Matematisk Statistik och Disktret Matematik, MVE051/MSG810, VT20

Föreläsning 6

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Example

The Land of Oz is blessed by many things, but not by good weather. They never have two nice days in a row. If they have a nice day, they are just as likely to have snow as rain the next day. If they have snow or rain, they have an even chance of having the same the next day. If there is change from snow or rain, only half of the time is this a change to a nice day. We take as states the kinds of weather R, N, and S. From the above information we can represent the probabilities by a matrix

$$\begin{array}{cc} & \begin{array}{ccc} R & N & S \end{array} \\ \begin{array}{c} R \\ N \\ S \end{array} & \begin{pmatrix} 0.5 & 0.25 & 0.25 \\ 0.5 & 0 & 0.5 \\ 0.25 & 0.25 & 0.5 \end{pmatrix} \end{array}$$

This is an example of a Markov chain.

Markov Chain

A Markov chain consists of

- A set of states *(sv. tillstånd)* $\{s_1, \cdots, s_n\}$.
- Successive moves from a state to another where each move is called a **step.**
- The probability to move from state s_i to state s_j is denoted by p_{ij} and is equal to the conditional probability of being at state s_j given that the previous step was s_i. **This probability does not depend on the states before step** i**.**

Transition Matrix

- p_{ij} are called **transition probabilities**, *(sv. övergångssannolikhet)*, and the matrix $\mathbf{P} = (p_{ij})$ is called the **transition matrix** *(sv. övergångsmatrisen)*.
- The transition probabilities verify the following:
  1. For all i, j $p_{ij} \geq 0$.
  2. $\sum_{j=1}^{n} p_{ij} = 1$, i.e the sum of each row is equal to 1.
- The ij^{th} entry $p_{ij}^{(n)}$ of the matrix $\mathbf{P}^n$ gives the probability that the Markov chain, starting in states s_i, will be in state s_j after n steps.

Example

Consider the previous example. Suppose we want to compute the probability that, given that it is rainy today, the weather will be snowy in two days. Then,
$p_{13}^{(2)} = p_{11}p_{13} + p_{12}p_{23} + p_{13}p_{33} = 0.5(0.25) + 0.25(0.5) + 0.25(0.5) = 0.375.$
The matrix

$$\mathbf{P}^2 = \begin{array}{c c} & \begin{array}{ccc} R & N & S \end{array} \\ \begin{array}{c} R \\ N \\ S \end{array} & \begin{pmatrix} 0.4375 & 0.1875 & 0.375 \\ 0.375 & 0.25 & 0.375 \\ 0.375 & 0.1875 & 0.4375 \end{pmatrix} \end{array}$$

gives all the probabilities of going from state i to state j in two steps.

- A **probability vector** is a row vector that gives the probabilities of being at each state at a certain step.
- The probability vector which represents the initial state of a Markov chain is **starting vector** and is denoted by $\mathbf{u}^{(0)}$ or simply $\mathbf{u}$. The probability vector at step k is denoted by $\mathbf{u}^{(k)}$.
- If $\mathbf{u}_k$ is the probability vector at step k, then the vector

$$\mathbf{u}^{(k+1)} = \mathbf{u}^{(k)}\mathbf{P}$$

  is the probability vector at step $k+1$.
- If $\mathbf{u}$ is the starting vector of a Markov Chain, then the probability vector at step n is given by

$$\mathbf{u}^{(n)} = \mathbf{u}\mathbf{P}^n$$

Example

In the previous example, if the initial probability vector is $\mathbf{u} = (1/3, 2/3, 0)$, then the probability vector on day 2 will be

$$\mathbf{u}^{(2)} = \mathbf{u}\mathbf{P}^2 = \begin{pmatrix} 1/3 & 2/3 & 0 \end{pmatrix} \begin{pmatrix} 0.4375 & 0.1875 & 0.375 \\ 0.375 & 0.25 & 0.375 \\ 0.375 & 0.1875 & 0.4375 \end{pmatrix}$$

$$= \begin{pmatrix} 0.3958 & 0.2292 & 0.3750 \end{pmatrix}$$

This means that on day 2, there is a 39.58% chance of rain, 22.92% chance that the weather will be nice and 37.5% chance that it will snow.

Regular Markov chains

- A Markov chain is said to be **regular** if there exists n such that all the elements of the matrix $\mathbf{P}^n$ are nonzero.
  The Markov chain of the previous example is regular since

$$\mathbf{P}^2 = \begin{pmatrix} 0.4375 & 0.1875 & 0.375 \\ 0.375 & 0.25 & 0.375 \\ 0.375 & 0.1875 & 0.4375 \end{pmatrix}$$

  *(all the values are **strictly** positive)*
- If the Markov chain is regular then, $\mathbf{P}^n \rightarrow \mathbf{Q}$ where

$$\mathbf{Q} = \begin{pmatrix} q_1 & q_2 & \dots & q_n \\ q_1 & q_2 & \dots & q_n \\ \vdots & \vdots & \ddots & \vdots \\ q_1 & q_2 & \dots & q_n \end{pmatrix}$$

  q_j is the probability to be at state s_j on the long run.

Absorbing Markov Chains

- A state is said to be **absorbing** if it is impossible to leave it, that is $p_{ii} = 1$.
- A Markov chain is called **absorbing** if it has at least one absorbing state, and if from every state it is possible to go to an absorning state.
- In an absorbing Markov chain, as state that is not absorbing is called **transient**.
- Example:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{4} & \frac{1}{2} & 0 & \frac{1}{4} \end{pmatrix}$$

- The transition matrix of an absorbing Markov chain with r absorbing states and t transient states can be written as

$$\mathbf{P} = \begin{pmatrix} \mathbf{Q} & \mathbf{R} \\ 0 & \mathbf{I}_r \end{pmatrix}$$

  where $\mathbf{I}_r$ is the identity matrix, 0 is the zero matrix (all elements are zeros), $\mathbf{Q}$ is a $t \times t$-matrix and $\mathbf{R}$ is a $t \times r$ nonzero matrix. This form is called the **canonical form**.
- $\mathbf{P}^n = \begin{pmatrix} \mathbf{Q}^n & \star \\ 0 & \mathbf{I}_r \end{pmatrix}$ where $\star$ is a $t \times r$ matrix.
- $\mathbf{Q}^n$ gives the probability for being in each of the transient states after n steps for each possible transien starting state.

Suppose we have an absorbing Markov chain with transition matrix $\mathbf{P} = \begin{pmatrix} \mathbf{Q} & \mathbf{R} \\ 0 & \mathbf{I}_r \end{pmatrix}$

- The probability that the process will be absorbed is 1, i.e $\mathbf{Q}^n \to 0$ as $n \to \infty$.
- The matrix $\mathbf{I}_t - \mathbf{Q}$ is invertable. Let $\mathbf{N} = (\mathbf{I}_t - \mathbf{Q})^{-1}$. $\mathbf{N}$ is called the **fundamental matrix.**
- The entry n_{ij} of $\mathbf{N}$ gives the expected number of times that the process is in the transient state s_j if it started in the transient state s_i.

Time to absorption

- Let $\mathbf{c} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$. $\mathbf{Nc} = \begin{pmatrix} d_1 \\ \vdots \\ d_t \end{pmatrix}$ where d_i is the expected number of steps before the chain is absorbed, given that the chain starts in state s_i.
- $d_i = \sum_{j=1}^{t} n_{ij}$. *(sum of the entries of row i)*.

Absorption probabilities

- As $n \to \infty$ the matrix $\star$ tends to the matrix $\mathbf{NR}$, therefore,

$$\mathbf{P}^n \to \begin{pmatrix} 0 & \mathbf{NR} \\ 0 & \mathbf{I}_r \end{pmatrix}$$

- Let $\mathbf{B} = \mathbf{NR}$. The entry b_{ij} of $\mathbf{B}$ gives the probability that an absorbing chain will be absorbed in the absorbing state s_j if it starts in the transient state s_i.

Example

Let

$$\mathbf{P} = \begin{pmatrix} 0 & 1/2 & 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

be a transition matrix of an absorbing Markov chain. We have

$$\mathbf{Q} = \begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1/2 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{R} = \begin{pmatrix} 1/2 & 0 \\ 0 & 0 \\ 0 & 1/2 \end{pmatrix}$$

Example

$$\mathbf{I} - \mathbf{Q} = \begin{pmatrix} 1 & -1/2 & 0 \\ -1/2 & 1 & -1/2 \\ 0 & -1/2 & 1 \end{pmatrix}$$

Therefore,

$$\mathbf{N} = (\mathbf{I} - \mathbf{Q})^{-1} = \begin{pmatrix} 3/2 & 1 & 1/2 \\ 1 & 2 & 1 \\ 1/2 & 1 & 3/2 \end{pmatrix}$$

Example

$$N\mathbf{c} = \begin{pmatrix} 3/2 & 1 & 1/2 \\ 1 & 2 & 1 \\ 1/2 & 1 & 3/2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 3 \end{pmatrix}.$$

Hence, starting in states 1,2 and 3, the expected value to absorption are 3, 4, and 3, respectively.

$$NR = \begin{pmatrix} 3/2 & 1 & 1/2 \\ 1 & 2 & 1 \\ 1/2 & 1 & 3/2 \end{pmatrix} \begin{pmatrix} 1/2 & 0 \\ 0 & 0 \\ 0 & 1/2 \end{pmatrix} = \begin{pmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \\ 1/4 & 3/4 \end{pmatrix}$$

Starting in state 1, the probability to be absorbed in state 4 is $\frac{3}{4}$ and the probability to be absorbed in state 5 is $\frac{1}{4}$.