Matematisk Statistik och Disktret Matematik, MVE051/MSG810, VT20

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Probability laws

Let S be the sample space and A and B be two events.

- **Complement:** The complement event to A is the event A'="A does not occur." (denoted by A' or $\bar{A}$ or A^c)

$$p(A') = 1 - p(A).$$

- **Addition rule**

$$p(A \cup B) = p(A) + p(B) - p(A \cap B)$$

- If A and B are disjoint then, $\Rightarrow p(A \cap B) = 0$ and $p(A \cup B) = p(A) + p(B)$

Example

Let A and B be two events such that $p(A) = 0.5$, $P(B) = 0.7$ *and* $p(A \cap B) = 0.4$. *Find* $p(A \cup B)$, $p(A \cap B')$, $p(A' \cap B)$ *and* $p(A' \cap B')$.

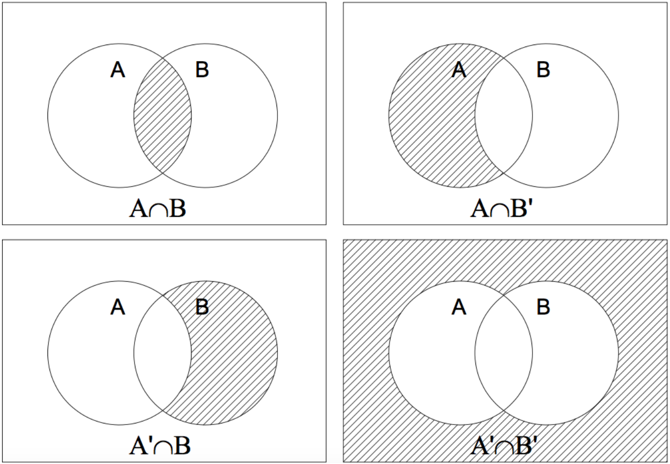
Solution

$p(A \cup B) = p(A) + p(B) - p(A \cap B) = 0.5 + 0.7 - 0.4 = 0.8$

$p(A \cap B') = p(A) - p(A \cap B) = 0.5 - 0.4 = 0.1$

$p(A' \cap B) = p(B) - p(A \cap B) = 0.7 - 0.4 = 0.3$

$p(A' \cap B') = p((A \cup B)') = 1 - p(A \cup B) = 1 - 0.8 = 0.2$


A
B
A∩B
A
B
A∩B'
A
B
A'∩B
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B
A'∩B'

Conditional probability

Example

300 products in a factory has been chosen randomly for a quality control and they were then classified either as "defected" or "good". Some of the products were produced by an old machine and the others by a new one. The following table gives the result of the experiment.

	Good	Defected	Total
Old machine	170	10	180
New machine	115	5	120
Total	285	15	300

Example

A product is chosen randomly. Let A, B, and C be three events defined as follows A="The chosen product is good".
B="The chosen product is produced by an old machine.
C="The chosen product is good knowing that (given that) is was produced by an old machine."
$p(A) = \frac{285}{300}$, $p(B) = \frac{180}{300}$
The event C invovles both events A and B, and is written as $C = A|B$ *(A given B, or A knowing B). From the table we can get the probability of C by considering only the row of the old machine, namely,*

$$p(C) = p(A|B) = \frac{170}{180} = \frac{p(A \cap B)}{p(B)}$$

Conditional probability

Given two events A and B with $p(B) \neq 0$. The **conditional probability** that A occurs given that B has occured is defined by

$$p(A|B) = \frac{p(A \cap B)}{p(B)}.$$

Multiplication rule

Multiplication rule: Suppose that $p(A) \neq 0$ and $p(B) \neq 0$.

$$p(A \cap B) = p(A|B)p(B) = p(B|A)p(A).$$

The conditional probability can therefore be written as

$$p(A|B) = \frac{p(B|A)p(A)}{p(B)}$$

Independent events

Independent events: Assume that the information "B has occured" has no influence on the probability that A occur, then $p(A|B) = p(A)$, and the multiplication rule can be written as $p(A \cap B) = p(A)p(B)$.

Theorem

A and B are said to be independent if and only if

$$p(A \cap B) = p(A)p(B).$$

This is equivalent to say that $p(A|B) = p(A)$ if $p(B) \neq 0$ and $p(B|A) = p(B)$ if $p(A) \neq 0$.

Example

At the entrance to a casino, there are two slot machines. Machine A is programmed so that in the long run it will produce a winner in 10% of the plays. Machine B is programmed so that in the long run it will produce a winner in 15% of the plays. The two machines run independently of each other. If we play each machine once, what is the probability that we will win on at least one play?

$$p(A \cup B) = p(A) + p(B) - p(A \cap B) = p(A) + p(B) - p(A)p(B) = 0.235$$

Bayes' theorem

Example

The fire alarm in a certain company seems to be reliable. However, a false alarm can sometimes occur or a fire could be missed. Let F="A fire has occured" och A="The alarm starts beeping.", $p(F) = 0.05$, $p(A|F) = 0.98$, and $p(A|F') = 0.10$. Find $p(F|A)$.
Lösning:

$$p(F|A) = \frac{p(F \cap A)}{p(A)} = \frac{p(A|F)p(F)}{p(A)}$$

Example

To compute $p(A)$ we can use the formula

$$A = (A \cap F) \cup (A \cap F')$$

Since $L \cap B$ and $L \cap B'$ are disjoint, then

$$\begin{aligned} p(A) &= p(A \cap F) + p(A \cap F') \\ &= p(A|F)p(F) + p(A|F')p(F') \end{aligned}$$

Therefore

$$\begin{aligned} p(F|A) &= \frac{p(A|F)p(F)}{p(A|F)p(F) + p(A|F')p(F')} \\ &= \frac{0.98 \cdot 0.05}{0.98 \cdot 0.05 + 0.10(1 - 0.05)} \\ &= 0.34 \end{aligned}$$

Bayes' theorem

Let $A_1, A_2, \cdots A_n$ be mutually disjoint events such that their union is S and $B \neq \emptyset$ be an event. For all A_j, $j = 1, \cdots n$

$$p(A_j|B) = \frac{p(B|A_j)p(A_j)}{\sum_{i=1}^{n} p(B|Ai)p(A_i)}$$