TMA026/MMA430
Exercises and solutions

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May 15, 2020

Exercise session 6: 10.1, 10.3, 10.4.

Exercise 10.1

Consider the problem

$$\begin{aligned} u_t - \Delta u &= f, \ \text{ in } \Omega \times \mathbb{R}_+, \\ u &= 0, \ \text{ on } \Omega \times \mathbb{R}_+, \\ u(\cdot, 0) &= v, \ \text{ in } \Omega, \end{aligned}$$

in the one-dimensional case $\Omega = (0,1)$. For the numerical solution, we use the piecewise linear functions based on the partition

$$0 < x_1 < x_2 < \cdots < x_M < 1, \quad x_j = jh, \quad h = \frac{1}{M+1}.$$

Determine the mass matrix B and the stiffness matrix A and write down the semi-discrete problem, the backward Euler equations, and the Crank–Nicholson equations.

Solution: In a standard way, we find the mass matrix elements as $B = (B_{ij})$ for $1 \leq i, j \leq M$, with

$$B_{ij} = (\varphi_j, \varphi_i),$$

where $\{\varphi_i\}_{i=1}^M$ are the hat functions of the partition, i.e. the basis functions for S_h. Since the hat functions fulfill the property

$$\operatorname{supp}(\varphi_i) \cap \operatorname{supp}(\varphi_j) \neq \emptyset$$

only for $j \in \{i-1, i, i+1\}$, these are the only indices that will give some contribution to the matrix. Hence, B is tri-diagonal. The matrix consists of two types of elements, namely (φ_i, φ_i) as well as $(\varphi_{i-1}, \varphi_i) = (\varphi_{i+1}, \varphi_i)$, due to the symmetry of the L_2-product. In the one-dimensional case, a hat function is defined by

$$\varphi_i(x) = \begin{cases} \frac{x - x_{i-1}}{h}, & \text{if } x \in [x_{i-1}, x_i], \\ \frac{x_{i+1} - x}{h}, & \text{if } x \in [x_i, x_{i+1}], \end{cases}$$

and 0 otherwise. The first type of element is thus computed as

$$\begin{aligned} (\varphi_i, \varphi_i) &= \int_0^1 \varphi_i(x)^2 \, \mathrm{d}x = \int_{x_{i-1}}^{x_i} \left(\frac{x - x_{i-1}}{h} \right)^2 \mathrm{d}x + \int_{x_i}^{x_{i+1}} \left(\frac{x_{i+1} - x}{h} \right)^2 \mathrm{d}x \\ &= 2 \int_{x_{i-1}}^{x_i} \left(\frac{x - x_{i-1}}{h} \right)^2 \mathrm{d}x = \frac{2}{h^2} \int_{x_{i-1}}^{x_i} (x - x_{i-1})^2 \, \mathrm{d}x \\ &= \frac{2}{h^2} \left[\frac{1}{3} (x - x_{i-1})^3 \right]_{x_{i-1}}^{x_i} = \frac{2}{3h^2} (x_i - x_{i-1})^3 = \frac{2h}{3}. \end{aligned}$$

The second type of element is evaluated as

$$\begin{aligned}
(\varphi_{i-1},\varphi_i) &= \int_0^1 \varphi_{i-1}(x)\varphi_i(x)\,\mathrm{d}x = \int_{x_{i-1}}^{x_i} \Big(\frac{x_i - x}{h}\Big)\Big(\frac{x - x_{i-1}}{h}\Big)\,\mathrm{d}x \\
&= \frac{1}{h^2}\Big[(x_i - x)\frac{(x - x_{i-1})^2}{2}\Big]_{x_{i-1}}^{x_i} + \frac{1}{h^2}\int_{x_{i-1}}^{x_i} \frac{(x - x_{i-1})^2}{2}\,\mathrm{d}x \\
&= \frac{1}{2h^2}\Big[\frac{1}{3}(x - x_{i-1})^3\Big]_{x_{i-1}}^{x_i} = \frac{1}{6h^2}h^3 = \frac{h}{6}.
\end{aligned}$$

The mass matrix thus becomes as

$$B = \frac{h}{6}\begin{bmatrix} 4 & 1 & 0 & \cdot & \cdot & 0 \\ 1 & 4 & 1 & \cdot & \cdot & 0 \\ 0 & 1 & 4 & \cdot & \cdot & 0 \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & 4 & 1 \\ 0 & 0 & 0 & \cdot & 1 & 4 \end{bmatrix}.$$

The stiffness matrix is found in a similar way, but as

$$A = (A_{ij}),\ 1 \le i, j \le M, \quad \text{with } A_{ij} = (\varphi_j', \varphi_i').$$

By same reason as for the mass matrix, only $j \in \{i-1, i, i+1\}$ will contribute, and hence there are two types of elements for this matrix as well (since in our case $a(\cdot,\cdot)$ is symmetric). Note that

$$\varphi_i'(x) = \begin{cases} \frac{1}{h}, & \text{if } x \in [x_{i-1}, x_i], \\ -\frac{1}{h}, & \text{if } x \in [x_i, x_{i+1}], \end{cases}$$

and 0 otherwise. Hence, the first type of element becomes as

$$\begin{aligned}
a(\varphi_i,\varphi_i) &= \int_0^1 \varphi_i'(x)^2\,\mathrm{d}x = \int_{x_{i-1}}^{x_i} \frac{1}{h}\cdot\frac{1}{h}\,\mathrm{d}x + \int_{x_i}^{x_{i+1}} \Big(-\frac{1}{h}\Big)\cdot\Big(-\frac{1}{h}\Big)\,\mathrm{d}x \\
&= \frac{1}{h^2}\int_{x_{i-1}}^{x_{i+1}} \mathrm{d}x = \frac{2h}{h^2} = \frac{2}{h}.
\end{aligned}$$

The second type evaluates as

$$a(\varphi_{i-1},\varphi_i) = \int_{x_{i-1}}^{x_i} -\frac{1}{h}\cdot\frac{1}{h}\,\mathrm{d}x = -\frac{1}{h^2}\int_{x_{i-1}}^{x_i} \mathrm{d}x = -\frac{1}{h}.$$

Hence, the stiffness matrix becomes as

$$A = \frac{1}{h}\begin{bmatrix} 2 & -1 & 0 & \cdot & \cdot & 0 \\ -1 & 2 & -1 & \cdot & \cdot & 0 \\ 0 & -1 & 2 & \cdot & \cdot & 0 \\ \cdot & & & \cdot & & \cdot \\ \cdot & & & & 2 & -1 \\ 0 & 0 & 0 & \cdot & -1 & 2 \end{bmatrix}.$$

The semi-discrete problem is given as: Find $u_h \in S_h$ such that

$$\begin{aligned}(u_{h,t}, \chi) + a(u_h, \chi) &= (f, \chi), \quad \forall \chi \in S_h,\ t > 0,\\ u_h(0) &= v_h,\end{aligned}$$

where the bilinear form is defined as $a(\cdot,\cdot) = ((\cdot)', (\cdot)')$. For the remaining tasks, we begin by writing

$$u_h(x,t) = \sum_{j=1}^{M} \alpha_j(t)\varphi_j(x),$$

since $u_h \in S_h = \operatorname{span}(\{\varphi_j\}_{j=1}^M)$. Take $\chi = \varphi_i(x)$ in the semi-discrete problem to arrive at

$$\sum_{j=1}^{M} \alpha_j'(t) \underbrace{(\varphi_j, \varphi_i)}_{=B_{ij}} + \sum_{j=1}^{M} \alpha_j(t) \underbrace{a(\varphi_j, \varphi_i)}_{=A_{ij}} = (f, \varphi_i),$$

for $i = 1, ..., M$. We thus arrive at the $M \times M$ matrix system

$$B\dot{\alpha}(t) + A\alpha(t) = b,$$

where $\alpha(t) = \begin{bmatrix}\alpha_1(t) & \cdot & \cdot & \cdot & \alpha_M(t)\end{bmatrix}^T$. For a standard ODE $\dot{u}(t) = f(u,t)$, we define the θ-scheme as

$$\frac{u^n - u^{n-1}}{k_n} = \theta f^n + (1-\theta) f^{n-1},$$

so that

$$\begin{aligned}\theta = 0 &\implies \text{Forward Euler},\\ \theta = \frac{1}{2} &\implies \text{Crank–Nicholson},\\ \theta = 1 &\implies \text{Backward Euler}.\end{aligned}$$

The θ-scheme for our matrix system thus becomes

$$B\Big(\frac{\alpha^n - \alpha^{n-1}}{k_n}\Big) + A\big(\theta\alpha^n + (1-\theta)\alpha^{n-1}\big) = \theta b^n + (1-\theta)b^{n-1},$$

which after rearranging the terms is written as

$$(B + \theta k_n A)\alpha^n = (B - (1-\theta)k_n A)\alpha^{n-1} + k_n(\theta b^n + (1-\theta)b^{n-1}).$$

The backward Euler–Galerkin is given by $\theta = 1$, i.e.

$$(B + k_n A)\alpha^n = B\alpha^{n-1} + k_n b^n,$$

and the Crank–Nicholson by $\theta = \frac{1}{2}$, i.e.

$$\Big(B + \frac{1}{2}k_n A\Big)\alpha^n = \Big(B - \frac{1}{2}k_n A\Big)\alpha^{n-1} + \frac{1}{2}k_n(b^n + b^{n-1}).$$

Exercise 10.3

(a) Show that the operator $-\Delta_h : S_h \to S_h$ defined by

$$(-\Delta_h \psi, \chi) = a(\psi, \chi), \quad \forall \chi \in S_h$$

is self-adjoint positive definite with respect to $(\cdot, \cdot)$.

(b) Show that (with the notation of Theorem 6.7)

$$-\Delta_h v_h = \sum_{i=1}^{M_h} \lambda_{i,h}(v_h, \varphi_{i,h})\varphi_{i,h}$$

and $\|\Delta_h\| = \lambda_{M_h,h}$.

(c) Assume that the family of finite element spaces $\{S_h\}$ satisfies the inverse inequality

$$\|\nabla \chi\| \le Ch^{-1}\|\chi\|, \quad \chi \in S_h.$$

Show that

$$\|\Delta_h\| \le Ch^{-2}.$$

Solution: (a) For the self-adjointness of the discrete Laplacian, we find by using the definition och $-\Delta_h$ as well as the symmetry of $a(\cdot, \cdot)$ and $(\cdot, \cdot)$ that

$$(-\Delta_h \psi, \chi) = a(\psi, \chi) = a(\chi, \psi) = (-\Delta_h \chi, \psi) = (\psi, -\Delta_h \chi).$$

For the positive definiteness, we recall that an operator A on a Hilbert space H is positive definite if

$$\inf_{0 \ne x \in H} \frac{(Ax, x)}{(x, x)} > 0.$$

We see this straight away by applying the definition, as

$$(-\Delta_h \psi, \psi) = a(\psi, \psi) = |\psi|_1^2 \ge 0,$$

where equality holds if and only if $\psi = 0$, and hence $-\Delta_h$ is positive definite with respect to $(\cdot, \cdot)$.

(b) We know that the set of discrete eigenfunctions $\{\varphi_{i,h}\}_{i=1}^{M_h}$ forms an ON-basis for S_h. By the definition of the discrete Laplacian, as well as the discrete eigenvalue problem, we also have

$$(-\Delta_h v_h, \varphi_{i,h}) = a(v_h, \varphi_{i,h}) = \lambda_{i,h}(v_h, \varphi_{i,h}).$$

Thus, if we expand $-\Delta_h v_h \in S_h$ in the basis $\{\varphi_{i,h}\}_{i=1}^{M_h}$ we see that

$$-\Delta_h v_h = \sum_{i=1}^{M_h}(-\Delta_h v_h, \varphi_{i,h})\varphi_{i,h} = \sum_{i=1}^{M_h} \lambda_{i,h}(v_h, \varphi_{i,h})\varphi_{i,h}.$$

To show that $\|\Delta_h\| = \lambda_{M_h,h}$, we need to show that

$$\|\Delta_h v_h\| \le \lambda_{M_h,h}\|v_h\|, \quad \forall v_h \in S_h,$$

with equality for some $v_h \in S_h$. To show the inequality, consider

$$\begin{aligned}
\|\Delta_h v_h\|^2 &= (-\Delta_h v_h, -\Delta_h v_h) \\
&= \Big(\sum_{i=1}^{M_h} \lambda_{i,h}(v_h, \varphi_{i,h})\varphi_{i,h}, \sum_{j=1}^{M_h} \lambda_{j,h}(v_h, \varphi_{j,h})\varphi_{j,h}\Big) \\
&= \sum_{i=1}^{M_h}\sum_{j=1}^{M_h} \lambda_{i,h}\lambda_{j,h}(v_h, \varphi_{i,h})(v_h, \varphi_{j,h})(\varphi_{i,h}, \varphi_{j,h}) \\
&= \sum_{i=1}^{M_h} \lambda_{i,h}^2 (v_h, \varphi_{i,h})^2 \\
&\le \max_{1\le i\le M_h} \lambda_{i,h}^2 \sum_{i=1}^{M_h} (v_h, \varphi_{i,h})^2 \\
&= \lambda_{M_h,h}^2 \sum_{i=1}^{M_h}\sum_{j=1}^{M_h} (v_h, \varphi_{i,h})(v_h, \varphi_{j,h})(\varphi_{i,h}, \varphi_{j,h}) \\
&= \lambda_{M_h,h}^2 \Big(\sum_{i=1}^{M_h} (v_h, \varphi_{i,h})\varphi_{i,h}, \sum_{j=1}^{M_h} (v_h, \varphi_{j,h})\varphi_{j,h}\Big) \\
&= \lambda_{M_h,h}^2 \|v_h\|^2.
\end{aligned}$$

Remains to find a function in S_h such that equality holds. Consider $-\Delta_h\varphi_{i,h} \in S_h$. For this function we can apply the definition of the discrete Laplacian and the discrete eigenvalue problem to find that

$$(-\Delta_h\varphi_{i,h}, \chi) = a(\varphi_{i,h}, \chi) = \lambda_{i,h}(\varphi_{i,h}, \chi).$$

Rearranging the terms and using linearity of L_2-product we thus have that

$$(-\Delta_h\varphi_{i,h} - \lambda_{i,h}\varphi_{i,h}, \chi) = 0, \quad \forall \chi \in S_h.$$

Take $\chi = -\Delta_h\varphi_{i,h} - \lambda_{i,h}\varphi_{i,h} \in S_h$ and insert this to get

$$\| - \Delta_h\varphi_{i,h} - \lambda_{i,h}\varphi_{i,h}\|^2 = 0 \implies -\Delta_h\varphi_{i,h} = \lambda_{i,h}\varphi_{i,h}.$$

For $\varphi_{M_h,h} \in S_h$ we thus see that

$$\|\Delta_h\varphi_{M_h,h}\| = \|\lambda_{M_h,h}\varphi_{M_h,h}\| = \lambda_{M_h,h}\|\varphi_{M_h,h}\|,$$

which shows the equality, and hence $\|\Delta_h\| = \lambda_{M_h,h}$.

(c) Take $\chi \in S_h$ and note that since $-\Delta_h\chi \in S_h$

$$\begin{aligned}
\|\Delta_h\chi\|^2 &= (-\Delta_h\chi, -\Delta_h\chi) = a(\chi, -\Delta_h\chi) = (\nabla\chi, \nabla(-\Delta_h\chi)) \\
&\le \|\nabla\chi\|\|\nabla(-\Delta_h\chi)\| \le Ch^{-1}\|\chi\|Ch^{-1}\|\Delta_h\chi\|.
\end{aligned}$$

Cancel one factor on each side to end up with the inequality

$$\|\Delta_h \chi\| \leq C h^{-2} \|\chi\|, \quad \forall \chi \in S_h,$$

which by the definition of the operator-norm gives

$$\|\Delta_h\| \leq C h^{-2}.$$

Exercise 10.4

Let u and u_h be the solutions of

$$\begin{aligned} u_t - \Delta u &= 0, \quad \text{in } \Omega \times \mathbb{R}_+, \\ u &= 0, \quad \text{on } \Gamma \times \mathbb{R}_+, \\ u(\cdot, 0) &= v, \quad \text{in } \Omega, \end{aligned}$$

and

$$\begin{aligned} (u_{h,t}, \chi) + a(u_h, \chi) &= (f, \chi), \quad \forall \chi \in S_h,\ t > 0, \\ u_h(0) &= v_h, \end{aligned}$$

respectively, with $v_h = P_h v$.

(a) Assume that $v \in H^2 \cap H_0^1$. Show that

$$\|u_h(t) - u(t)\| \le Ch^2 \|v\|_2, \text{ for } t \ge 0.$$

(b) Assume that $v \in L_2$. Show that

$$\|u_h(t) - u(t)\| \le Ch^2 t^{-1} \|v\|, \text{ for } t > 0.$$

Solution: (a) Decompose the error by using the Ritz-projection, i.e.

$$e(t) = u_h(t) - u(t) = \underbrace{u_h(t) - R_h u(t)}_{=:\theta(t)} + \underbrace{R_h u(t) - u(t)}_{=:\rho(t)},$$

so that the error can be bounded as

$$\|e\| \le \|\theta\| + \|\rho\|.$$

We begin with the error from the elliptic projection ρ, and recall the error estimates

$$\|R_h v - v\| + h|R_h v - v|_1 \le Ch^s \|v\|_s, \text{ for } s = 1, 2,$$

from Theorem 5.5. Hence, we can bound ρ as

$$\|\rho\| \le Ch^2 \|u(t)\|_2 \le Ch^2 \|v\|_2,$$

where the last inequality follows from Problem 8.10. For θ we make the same calculation as in (10.14) in the course literature and get

$$(\theta_t, \chi) + \underbrace{a(\theta, \chi)}_{(-\Delta_h \theta, \chi)} = -\underbrace{(\rho_t, \chi)}_{(P_h \rho_t, \chi)},$$

which in turn yields

$$(\theta_t - \Delta_h \theta + P_h \rho_t, \chi) = 0, \quad \forall \chi \in S_h.$$

Take $\chi = \theta_t - \Delta_h\theta + P_h\rho_t \in S_h$ and insert this to get

$$\theta_t - \Delta_h\theta = -P_h\rho_t.$$

With θ satisfying this equation, we can apply the discrete version of Duhamel's principle ((10.8) in the course literature) to get

$$\theta(t) = E_h(t)\theta(0) - \int_0^t E_h(t-s)P_h\rho_s(s)\,\mathrm{d}s.$$

We now follow the hint given in the problem formulation (as stated in the course literature) and split the integral as

$$\int_0^t \ldots\,\mathrm{d}s = \int_0^{t/2} \ldots\,\mathrm{d}s + \int_{t/2}^t \ldots\,\mathrm{d}s$$

and apply integration by parts on the first integral to get

$$\begin{aligned}
-&\int_0^{t/2} E_h(t-s)D_s(P_h\rho)(s)\,\mathrm{d}s \\
&\qquad= -\Big[E_h(t-s)P_h\rho(s)\Big]_0^{t/2} + \int_0^{t/2} D_sE_h(t-s)P_h\rho(s)\,\mathrm{d}s \\
&\qquad= -E_h(t/2)P_h\rho(t/2) + E_h(t)\rho(0) + \int_0^{t/2} D_sE_h(t-s)P_h\rho(s)\,\mathrm{d}s.
\end{aligned}$$

Note here that we wrote $P_h\rho_s(s) = D_s(P_h\rho)(s)$, which works since D_s commutes with P_h. Consequently, we can write θ as

$$\begin{aligned}
\theta(t) &= \overbrace{E_h(t)\theta(0) + E_h(t)P_h\rho(0)}^{(I)} - \overbrace{E_h(t/2)P_h\rho(t/2)}^{(II)} \\
&\quad+ \underbrace{\int_0^{t/2} D_sE_h(t-s)P_h\rho(s)\,\mathrm{d}s}_{(III)} - \underbrace{\int_{t/2}^t E_h(t-s)P_h\rho_s(s)\,\mathrm{d}s}_{(IV)},
\end{aligned}$$

so that we furthermore can bound it as

$$\|\theta\| \le \|(I)\| + \|(II)\| + \|(III)\| + \|(IV)\|.$$

For the first one we find that

$$\begin{aligned}
(I) &= E_h(t)(\theta(0) + P_h\rho(0)) \\
&= E_h(t)P_h(\theta(0) + \rho(0)) \\
&= E_h(t)(P_hu_h(0) - P_hu(0)) \\
&= E_h(t)(v_h - v_h) = 0.
\end{aligned}$$

Here we used the fact that $\theta(0) \in S_h$ so that $\theta(0) = P_h\theta(0)$ in the second equality, and the rest follows from the projection property of P_h. For the second one we get, since $\|P_h\| = 1$,

$$\|(II)\| = \|E_h(t/2)P_h\rho(t/2)\| = \|P_h\rho(t/2)\| \leq \|\rho(t/2)\| \leq Ch^2\|v\|_2,$$

where the last inequality follows from the ρ-part. A similar bound is found for the third one by

$$\begin{aligned}
\|(III)\| &= \left\| \int_0^{t/2} D_s E_h(t-s)P_h\rho(s)\,\mathrm{d}s \right\| \\
&\leq \int_0^{t/2} \|D_s E_h(t-s)P_h\rho(s)\|\,\mathrm{d}s \\
&\leq C\int_0^{t/2} (t-s)^{-1}\|P_h\rho(s)\|\,\mathrm{d}s \\
&\leq Ch^2\|v\|_2 \int_0^{t/2} \frac{1}{t-s}\,\mathrm{d}s \\
&= Ch^2\|v\|_2\Big[-\log(t-s)\Big]_0^{t/2} \\
&= Ch^2\|v\|_2(\log(t) - \log(t/2)) \\
&= C\log(2)h^2\|v\|_2 = Ch^2\|v\|_2.
\end{aligned}$$

Here we applied inequality (10.18) from the course literature in the third line, followed by the previously found bound for $P_h\rho$ in the subsequent step. Remains to show similar bound for (IV). At first we note that

$$\|E_h(t-s)P_h\rho_s(s)\| \leq \|\rho_s(s)\| = \|R_h u_s(s) - u_s(s)\| \leq Ch^2\|u_s(s)\|_2,$$

since $\|E_h(t-s)\| \leq 1$ and $\|P_h\| = 1$. Thus, we find that

$$\begin{aligned}
\|(IV)\| &\leq \int_{t/2}^{t} \|E_h(t-s)P_h\rho_s(s)\|\,\mathrm{d}s \\
&\leq Ch^2\int_{t/2}^{t} \|u_s(s)\|_2\,\mathrm{d}s \\
&= Ch^2\int_{t/2}^{t} \|D_sE(s)v\|_2\,\mathrm{d}s \\
&\leq Ch^2\int_{t/2}^{t} Cs^{-1-2/2}\|v\|\,\mathrm{d}s \\
&= Ch^2\|v\|\int_{t/2}^{t} s^{-2}\,\mathrm{d}s \\
&\leq Ch^2\|v\|_2\Big[-s^{-1}\Big]_{t/2}^{t} \\
&= \frac{C}{t}h^2\|v\|_2 \leq Ch^2\|v\|_2.
\end{aligned}$$

In these calculations, we used the fact that $u(s) = E(s)v$ in the third step, followed by the property (8.18) from the course literature. Moreover, the last step assumes that $t > 0$. In the case $t = 0$ it holds that $(IV) = 0$, so the estimate holds regardless. Summing these results now gives the bound

$$\|\theta(t)\| \leq Ch^2 \|v\|_2,$$

which in turn yields the desired estimate.

(b) Assume $t > 0$. We decompose the error in the same ρ-θ way as in (a). For the elliptic projection we have

$$\|\rho(t)\| = \|R_h u(t) - u(t)\| \leq Ch^2 \|u(t)\|_2 = Ch^2 \|E(t)v\|_2 \leq Ch^2 t^{-1} \|v\|.$$

Here, the first inequality followed from the error estimate results for the Ritz-projection, and the last inequality from the identity (8.18) in the course literature. For the θ-part, we follow the hint given in the course literature and write

$$\tilde{\rho}(t) = \int_0^t \rho(s)\,\mathrm{d}s,$$

so that $D_t \tilde{\rho}(t) = \rho(t)$. The results given in the hint follows since

$$\|\tilde{\rho}(t)\| = \|R_h \tilde{u}(t) - \tilde{u}(s)\| \leq Ch^2 \|\tilde{u}\|_2,$$

and by elliptic regularity $\|\tilde{u}\|_2 \leq C\|\Delta \tilde{u}\|$. Moreover, by the heat equation, it follows that

$$\Delta \tilde{u} = \int_0^t u_s\,\mathrm{d}s = u(t) - v,$$

so we end up with the estimate

$$\|\tilde{\rho}\| \leq Ch^2 \|u(t) - v\| = Ch^2 \|(E(t) - I)v\| \leq Ch^2 \|E(t) - I\| \|v\| \leq Ch^2 \|v\|.$$

For the estimate of θ, we will decompose the error in a similar way as in task (a), but this time we apply integration by parts once more on (III), so that

$$\begin{aligned}
(III) &= \int_0^{t/2} D_s E_h(t-s) P_h \rho(s)\,\mathrm{d}s \\
&= \int_0^{t/2} D_s E_h(t-s) P_h D_s \tilde{\rho}(s)\,\mathrm{d}s \\
&= \Big[D_s E_h(t-s) P_h \tilde{\rho}(s) \Big]_0^{t/2} - \int_0^{t/2} D_s^2 E_h(t-s) P_h \tilde{\rho}(s)\,\mathrm{d}s \\
&= D_t E_h(t/2) P_h \tilde{\rho}(t/2) - 0 - \int_0^{t/2} D_s^2 E_h(t-s) P_h \tilde{\rho}(s)\,\mathrm{d}s,
\end{aligned}$$

where the second boundary term vanishes since $\tilde{\rho}(0) = 0$. Moreover, recall that $(I) = 0$ in (a), so we can neglect that term this time. The decomposition becomes

$$\theta(t) = -\overbrace{E_h(t/2)P_h\rho(t/2)}^{(i)} + \overbrace{D_tE_h(t/2)P_h\tilde{\rho}(t/2)}^{(ii)} \\ -\underbrace{\int_0^{t/2} D_s^2E_h(t-s)P_h\tilde{\rho}(s)\,\mathrm{d}s}_{(iii)} - \underbrace{\int_{t/2}^{t} E_h(t-s)P_h\rho_s(s)\,\mathrm{d}s}_{(iv)},$$

so that we bound the θ-error as

$$\|\theta\| \leq \|(i)\| + \|(ii)\| + \|(iii)\| + \|(iv)\|.$$

For the first term, it suffices to bound the operators by their corresponding norms and then apply the results for ρ, i.e.

$$\|(i)\| = \|E_h(t/2)P_h\rho(t/2)\| \leq \|\rho(t/2)\| \leq 2Ch^2t^{-1}\|v\|.$$

For the second term, which includes a derivative on the discrete solution operator, we can once again apply the identity (10.18) from the course literature along with the previously derived results for $\tilde{\rho}$ to get

$$\begin{aligned}\|(ii)\| &= \|D_tE_h(t/2)P_h\tilde{\rho}(t/2)\| \\ &\leq Ct^{-1}\|P_h\tilde{\rho}(t/2)\| \\ &\leq Ct^{-1}\|\tilde{\rho}(t/2)\| \\ &\leq Ch^2t^{-1}\|v\|.\end{aligned}$$

For the third term, we once again apply (10.18) for the derivative on the discrete solution operator and the results for $\tilde{\rho}$ to get

$$\begin{aligned}\|(iii)\| &\leq \int_0^{t/2} \|D_s^2E_h(t-s)P_h\tilde{\rho}(s)\|\,\mathrm{d}s \\ &\leq \int_0^{t/2} C(t-s)^{-2}\|P_h\tilde{\rho}(s)\|\,\mathrm{d}s \\ &\leq Ch^2\|v\|\int_0^{t/2}(t-s)^{-2}\,\mathrm{d}s \\ &= Ch^2t^{-1}\|v\|.\end{aligned}$$

For (iv), we repeat the calculations from (IV) in (a), but skip the part where we bound $\|v\|$ by $\|v\|_2$ and just leave it as it is, which gives the bound

$$\|(iv)\| \leq Ch^2t^{-1}\|v\|.$$

In total, we thus get

$$\|\theta\| \leq Ch^2t^{-1}\|v\|,$$

which yields the desired estimate.