Moment generating functions

Let X be a random variable. Recall that the moments of X are defined by :

(k^{th}- moment) : $E[X^k] = \sum_{\text{all } x} x_i^k \; p(X = x_i)$ (if X is discrete)

$E[X^k] = \int_{\mathbb{R}} x^k f(x)\, dx$ (if X is continuous)

The moment generating function (mgf) of X is defined by

$$m_X(t) = E[e^{tX}].$$

Theorem : the k-th derivative of $m_X(t)$ evaluated at 0 is the k-th moment of X, i.e. $m_X^{(k)}(0) = E[X^k]$.

Pf: Mc-Laurin series of a function $f(t)$:

$$f(t) = \sum_{k \geq 0} f^{(k)}(0) \frac{t^k}{k!}$$ where $f^{(k)}(t)$ is the k-th derivative of f

$$= f(0) + f'(0) \cdot t + f''(0) \frac{t^2}{2!} + f^{(3)}(0) \frac{t^3}{3!} + \cdots$$

the Mc. Laurin series of e^z is :

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots$$

$$\Rightarrow e^{tX} = 1 + tX + \frac{(tX)^2}{2!} + \frac{(tX)^3}{3!} + \frac{(tX)^4}{4!} + \cdots$$

$$\Rightarrow m_X(t) = E\left[1 + tX + \frac{(tX)^2}{2!} + \frac{(tX)^3}{3!} + \frac{(tX)^4}{4!} + \cdots\right]$$

$$= 1 + t\,E[X] + \frac{t^2}{2!} E[X^2] + \frac{t^3}{3!} E[X^3] + \cdots$$

On the other hand, the Mc-Laurin series of $m_X(t)$ is:

$$m_X(t) = m_X(0) + m_X'(0)\,t + m_X''(0)\frac{t^2}{2!} + m_X^{(3)}(0)\frac{t^3}{3!} + \cdots$$

By identification we get $m_X^{(k)}(0) = E[X^k]$.

<u>Ex.</u> Find the moment generating function for the Binomial distribution and find $E[X]$ and $V[X]$. $(Bin(n,p))$

<u>Solution</u>: the density function of a binomial distribution is $f(k) = \binom{n}{k} p^k (1-p)^{n-k}$.

$$\begin{aligned} m_X(t) = E[e^{tX}] &= \sum_{k=0}^{n} e^{tk} f(k) \\ &= \sum_{k=0}^{n} (e^t)^k \binom{n}{k} p^k (1-p)^{n-k} \\ &= \sum_{k=0}^{n} \binom{n}{k} (\underbrace{pe^t}_{a})^k (\underbrace{1-p}_{b})^{n-k} \\ &= (a+b)^n = (pe^t + 1-p)^n \end{aligned}$$

$$m_X(t) = (pe^t + 1-p)^n$$

$$E[X] = m_X'(0).$$

$$m_X'(t) = n(pe^t + 1-p)^{n-1} \cdot pe^t \Rightarrow m_X'(0) = n(pe^0 + 1-p)\cdot pe^0$$
$$= n(\underbrace{p+1-p}_{1}) \cdot p = np$$

$$\Rightarrow E[X] = np$$

$$m_X''(t) = n(n-1)(pe^t + 1-p)^{n-2} p^2 e^{2t} + n(pe^t + 1-p)^{n-1} p e^t$$

$$\Rightarrow m_X''(0) = n(n-1)p^2 + np = n^2p^2 - np^2 + np = E[X^2]$$

$$\Rightarrow Var\, X = E[X^2] - E[X]^2 = n^2p^2 - np^2 + np - n^2p^2 = np - np^2 = np(1-p).$$

Thm 1 Let X and Y be two random variables with mgf $m_X(t)$ and $m_Y(t)$ respectively. If $m_X(t) = m_Y(t)$ for all t in some open interval around 0, then X and Y have the same distribution.

Thm 2 Let X be a random variable with mgf $m_X(t)$, and let $Y = \alpha + \beta X$ (α, β constants). Then the mgf of Y is

$$m_Y(t) = e^{\alpha t} m_X(\beta t).$$

Corollary if $X \sim N(\mu, \sigma^2) \Rightarrow \frac{X-\mu}{\sigma} \sim N(0,1)$.

Pf: $m_X(t) = e^{\mu t + \sigma^2 t^2/2}$. Let $Z \sim N(0,1) \Rightarrow m_Z(t) = e^{\frac{t^2}{2}}$.

Let $Y = \frac{X-\mu}{\sigma} = \underbrace{-\frac{\mu}{\sigma}}_{\alpha} + \underbrace{\frac{1}{\sigma}}_{\beta} X$

$$\stackrel{\text{Thm 2}}{\Rightarrow} m_Y(t) = e^{\alpha t} m_X(\beta t) = e^{-\frac{\mu t}{\sigma}} e^{\mu \cdot \frac{1}{\sigma} t + \sigma^2 \cdot \frac{1}{\sigma^2} t^2/2}$$

$$= \underbrace{e^{-\frac{\mu t}{\sigma}} \cdot e^{\frac{\mu t}{\sigma}}}_{=1} \cdot e^{\frac{t^2}{2}}$$

$$= e^{t^2/2}$$

$\Rightarrow m_Y(t) = m_Z(t)$ for all $t \Rightarrow$ by Thm 1, $Y \sim N(0,1)$.

Thm 3 Let X_1 and X_2 be independent random variables with mgf $m_{X_1}(t)$ and $m_{X_2}(t)$ respectively. Let $Y = X_1 + X_2$. The mgf of is then given by

$$m_Y(t) = m_{X_1}(t) \cdot m_{X_2}(t).$$

Corollary 1: If $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$ are independent

then $X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$

Pf: $$m_{X_1+X_2}(t) = m_{X_1}(t) \cdot m_{X_2}(t)$$
$$= e^{\mu_1 t + \sigma_1^2 t^2/2} \cdot e^{\mu_2 t + \sigma_2^2 t^2/2}$$
$$= e^{(\mu_1+\mu_2)t + (\sigma_1^2+\sigma_2^2)\frac{t^2}{2}}$$
$$\Rightarrow X_1 + X_2 \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2).$$

Corollary 2: If $X_1 \sim \text{Poiss}(\lambda_1)$ and $X_2 \sim \text{Poiss}(\lambda_2)$ are indep.

then $X_1 + X_2 \sim \text{Poiss}(\lambda_1 + \lambda_2)$

Pf: $m_{X_1}(t) = e^{\lambda_1(e^t - 1)}$, $m_{X_2}(t) = e^{\lambda_2(e^t-1)}$

$$m_{X_1+X_2}(t) = m_{X_1}(t) \cdot m_{X_2}(t) = e^{\lambda_1(e^t-1)} \cdot e^{\lambda_2(e^t-1)}$$
$$= e^{(\lambda_1+\lambda_2)(e^t-1)}$$
$$\Rightarrow X_1 + X_2 \sim \text{Poiss}(\lambda_1+\lambda_2)$$