Financial Time Series – Trend and seasonality

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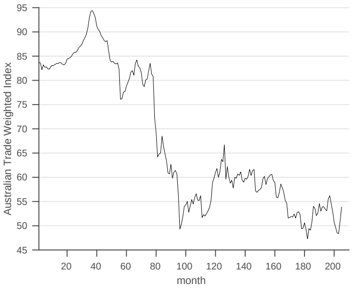
Classical decomposition model

- Data is a time series $X = (X_t, t \in \mathbb{Z})$ with

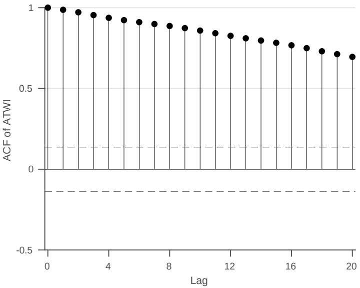
$$X_t = m_t + s_t + Y_t, t \in \mathbb{Z}$$

- Assume we have observed $X^n = (X_1, \ldots, X_n)$
- the *trend component* $m : \mathbb{Z} \to \mathbb{R}$ is a slowly changing function
- the *seasonal component* $s : \mathbb{Z} \to \mathbb{R}$ is a function with period d, i.e., $s_{t+d} = s_t$
- $Y = (Y_t, t \in \mathbb{Z})$ is a zero mean stationary time series

Let $x = (x_t)_{t=1}^{205}$ be the monthly observations of the Australian Trade Weighted Index (ATWI).

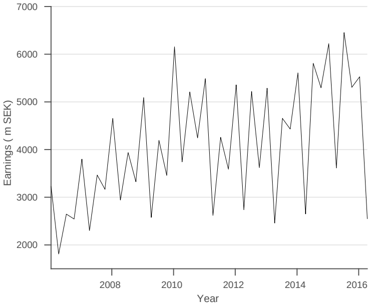


(a) Monthly observations of the ATWI.

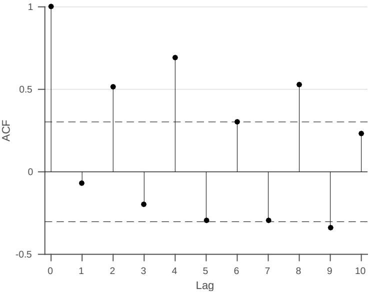


(b) ACF of raw ATWI data.

Let the set $(x_t)_{t=1}^{42}$ be the quarterly earnings of H&M.



(a) Quarterly earnings of H&M from January 2006 through April 2016.



(b) Sample autocorrelation function for the H&M data.

Estimation of trend by a moving average filter

- trend only: $s_t = 0 \, \forall t \in \mathbb{Z}$
- $q \in \mathbb{N}$ with $2q < n$ fixed
- *two-sided moving average*: $W_t := (2q+1)^{-1} \sum_{j=-q}^{q} X_{t-j}$ for all $t = q+1, \ldots, n-q$
- $W_t = (2q+1)^{-1} \sum_{j=-q}^{q} m_{t-j} + (2q+1)^{-1} \sum_{j=-q}^{q} Y_{t-j} \approx m_t$
- $\hat{m}_t := (2q+1)^{-1} \sum_{j=-q}^{q} X_{t-j}$ for $q+1 \leq t \leq n-q$

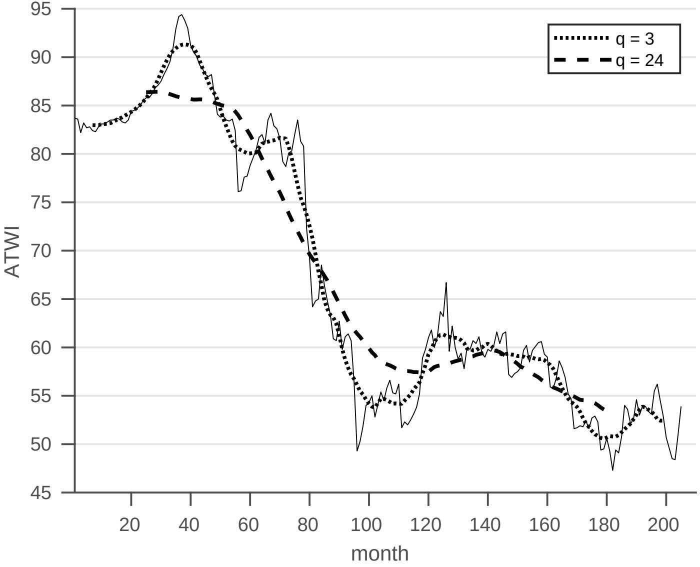


Figure: Moving average trends.

Estimation of trend by exponential smoothing

- trend only: $s_t = 0 \, \forall t \in \mathbb{Z}$
- for $\alpha \in [0,1]$ define *one-sided moving averages* $(\hat{m}_t, t = 1, \dots, n)$ by

$$\hat{m}_t := \alpha X_t + (1-\alpha)\hat{m}_{t-1}$$

  for $t = 2, \dots, n$ and

$$\hat{m}_1 := X_1.$$

- for $t \geq 2$:

$$\hat{m}_t = \sum_{j=0}^{t-2} \alpha(1-\alpha)^j X_{t-j} + (1-\alpha)^{t-1} X_1,$$

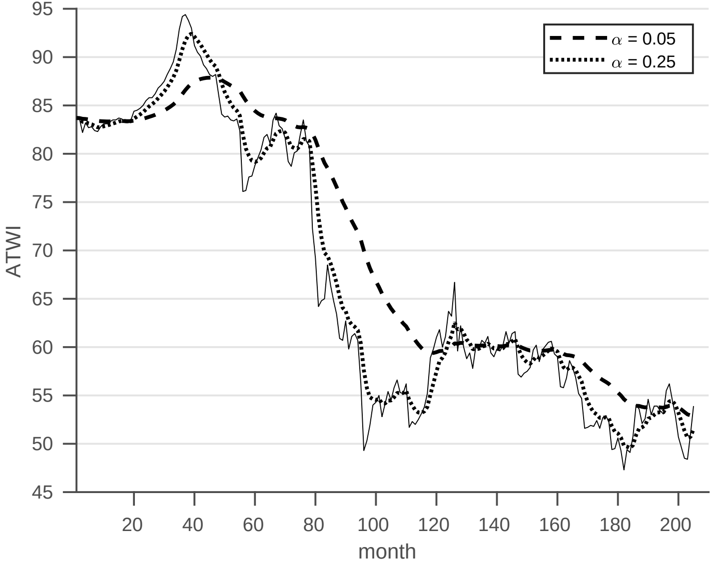


Figure: Exponential smoothing trends.

Estimation of trend and seasonality by linear least squares

Let m be given by $m_t := \sum_{j=0}^{q} a_j t^j$ for $t \in \mathbb{Z}, q \in \mathbb{N}$. Let s with known period d be given by $s_t := \sum_{k=0}^{p} b_k \cos(2\pi\lambda_k t/d) + c_k \sin(2\pi\lambda_k t/d)$ for $t \in \mathbb{Z}, p \in \mathbb{N}$ and some *known* coefficients $(\lambda_j)_{j=1}^{p} \subset \mathbb{N}$.

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$(a_j)_{j=1}^{q}$, $(b_j)_{j=1}^{p}$ and $(c_j)_{j=1}^{p}$ are obtained by

$$\underset{a_j, b_k, c_k}{\arg\min} \sum_{t=1}^{n} (X_t - m_t - s_t)^2,$$

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The design matrix:

$$C := \begin{pmatrix} 1 & \cdots & 1^q & \cos(2\pi\lambda_1/d) & \cdots & \cos(2\pi\lambda_p/d) & \sin(2\pi\lambda_1/d) & \cdots & \sin(2\pi\lambda_p/d) \\ & & & & \vdots & & & & \\ 1 & \cdots & t^q & \cos(2\pi\lambda_1 t/d) & \cdots & \cos(2\pi\lambda_p t/d) & \sin(2\pi\lambda_1 t/d) & \cdots & \sin(2\pi\lambda_p t/d) \\ & & & & \vdots & & & & \\ 1 & \cdots & n^q & \cos(2\pi\lambda_1 n/d) & \cdots & \cos(2\pi\lambda_p n/d) & \sin(2\pi\lambda_1 n/d) & \cdots & \sin(2\pi\lambda_p n/d) \end{pmatrix}$$

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If $C'C$ is non-singular, then the minimum is given by $(\hat{a}_0, \hat{a}_1, \ldots, \hat{a}_q, \hat{b}_1, \ldots, \hat{b}_p, \hat{c}_1, \ldots, \hat{c}_p)' = (C'C)^{-1}C'X$, where $X = (X_1, X_2, \ldots, X_n)'$.

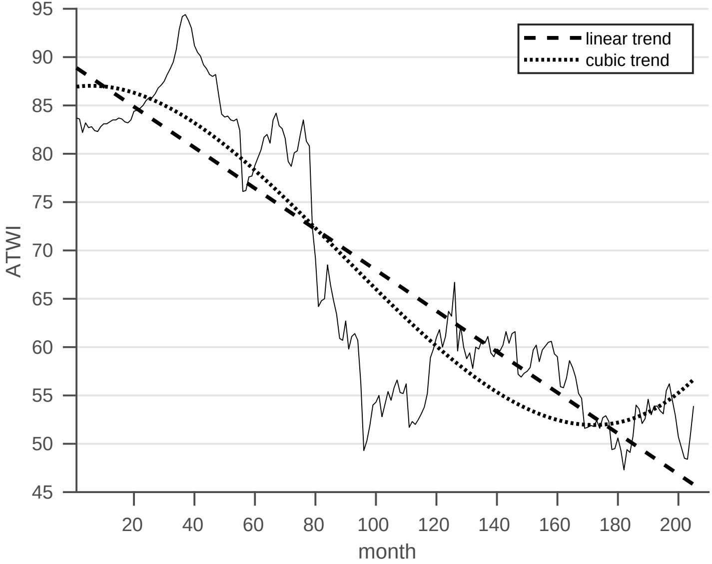


Figure: Trend estimation by linear least squares.

Estimation of trend and seasonality by moving averages

- Assume $n/d \in \mathbb{N}$
- For even $d = 2q$ and $q < t \leq n - q$

$$\hat{m}_t := d^{-1}(2^{-1}x_{t-q} + x_{t-q+1} + \cdots + x_{t+q-1} + 2^{-1}x_{t+q})$$

- For an odd period $d = 2q + 1$ and $q < t \leq n - q$

$$\hat{m}_t := d^{-1} \sum_{j=-q}^{q} x_{t-j}.$$

- for $k = 1, \ldots, d$ and $q < k + jd \leq n - q$

$$w_k := |\{j \in \mathbb{N}_0, q < k+jd \leq n-q\}|^{-1} \sum_{q<k+jd\leq n-q} (x_{k+jd} - \hat{m}_{k+jd}),$$

- $\hat{s}_k := w_k - d^{-1} \sum_{j=1}^{d} w_j$, extend it and reestimate trend on $(x_t - \hat{s}_t, t = 1, \ldots, n)$

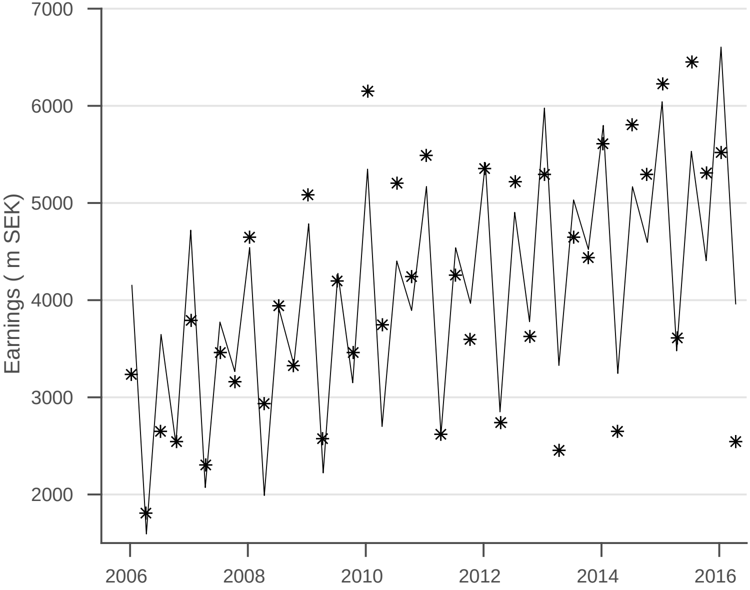
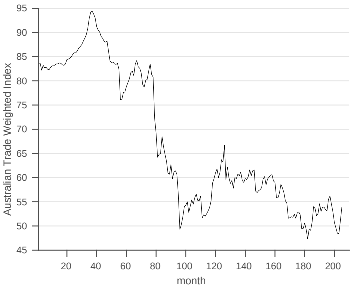


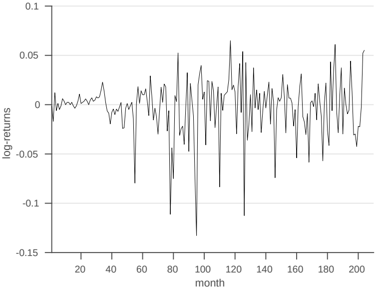
Figure: The H&M data (stars) with an estimated linear trend and seasonal component with period 4, $\hat{m}_t + \hat{s}_t$ (line).

Elimination of trend and seasonality by differencing

- Recall: $X_t = m_t + s_t + Y_t, t \in \mathbb{Z}$
- $BX_t := X_{t-1}$ and $B^j X_t := B^{j-1} B X_t = B^{j-1} X_{t-1} = \cdots = X_{t-j}$
- $\nabla X_t := X_t - X_{t-1} = (1-B) X_t$
- Trend:
    - If $m_t := \sum_{j=0}^{q} a_j t^j$ then $\nabla^q m_t = q!\, a_q$
    - For $s = 0$, $\nabla^q X_t = q!\, a_q + \nabla^q Y_t$
- $\nabla_d X_t := X_t - X_{t-d} = (1 - B^d) X_t$
- Seasonality:
    - $\nabla_d X_t = m_t - m_{t-d} + s_t - s_{t-d} + Y_t - Y_{t-d} = \nabla_d m_t + \nabla_d Y_t$



(a) Monthly observations of the ATWI.



(b) Log-returns of the ATWI.

Forecasting a differenced time series

- Suppose $\nabla^N \nabla_d^M X_t = \tilde{Y}_t$ where $\tilde{Y} = (\tilde{Y}_t, t \in \mathbb{Z})$ is stationary so

$$\tilde{Y}_t = \nabla^N \nabla_d^M X_t = (1-B)^N (1-B^d)^M X_t$$
$$= \sum_{k=0}^{N+Md} b_k B^k X_t = \sum_{k=0}^{N+Md} b_k X_{t-k}$$

- $X_{n+h} = \tilde{Y}_{n+h} - \sum_{k=1}^{N+Md} b_k X_{n+h-k}$
- Observations $X^{n+N+Md} := (X_{-N-Md+1}, \ldots, X_n)$ and $\tilde{Y}^n := (\tilde{Y}_1, \ldots, \tilde{Y}_n)$
- If $X_{-N-Md+1}, \ldots, X_0$ are uncorrelated with $\tilde{Y}^n$ then

$$b_{n+h}^{\ell}(X^{n+N+Md}) = b_{n+h}^{\ell}(\tilde{Y}^n) - \sum_{k=1}^{N+Md} b_k b_{n+h-k}^{\ell}(X^{n+N+Md}).$$

- $b_{n+h-k}^{\ell}(X^{n+N+Md}) = X_{n+h-k}$ if $h \le k$