Matematisk Statistik och Disktret Matematik, MVE051/MSG810, VT20

Föreläsning 2

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Random variables

- A **random variable** *(sv. stokastisk variabel)* is a function that associates a real number to every outcome of a random trial ($X : S \to \mathbb{R}$ where S is the sample space).

Example

A coin is flipped twice. A person gets 0 kr if the result is 2 Tail, 1 kr if the result is one Tail and one Head and 2 kr if it is two Head. $S = \{00, 01, 10, 11\}$ *where 0 and 1 correspond to Tail and Head respectively. The experiment can be modelled with a random variable* $X : S \to \{0, 1, 2\}$ *such that* $S(00) = 0, S(01) = S(10) = 1, S(11) = 2)$

- A random variable is said to be **discrete** if it can assume at most a finite or a countable infinite number of possible values.
- A random variable is said to be **continuous** if it can assume any value in some interval or intervals of real numbers and the probability that it assumes a specific value is 0.

Density and distribution functions

Let X be a discrete random variable.

- The function
  $$f(x) = P(X = x)$$
  for x real is called the **density function** *(sv. täthetsfunktion* for X.
- *(Important)* A funktion f is a density function for a discrete random variable X if and only if $f(x) \geq 0$ and $\sum_{all\ x} f(x) = 1$.
- The function
  $$F(x) = p(X \leq x)$$
  for x real is called the cumulative distribution function *(sv. kumulativ fördelningsfunktion)* for X.

Geometric sequence

Let $(ar^k)_{k\in\mathbb{N}}$ be a geometric sequence with $r \neq 1$.
Sum of the first n terms:

$$\sum_{k=1}^{n} ar^k = \frac{ar(1-r^n)}{1-r} = \frac{(\text{first term})\cdot(1-r^{\text{number of terms}})}{1-r}$$

Infinite sum for $|r| < 1$

$$\sum_{k=1}^{\infty} ar^k = \frac{ar}{1-r}$$

Example

Show that the function

$$f(x) = \left(\frac{1}{2}\right)^x \quad x = 1, 2, 3, \ldots$$

and $f(x) = 0$ *otherwise, is a density function and find* $p(X \geq 4)$ *and* $F(10)$ *where* F *is the cumulative distribution function.*

Solution:

$f(x) \geq 0$ *for all* $x \in \mathbb{R}$*, and* $\sum_{all\ x} f(x) = \sum_{x=1}^{\infty} \left(\frac{1}{2}\right)^x = \frac{1}{2} \cdot \frac{1}{1-\frac{1}{2}} = 1$

Therefore f *is a density function.*

$$\begin{aligned} p(X \geq 4) &= 1 - p(X < 4) = 1 - p(X \leq 3) = 1 - F(3) \\ &= 1 - (1/2 + (1/2)^2 + (1/2)^3) = \frac{1}{8}. \end{aligned}$$

Example

$$
\begin{aligned}
F(10) &= p(X \leq 10) = p(X = 1) + p(X = 2) + \cdots p(X = 10) \\
&= \frac{1}{2} + (\frac{1}{2})^2 + \cdots + (\frac{1}{2})^{10} \\
&= \frac{1}{2} \cdot \frac{1 - (1/2)^{10}}{1 - (1/2)} \\
&= 1 - (1/2)^{10}.
\end{aligned}
$$

Expected Value

Let X be a discrete random variable with density $f(x)$.

- The **expected value** *(sv. väntevärde)* of X is given by

$$E[X] = \sum_{\text{all } x} x f(x).$$

provided that $\sum_{\text{all } x} x f(x) < \infty$. $E[X]$ is also denoted by μ.

- In general, if $H(X)$ is a random variable, the expected value of $H(X)$, denoted by $E[H(X)]$, is given by

$$E[H(X)] = \sum_{\text{alla } x} H(x) f(x)$$

provided that $\sum_{\text{alla } x} |H(x)| f(x) < \infty$.

Variance and standard deviation

Let X be a discrete random variable with density $f(x)$ and $E[X] = \mu$.

- The **variance** *(sv. varians)* of X is defined by

$$Var[X] = E[(X-\mu)^2] = \sum_{all\ x}(x-\mu)^2 f(x)$$

  It is usually denoted by σ^2.

- The **standard deviation** *(sv. standardavvikelse)* of X is defined by

$$\sigma = \sqrt{Var[X]}$$

Rules

Let X and Y be two random variables and c a constant real number.

Rules for expected value

- $E[c] = c$
- $E[cX] = cE[X]$
- $E[X + Y] = E[X] + E[Y]$

Rules for the variance:

- $Var[c] = 0$
- $Var[cX] = c^2 Var[X]$
- $Var[X + Y] = Var[X] + Var[Y]$ if X and Y are independent.

Theorem

Let $E[X]$ and $V[X]$ be the expected value and respectively the variance of a random variable X, then

$$V[X] = E[X^2] - E[X]^2$$

Example (p.1, coin flipping)

$f(0) = \frac{1}{4}$, $f(1) = \frac{1}{2}$ *and* $f(2) = \frac{1}{4}$.

$E[X] = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 1$.

This means that if we repeat the experiment infinitely many times, the average value of the money one would get will be 1.

$E[X^2] = 0^2 \cdot \frac{1}{4} + 1^2 \cdot \frac{1}{2} + 2^2 \cdot \frac{1}{4} = \frac{3}{2}$.

$Var[X] = E[X^2] - E[X]^2 = \frac{3}{2} - 1 = \frac{1}{2}$.

$\sigma = \frac{1}{\sqrt{2}}$.

Remark

- *The variance and the standard deviation describe how much the values of X deviate from μ.*
- *The unit of the variance is meaningless and usually omitted. The standard deviation has the same unit as the original data.*
- *The values of the variance or the standard deviation are not informative in themselves. They are often used for comparative purposes. For instance, if X and Y are two similar random variables with $E[X] = E[Y] = 70$, $\sigma_X = 5$ and $\sigma_Y = 30$, then the values that X takes are closer to the mean than the values that Y takes.*

Bernoulli distribution

- X takes two possible values 0 and 1, and the density function is given by

$$f(x) = \begin{cases} p & \text{if } x = 1 \\ 1 - p & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

- 0 is called *failure* and 1 is called *success*.
- $E[X] = p$ and $V[X] = p(1 - p)$ (Prove it!).

Geometric distribution *(sv. för första gång fördelning)*

- The experiment consists of a series of Bernoulli trials with probability of success equals to p.
- The trials are identical and independent of each other. This means that the probability of success will remain the same in all trials.
- The random variable X denotes the number of trials needed to get the first success.
- p is called the parameter of X.
- A random variable X that follows a geometric distribution with parameter p is denoted by $X \sim Geom(p)$.

Geometric distribution *(sv. för första gång fördelning)*

- The density function of X is given by
  $f(x) = \begin{cases} (1-p)^{x-1}p & \text{om } x = 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}$
- The distribution function of X is given by

  $$F(x) = 1 - q^{\lfloor x \rfloor}$$

  where $q = 1 - p$ and $\lfloor x \rfloor$ is the floor function of x, i.e. the highest integer less than or equal to x.
- $E[X] = \frac{1}{p}$ and $Var[X] = \frac{1-p}{p^2}$.

Moment generating function (m.g.f.)

Let X be a random variable (discrete or continuous)

- The k^{th} moment for X is defined by $E[X^k]$.
- The moment generating function for X is defined by

$$m_X(t) = E[e^{tX}]$$

- Let $m_X(t)$ be the m.g.f for X. Then

$$\left.\frac{d^k m_X(t)}{dt^k}\right|_{t=0} = E[X^k]$$

Let $X \sim Geom(p)$. The m.g.f for X is given by

$$m_X(t) = \frac{pe^t}{1 - qe^t}$$

where $q = 1 - p$.
$m'_X(t) = \frac{pe^t}{(1-qe^t)^2}$, then $m'_X(0) = \frac{1}{p}$. Therefore, $E[X] = \frac{1}{p}$.
$m''_X(t) = \frac{pe^t(1+qe^t)}{(1-qe^t)^3}$, then $E[X^2] = m''_X(0) = \frac{p(1+q)}{p^3} = \frac{1+q}{p^2}$.
Therefore

$$Var[X] = E[X^2] - E[X]^2 = \frac{1+q}{p^2} - \frac{1}{p^2} = \frac{q}{p^2} = \frac{1-p}{p^2}$$

Binomial distribution

- The experiment consists of a fixed number n of Bernoulli trials with probability of success equals to p.
- The trials are identical and independent.
- The random variable X denotes the number of success obtained in n trials.
- n and p are called the parameters of X
- A random variable X that follows a binomial distribution with parameters n and p is denoted by $X \sim Bin(n, p)$.

Binomial distribution

- The density function of X is
  $f(x) = \begin{cases} \binom{n}{x}(1-p)^{n-x}p^x & \text{om } x = 0, 1 \cdots, n \\ 0 & \text{annars} \end{cases}$
- For the cumulative distribution function we use the table in the book (pp. 687-691).
- The m.g.f. for X is

$$m_X(t) = (q + pe^t)^n$$

  where $q = 1 - p$.
- $E[X] = \mu = np$ and $Var[X] = \sigma^2 = npq$.