Chapter 5

Exercise 4 p.181

$$f_{XY}(x,y) = \frac{2}{n(n+1)} \qquad 1 \le y \le x \le n \qquad n \in \mathbb{N}_{>0}$$

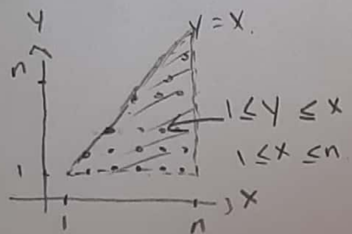
(a) (i) $f_{XY}(x,y) \ge 0 \quad \forall\, x,y$

(ii) $\displaystyle\sum_x \sum_y f_{XY}(x,y) = \sum_{x=1}^{n} \sum_{y=1}^{x} \frac{2}{n(n+1)}$ ← independent of y

$$= \sum_{x=1}^{n} \underbrace{\left(\frac{2}{n(n+1)} + \frac{2}{n(n+1)} + \dots + \frac{2}{n(n+1)} \right)}_{x \text{ times}}$$

$$= \sum_{x=1}^{n} \frac{2x}{n(n+1)} = \frac{2}{n(n+1)} \sum_{x=1}^{n} x = \frac{2}{n(n+1)} \cdot \frac{n(n+1)}{2} = 1.$$

(b)



$$* \; f_X(x) = \sum_y f_{XY}(x,y) = \sum_{1}^{x} \frac{2}{n(n+1)} = \frac{2x}{n(n+1)}$$

$$* \; f_Y(y) = \sum_x f_{XY}(x,y) \overset{(1)}{=} \sum_{y}^{n} \frac{2}{n(n+1)} = \frac{2}{n(n+1)} (n-y+1)$$

nb. of terms.

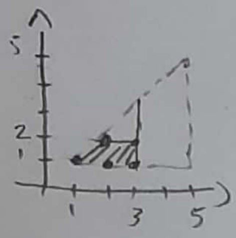
(1) the sum goes from y to n since $y \le x \le n$

(c) $\displaystyle f_X(x) \cdot f_Y(y) = \frac{2x}{n(n+1)} \cdot \frac{2}{n(n+1)} (n-y+1) = \frac{4x(n-y+1)}{n^2(n+1)^2}$

$\displaystyle f_X(1) = \frac{2}{n(n+1)} \qquad f_Y(1) = \frac{2n}{n(n+1)} = \frac{2}{n+1} \Rightarrow f_X(1) \cdot f_Y(1) = \frac{4}{n(n+1)^2}$

$\displaystyle f_{XY}(1,1) = \frac{2}{n(n+1)} \ne \frac{4}{n(n+1)^2} \Rightarrow$ not independent.

(d) $n=5$ $\quad X \leq 3$ and $Y \leq 2 = \{(1,1),(2,1),(2,2),(3,1),(3,2),(3,3)\}$



$$\Rightarrow P(X \leq 3 \ \& \ Y \leq 2) = f(1,1) + f(2,1) + f(2,2) + f(3,1) + f(3,2) + f(3,3)$$

$$= 6 \cdot \frac{2}{5(6)} = \frac{2}{5}$$

(since f is constant)

$$P(X \leq 3) = f_X(1) + f_X(2) + f_X(3) = \frac{2}{n(n+1)} + \frac{4}{n(n+1)} + \frac{6}{n(n+1)} = \frac{12}{5 \cdot 6} = \frac{2}{5}$$

$$P(Y \leq 2) = f_Y(1) + f_Y(2) = \frac{2}{5(6)} \cdot (5) + \frac{2}{5(6)}(5-1) = \frac{18}{5 \cdot 6} = \frac{3}{5}$$

Note that $P(X \leq 3) \cdot P(Y \leq 2) = \frac{2}{5} \cdot \frac{3}{5} = \frac{6}{25} \neq \frac{2}{5} = P(X \leq 3$ and $Y \leq 2)$

<u>Ex.10</u> $\quad f_{XY}(x,y) = \dfrac{x^3 y^3}{16} \qquad 0 \leq x \leq 2, \ 0 \leq y \leq 2$

(a) $$f_X(x) = \int_0^2 f_{XY}(x,y)\,dy = \int_0^2 \frac{x^3 y^3}{16}\,dy = \frac{x^3}{16} \cdot \frac{y^4}{4}\Big]_{y=0}^{y=2}$$

$$= \frac{x^3}{16}\left(\frac{16}{4}\right) = \frac{x^3}{4}.$$

$$f_Y(y) = \int_0^2 f_{XY}(x,y)\,dx = \int_0^2 \frac{x^3 y^3}{16}\,dx = \frac{x^4}{4} \frac{y^3}{16}\Big]_{x=0}^{x=2}$$

$$= \frac{y^3}{16} \cdot \frac{16}{4} = \frac{y^3}{4}.$$

(b) $$f_X(x) \cdot f_Y(y) = \frac{x^3}{4} \cdot \frac{y^3}{4} = \frac{x^3 y^3}{16} = f_{XY}(x,y)$$

Yes they are independent

(c) $$P(X \leq 1) = \int_0^1 f_X(x)\,dx = \int_0^1 \frac{x^3}{4}\,dx = \frac{x^4}{16}\Big]_0^1 = \frac{1}{16}$$

(d) Since they are independent $\Rightarrow$ $P(X \leq 1) = \frac{1}{16}$ independently of the value of y.