Övning 3   3 - 6 - 9 39 . 41 . 42 . 52   Kapitel 4

Ex. 3 $f(x) = \frac{1}{10} e^{-\frac{x}{10}} \quad x > 0$

(a) f is a density for a continuous distribution if

(i) $f(x) \geq 0 \quad \forall x \in \mathbb{R}$

(ii) $\int_{\mathbb{R}} f(x)dx = 1$

(i) $f(x) = \frac{1}{10} e^{-x/10} > 0 \quad \forall \ x > 0$

och $= 0 \geq 0 \quad \forall \ x \leq 0$

(ii) $\int_{\mathbb{R}} f(x)\,dx = \int_0^\infty \frac{1}{10} e^{-\frac{x}{10}} dx = \frac{1}{10} \left. \frac{e^{-\frac{x}{10}}}{-\frac{1}{10}} \right]_0^\infty = -e^{-\frac{x}{10}} \Big]_0^\infty = -e^{-\infty} + e^{0} = 1$

(b) $P(X \leq 7) = \int_0^7 \frac{1}{10} e^{-\frac{x}{10}} dx = -e^{-\frac{x}{10}} \Big|_0^7 = -e^{-\frac{7}{10}} + 1 = 1 - e^{-7/10} = 0,497$

$P(X \geq 7) = 1 - P(X < 7) = 1 - P(X \leq 7) = 1 - 0,497 = 0,503$

(eller $P(X \geq 7) = \int_7^\infty \frac{1}{10} e^{-x/10} dx$)

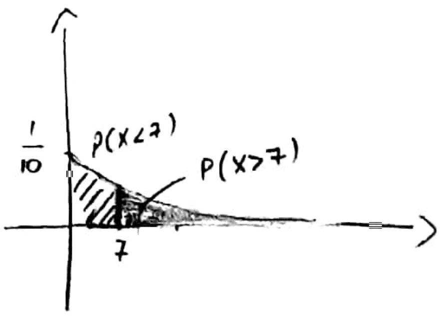
$P(X = 7) = 0$

(c) $P(1 < X < 2) = \int_1^2 f(x)dx = \int_1^2 \frac{1}{10} e^{-\frac{x}{10}} dx = -e^{-\frac{x}{10}} \Big]_1^2 = -e^{-\frac{2}{10}} + e^{-\frac{1}{10}}$

$= 0,086$

$P(1 < X < 2)$ is very close to 0 $\Rightarrow$ it's unusual for a call to last between 1 and 2 minuts.

(d)



arean av ▨ är $P(X < 7)$

arean av ■ är $P(X > 7)$

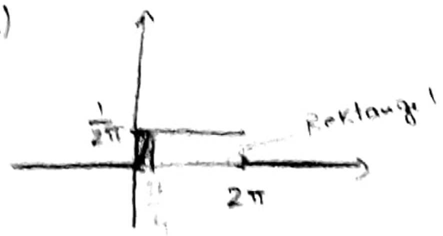
$P(X = 7)$ är arean av linjen | som är 0

Ex 6 $\Theta \sim U[0, 2\pi]$ $\frac{1}{2\pi - 0}$

$U(a,b)$ $f(x) = \begin{cases} \frac{1}{b-a} & a < x < b \\ 0 & \text{otherwise} \end{cases}$

(a) $f(\theta) = \begin{cases} \frac{1}{2\pi} & 0 \le \theta \le 2\pi \\ 0 & \text{otherwise} \end{cases}$

(b)



(c)

$P(0 < \theta < \frac{\pi}{4}) + P(2\pi - \frac{\pi}{4} < \theta < 2\pi)$ = arean av rektangel ① + arean av rektangel ②

$= \frac{\pi}{4} \cdot \frac{1}{2\pi} + \frac{\pi}{4} \cdot \frac{1}{2\pi} = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$.

(d) $P(0 < \theta < \frac{\pi}{4}) + P(2\pi - \frac{\pi}{4} < \theta < 2\pi) = \int_0^{\pi/4} \frac{1}{2\pi} d\theta + \int_{2\pi - \frac{\pi}{4}}^{2\pi} \frac{1}{2\pi} d\theta$

$= \frac{1}{2\pi}\theta \Big]_0^{\pi/4} + \frac{\theta}{2\pi}\Big]_{2\pi - \frac{\pi}{4}}^{2\pi} = \frac{1}{2\pi}(\frac{\pi}{4} - 0) + \frac{1}{2\pi}(2\pi - 2\pi + \frac{\pi}{4})$

$= \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$

(e) Let X: nb. of birds that orient within $\frac{\pi}{4}$ of home.

$\Rightarrow X \sim Bin(10, \frac{1}{4})$ (Success: a bird orients within $\frac{\pi}{4}$ of home
$p = \frac{1}{4}$ from (c). $n = 10$)

$P(X \ge 7) = 1 - P(X < 7) = 1 - 0,9996$ ← enligt tabellen.

$= 0,0004$ ← close to zero

Scanned by CamScanner

Ex. 9 $\quad f(x) = Kx \qquad 2 \leq x \leq 4$

Ex.1 (a) Find K

(i) $f(x) \geq 0 \Rightarrow K \geq 0$

(ii) $\int_{\mathbb{R}} f(x)\,dx = 1 \Rightarrow \int_2^4 Kx\,dx = K \left.\frac{x^2}{2}\right]_2^4 = K\left(\frac{16}{2} - \frac{4}{2}\right) = 6K = 1$

$\Rightarrow K = \frac{1}{6}. \qquad \Rightarrow f(x) = \frac{1}{6}x \qquad 2 \leq x \leq 4$

Ex.9 (a) $\quad F(x) = \int_{-\infty}^{x} f(t)\,dt$

If $x < 2 \Rightarrow f(t) = 0$ for all $t < x$

$\Rightarrow F(x) = \int_{-\infty}^{x} 0\,dt = 0$

$= \int_{-\infty}^{2} 0\,dt + \int_2^x \frac{1}{6} t\,dt$

If $2 < x < 4 \Rightarrow F(x) = \int_{-\infty}^{x} f(t)\,dt = \int_2^x \frac{1}{6} t\,dt = \frac{1}{6}\left.\frac{t^2}{2}\right]_2^x = \frac{x^2}{12} - \frac{4}{12}$

$= \frac{x^2-4}{12}$

If $x > 4 \Rightarrow F(x) = \int_{-\infty}^{x} f(t)\,dt = \underbrace{\int_{-\infty}^{2} f(t)\,dt}_{0} + \overbrace{\int_2^4 f(t)\,dt}^{1} + \overbrace{\int_4^x f(t)\,dt}^{0}$

$= 0 + 1 + 0 = 1$

$\Rightarrow F(x) = \begin{cases} 0 & x < 2 \\ \frac{x^2-4}{12} & 2 \leq x < 4 \\ 1 & x \geq 4 \end{cases}$

(b) $P(2{,}5 \leq X \leq 3) = F(3) - F(2{,}5) = \frac{9-4}{12} - \frac{2{,}5^2 - 4}{12} = \frac{5}{12} - \frac{2{,}25}{12}$

$= \frac{20}{48} - \frac{9}{48} = \frac{11}{48}$

(c). $F'(x) = \begin{cases} \frac{2x}{12} & 2 < x < 4 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{x}{6} & 2 < x < 4 \\ 0 & \text{otherwise} \end{cases} = f(x)$

Ex. 39 (a) $P(Z \leq 1{,}57) = 0{,}9418$ (according to the table)

(b) $P(Z < 1{,}57) = P(Z \leq 1{,}57) = 0{,}9418$

(c) $P(Z = 1{,}57) = 0$

(d) $P(Z > 1{,}57) = 1 - P(Z \leq 1{,}57) = 1 - 0{,}9418 = 0{,}0582$

(e) $P(-1{,}25 \leq Z \leq 1{,}75) = P(Z \leq 1{,}75) - P(Z \leq -1{,}25)$

$= 0{,}9418 - 0{,}1056 = 08362$

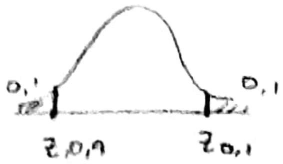
(f) $z_{0,10}$ $P(Z > z_{0,10}) = 0{,}10$

$\Rightarrow P(Z \leq z_{0,10}) = 1 - 0{,}10 = 0{,}90 \Rightarrow z \simeq 1{,}28$

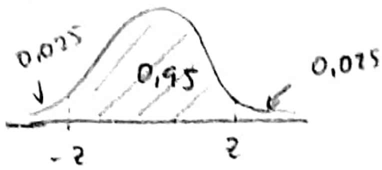
(g) $z_{0,90}$: $P(Z > z_{0,90}) = 0{,}90$

$\Rightarrow P(Z < z_{0,9}) = 0{,}10$

$\Rightarrow z_{0,9} = -z_{0,1} = -1{,}28$



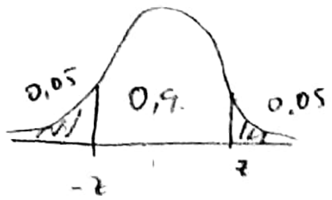
(h) $P(-z \leq Z \leq z) = 0{,}95$



$\Rightarrow P(Z \leq -z) = 0{,}025$ och $P(Z > z) = 0{,}025 \Rightarrow P(Z \leq z) = 1 - 0{,}025 = 0{,}975$

$\Rightarrow z = z_{0,025} = 1{,}96$

(i)

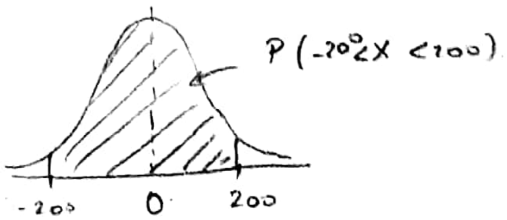


$\Rightarrow P(Z > z) = 0{,}05 \Rightarrow P(Z \leq z) = 0{,}95$

$\Rightarrow z = 1{,}645$ (mellan 1,64 och 1,65)

Ex. 41 $X \sim N(0, 100^2)$

(a)



$P(-200 < X < 200) = 1 - 2P(X < -200)$

$= 1 - 2 \cdot P\left(\frac{X-0}{100} < \frac{-200-0}{100}\right)$

$= 1 - 2P(Z < -2) = 1 - 2 \cdot 0{,}0228$

$= 0{,}9544$

(b) $P(X > 250) + P(X < -250) = 2P(X < -250) = 2P\left(Z < \frac{-250}{100}\right)$

$= 2P(Z < -2{,}5) = 2 \cdot 0{,}0062 = 0{,}0124$

(c) Let d be this distance

$P(X \geq d) + P(X \leq -d) = 0,20 \Rightarrow 2P(X \leq -d) = 0,20$

$\Rightarrow P(X \leq -d) = 0,10$

$\Rightarrow P\left(Z \leq \frac{-d}{100}\right) = 0,10$

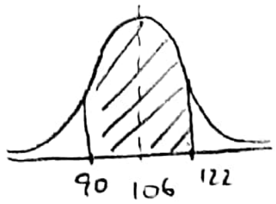
$\Rightarrow -\frac{d}{100} \simeq -1,28 \Rightarrow d = 128$

(d) $m_X(t) = e^{\mu t + \sigma^2 t^2/2}$ with $\mu = 0$ $\sigma = 100$

$\Rightarrow m_X(t) = e^{100^2 t^2/2} = e^{5000t^2}$

42. $X \sim N(106, 8^2)$

(a)



$P(90 < X < 122) = P\left(\frac{90-106}{8} < \frac{X-106}{8} < \frac{122-106}{8}\right)$

$= P(-2 < Z < 2) = 1 - 2P(Z < -2)$

$= 0,9544$

(b) $P(X \leq 120) = P\left(\frac{X-106}{8} \leq \frac{120-106}{8}\right) = P\left(Z \leq \frac{7}{4}\right) = P(Z \leq 1,75) = 0,9599$

(c) $P(X \leq x_0) = 0,25 \Rightarrow P\left(\frac{X-106}{8} \leq \frac{x_0-106}{8}\right) = 0,25$

Let $z_0 = \frac{x_0 - 106}{8}$, $Z = \frac{X-106}{8} \sim N(0,1)$

$\Rightarrow P(Z < z_0) = 0,25 \Rightarrow z_0 = -0,675$

$\Rightarrow \frac{x_0 - 106}{8} = -0,675 \Rightarrow x_0 = -8 \cdot (0,675) + 106 = 100,6$

52. $X \sim Bin(20, 0.3)$ $np = 20 \cdot 0.3 = 6 > 5 \Rightarrow X \sim Y = N(np, npq)$

$npq = 20 \cdot 0.3 \cdot 0.7 = 4,2 \Rightarrow Y \sim N(6, 4,2)$

(a) $P(X \leq 3) \simeq P(Y \leq 3,5) = P\left(\frac{Y-6}{\sqrt{4,2}} \leq \frac{3,5-6}{\sqrt{4,2}}\right) = P(Z \leq -1,22) = 0,1112$

Binomial Table: $P(X \leq 3) = 0,1071$

(b) $P(3 \leq X \leq 6) = P(X \leq 6) - P(X < 3) \doteq P(Y \leq 6,5) - P(Y \leq 2,5)$

$= P\left(\frac{Y-6}{\sqrt{4,2}} \leq \frac{6,5-6}{\sqrt{4,2}}\right) - P\left(\frac{Y-6}{\sqrt{4,2}} \leq \frac{2,5-6}{\sqrt{4,2}}\right)$

$= P(Z \leq 0,24) - P(-1,71) = 0,5948 - 0,0436 = 0,5512$

Binomial table $P(3 \leq X \leq 6) = P(X \leq 6) - P(X < 3)$

$= P(X \leq 6) - P(X \leq 2) = 0{,}6080 - 0{,}0355 = 0{,}5725$

(c) $P(X \geq 4) = 1 - P(X < 4) \approx 1 - P(Y \leq 3{,}5)$

$$= 1 - P\left(Z \leq \frac{3{,}5 - 6}{\sqrt{4{,}2}}\right) = 1 - P(Z \leq -1{,}22) = 1 - 0{,}1112 = 0{,}8888$$

Or $P(X \geq 4) = 1 - P(X < 4) = 1 - P(X \leq 3) \overset{(a)}{=} 1 - 0{,}1112 = 0{,}8888$

According to the binomial table:

$P(X \geq 4) = 1 - P(X \leq 3) = 1 - 0{,}1071 = 0{,}8929$

(d) $P(X = 4) = P(X \leq 4) - P(X \leq 3)$

$\approx P(Y \leq 4{,}5) - P(Y \leq 3{,}5)$

$$= P\left(Z \leq \frac{4{,}5 - 6}{\sqrt{4{,}2}}\right) - 0{,}1112 = P(Z \leq -0{,}71) - 0{,}1112$$

$= 0{,}2389 - 0{,}1112 = 0{,}1277$

Binomial table: $P(X = 4) = P(X \leq 4) - P(X \leq 3)$

$= 0{,}2375 - 0{,}1071$

$= 0{,}1304$

Ex. 73 $C \sim U(15, 21) \Rightarrow f_C(c) = \begin{cases} \frac{1}{21-15} = \frac{1}{6} & 15 < c < 21 \\ 0 & \text{otherwise} \end{cases}$ density for C

$F = \frac{9}{5} C + 32 \Rightarrow C = \frac{5}{9}(F - 32) = h(y)$

According to theorem 4.8.1 (see the equivalent formula on the slides)

$f_F(y) = f_C(h(y)) \, |h'(y)| = \frac{1}{6} \cdot \frac{5}{9} = \frac{5}{54}$

$15 < c < 21 \Rightarrow$

$\overset{\cdot 9/5}{\Leftrightarrow} 27 < \frac{9}{5} c < \frac{9}{5} \cdot 21$

$\overset{+32}{\Leftrightarrow} 59 < \frac{9}{5} c + 32 < \frac{9}{5} \cdot 21 + 32 = 69{,}8$

$\Rightarrow f_F(y) = \begin{cases} \frac{5}{54} & 59 < y < 69{,}8 \\ 0 & \text{otherwise} \end{cases}$