER}}\|\Delta u(t)\|^2.$$

In total, we thus have

$$\frac{1}{2C_{\mathrm{ER}}}\|u(t)\|_2^2 + \int_0^t |u_t(s)|_1^2 \,\mathrm{d}s \leq \frac{1}{2}\|v\|_2^2.$$

The desired result now follows as

$$\|u(t)\|_2^2 + \int_0^t |u_t(s)|_1^2 \,\mathrm{d}s \leq C\|v\|_2^2$$

where the constant is given by

$$C = \frac{1}{2\min\{\frac{1}{2C_{\mathrm{ER}}}, 1\}}.$$

Exercise 8.16

Let $u(x,t) = (E(t)v)(x)$ be the solution of

$$\begin{aligned} u_t - \Delta u &= 0, \ \text{ in } \Omega \times \mathbb{R}_+, \\ u &= 0, \ \text{ on } \Gamma \times \mathbb{R}_+, \\ u(\cdot, 0) &= v, \ \text{ in } \Omega, \end{aligned}$$

and let $\{\lambda_j\}_{j=1}^{\infty}$ and $\{\varphi_j\}_{j=1}^{\infty}$ be the eigenvalues and normalized eigenfunctions of

$$\begin{aligned} -\Delta \varphi_i &= \lambda_i \varphi_i, \text{ in } \Omega, \\ \varphi_i &= 0, \qquad \text{on } \Gamma. \end{aligned}$$

Show that

$$u(x,t) = (E(t)v)(x) = \int_{\Omega} G(x,y,t) v(y) \, \mathrm{d}y$$

where the Green's function is

$$G(x,y,t) = \sum_{j=1}^{\infty} e^{-\lambda_j t} \varphi_j(x) \varphi_j(y).$$

Solution: We first show that

$$u(x,t) = \sum_{j=1}^{\infty} \hat{v}_j e^{-\lambda_j t} \varphi_j,$$

and continue by showing that the integral of the Green's function evaluates to this as well. Since $\{\varphi_j\}_{j=1}^{\infty}$ form an ON-basis for L_2, we have

$$u(x,t) = \sum_{j=1}^{\infty} \underbrace{(u(\cdot,t), \varphi_j)}_{=\hat{u}_j(t)} \varphi_j(x) = \sum_{j=1}^{\infty} \hat{u}_j(t) \varphi_j(x).$$

For the derivatives, this expression becomes

$$\begin{aligned} u_t &= \sum_{j=1}^{\infty} \hat{u}_j'(t) \varphi_j(x), \\ -\Delta u &= \sum_{j=1}^{\infty} \hat{u}_j(t)(-\Delta \varphi_j(x)) = \sum_{j=1}^{\infty} \lambda_j \hat{u}_j(t) \varphi_j(x), \end{aligned}$$

where we applied the eigenvalue problem formulation in the last step. Insert these into the heat equation above and we get

$$\sum_{j=1}^{\infty} [\hat{u}_j'(t) + \lambda_j \hat{u}_j(t)] \varphi_j(x) = 0,$$

and since $\{\varphi_j\}_{j=1}^{\infty}$ is an ON-basis it hence holds that

$$\hat{u}_j'(t) + \lambda_j \hat{u}_j(t) = 0.$$

This a standard ODE to which the solution is

$$\hat{u}_j(t) = \hat{u}_j(0)e^{-\lambda_j t} = \hat{v}_j e^{-\lambda_j t},$$

where the last inequality holds since $u(x,0) = v(x)$, so

$$\sum_{j=1}^{\infty} \hat{u}_j(0)\varphi_j(x) = \sum_{j=1}^{\infty} \hat{v}_j \varphi_j(x) \implies \hat{u}_j(0) = \hat{v}_j$$

for all j, once again since $\{\varphi_j\}_{j=1}^{\infty}$ is an ON-basis. Consequently

$$u(x,t) = \sum_{j=1}^{\infty} \hat{v}_j e^{-\lambda_j t} \varphi_j.$$

The fact that this equals the integral of the Green's function now follows since

$$\begin{aligned}
\int_{\Omega} G(x,y,t)v(y)\,\mathrm{d}y &= (G(x,\cdot,t), v(\cdot)) \\
&= \Big(\sum_{j=1}^{\infty} e^{-\lambda_j t} \varphi_j(x)\varphi_j(\cdot), \sum_{i=1}^{\infty} \hat{v}_i \varphi_i(\cdot) \Big) \\
&= \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} e^{-\lambda_j t} \varphi_j(x) \hat{v}_i \underbrace{(\varphi_j, \varphi_i)}_{=\delta_{ij}} \\
&= \sum_{j=1}^{\infty} \hat{v}_j e^{-\lambda_j t} \varphi_j(x) = u(x,t).
\end{aligned}$$