Financial Time Series – (S)ARIMA processes and unit root tests

Andreas Petersson

TMS088/MSA410 – May 2020

CHALMERS
UNIVERSITY OF TECHNOLOGY

UNIVERSITY OF GOTHENBURG

Mathematical Sciences, Chalmers University of Technology & University of Gothenburg, Sweden

(S)ARIMA processes

We use $\nabla = 1 - B$ one or more times to delete m_t and $\nabla_s = 1 - B^s$ to delete s_t from

$$X_t = m_t + s_t + Y_t.$$

(S)ARIMA processes

We use $\nabla = 1 - B$ one or more times to delete m_t and $\nabla_s = 1 - B^s$ to delete s_t from

$$X_t = m_t + s_t + Y_t.$$

Definition

Let X be a stochastic process and let $d, D \in \mathbb{N}_0$. Then X is a $\mathrm{SARIMA}(p, d, q) \times (P, D, Q)_s$ *process* if the process Y defined by $Y_t := \nabla^d \nabla_s X_t$ is a causal ARMA process defined by

$$\phi(B)\Phi(B^s)Y_t = \theta(B)\Theta(B^s)Z_t,$$

where $\phi(z) = 1 - \phi_1 z - \ldots - \phi_p z^p$, $\Phi(z) = 1 - \Phi_1 z - \ldots - \Phi_P z^P$, $\theta(z) = 1 + \theta_1 z^1 + \ldots + \theta_q z^q$ and $\Theta(z) = 1 + \Theta_1 z^1 + \ldots + \Theta_Q z^Q$ for $p, q, P, Q \in \mathbb{N}_0$.

(S)ARIMA processes

We use $\nabla = 1 - B$ one or more times to delete m_t and $\nabla_s = 1 - B^s$ to delete s_t from

$$X_t = m_t + s_t + Y_t.$$

Definition

Let X be a stochastic process and let $d, D \in \mathbb{N}_0$. Then X is a $\mathrm{SARIMA}(p, d, q) \times (P, D, Q)_s$ *process* if the process Y defined by $Y_t := \nabla^d \nabla_s X_t$ is a causal ARMA process defined by

$$\phi(B)\Phi(B^s)Y_t = \theta(B)\Theta(B^s)Z_t,$$

where $\phi(z) = 1 - \phi_1 z - \ldots - \phi_p z^p$, $\Phi(z) = 1 - \Phi_1 z - \ldots - \Phi_P z^P$, $\theta(z) = 1 + \theta_1 z^1 + \ldots + \theta_q z^q$ and $\Theta(z) = 1 + \Theta_1 z^1 + \ldots + \Theta_Q z^Q$ for $p, q, P, Q \in \mathbb{N}_0$.

Y in the definition is a causal $\mathrm{ARMA}(p + sP, q + sQ)$ process.

Special (important) case: ARIMA processes

Definition

Let X be a stochastic process and d a nonnegative integer. Then X is an $\mathrm{ARIMA}(p, d, q)$ *process* if the process Y defined by $Y_t := (1 - B)^d X_t$ is a causal $\mathrm{ARMA}(p, q)$ process.

Special (important) case: ARIMA processes

Definition

Let X be a stochastic process and d a nonnegative integer. Then X is an $\mathrm{ARIMA}(p, d, q)$ *process* if the process Y defined by $Y_t := (1 - B)^d X_t$ is a causal $\mathrm{ARMA}(p, q)$ process.

- $\phi^*(B)X_t := \phi(B)(1 - B)^d X_t = \theta(B)Z_t$ where $Z \sim \mathrm{WN}(0, \sigma^2)$ and ϕ and θ are polynomials of degree p and q
- $\phi(z) \neq 0$ for $|z| \leq 1$ while ϕ^* has a zero of degree d at $z = 1$ $\rightarrow$ a *unit root*

Unit root tests for ARIMA processes

- When to difference a time series?
- Apply $(1-B)$ until sample ACF is no longer "slowly decreasing".
- More systematic: *Unit root tests* (Dickey and Fuller). Restrict to $\mathrm{ARIMA}(p,d,0)$ vs. $\mathrm{AR}(p)$.
- $H_0 = \{$there is a unit root$\}$, $H_1 = \{$there is no unit root$\}$
- To start: let $(X_1,\ldots,X_n)$ be observations of

$$X_t - \mu = \phi_1(X_{t-1} - \mu) + Z_t,$$

  where $Z \sim \mathrm{WN}(0,\sigma^2)$
- When $\phi_1 < 1$, this gives an $\mathrm{AR}(1)$ process with mean μ so set $H_0 = \{\phi_1 = 1\}$ and $H_1 = \{\phi_1 < 1\}$
- Rewrite to

$$\nabla X_t = X_t - X_{t-1} = \phi_0^* + \phi_1^* X_{t-1} + Z_t,$$

  where $\phi_0^* := \mu(1-\phi_1)$, and $\phi_1^* := \phi_1 - 1$

Unit root tests for ARIMA processes

- Let $\hat{\phi}_1^*$ be OLS estimator of ϕ_1^* by regressing ∇X_t on 1 and X_{t-1}, i.e.,

$$(\hat{\phi}_0^*, \hat{\phi}_1^*) = \underset{(\phi_0^*, \phi_1^*)}{\arg\min} \sum_{t=2}^{n} (\nabla X_t - \phi_0^* - \phi_1^* X_{t-1})^2.$$

Unit root tests for ARIMA processes

- Let $\hat{\phi}_1^*$ be OLS estimator of ϕ_1^* by regressing ∇X_t on 1 and X_{t-1}, i.e.,

$$(\hat{\phi}_0^*, \hat{\phi}_1^*) = \underset{(\phi_0^*, \phi_1^*)}{\arg\min} \sum_{t=2}^{n} (\nabla X_t - \phi_0^* - \phi_1^* X_{t-1})^2.$$

- Then

$$\widehat{\mathrm{SE}}(\hat{\phi}_1^*) = S \left(\sum_{t=2}^{n} (X_{t-1} - \bar{X}_{n-1})^2 \right)^{-1/2},$$

where

$$S^2 := (n-3)^{-1} \sum_{t=2}^{n} (\nabla X_t - \hat{\phi}_0^* - \hat{\phi}_1^* X_{t-1})^2$$

and $\bar{X}_{n-1} = \sum_{i=1}^{n-1} X_i$.

Unit root tests for ARIMA processes

- Let $\hat{\phi}_1^*$ be OLS estimator of ϕ_1^* by regressing ∇X_t on 1 and X_{t-1}, i.e.,

$$(\hat{\phi}_0^*, \hat{\phi}_1^*) = \underset{(\phi_0^*, \phi_1^*)}{\arg\min} \sum_{t=2}^{n} (\nabla X_t - \phi_0^* - \phi_1^* X_{t-1})^2.$$

- Then

$$\widehat{\mathrm{SE}}(\hat{\phi}_1^*) = S \left(\sum_{t=2}^{n} (X_{t-1} - \bar{X}_{n-1})^2 \right)^{-1/2},$$

  where

$$S^2 := (n-3)^{-1} \sum_{t=2}^{n} (\nabla X_t - \hat{\phi}_0^* - \hat{\phi}_1^* X_{t-1})^2$$

  and $\bar{X}_{n-1} = \sum_{i=1}^{n-1} X_i$.
- Dickey and Fuller derived the limit distribution for $n \to +\infty$ under $H_0 : \phi_1 = 1$:

$$\hat{\tau}_\mu := \frac{\hat{\phi}_1^*}{\widehat{\mathrm{SE}}(\hat{\phi}_1^*)}$$

- The 0.01, 0.05, and 0.10 quantiles are -3.43, -2.86, and -2.57.

Unit root tests for ARIMA processes

- Extend to $\mathrm{AR}(p)$ case:

$$X_t - \mu = \sum_{j=1}^{p} \phi_j (X_{t-j} - \mu) + Z_t,$$

- Rewrite to

$$\nabla X_t = \phi_0^* + \phi_1^* X_{t-1} + \sum_{j=2}^{p} \phi_j^* \nabla X_{t+1-j} + Z_t,$$

where

$$\phi_0^* := \mu \left(1 - \sum_{i=1}^{p} \phi_i\right),$$

$$\phi_1^* := \sum_{i=1}^{p} \phi_i - 1,$$

$$\phi_j^* := -\sum_{i=j}^{p} \phi_i$$

for $j = 2, \ldots, p$.

Unit root tests for ARIMA processes

Method (Dickey–Fuller test)

Estimate ϕ_1^* as the coefficient of X_{t-1} in the OLS regression of ∇X_t onto 1, X_{t-1}, ∇X_{t-1}, $\ldots$, ∇X_{t-1+p}. For large n the t-ratio

$$\hat{\tau}_\mu := \frac{\hat{\phi}_1^*}{\widehat{\mathrm{SE}}(\hat{\phi}_1^*)},$$

where $\widehat{\mathrm{SE}}(\hat{\phi}_1^*)$ is the estimated standard error of $\hat{\phi}_1^*$, has the same limit distribution as for the $\mathrm{AR}(1)$ process with 0.01, 0.05, and 0.10 quantiles -3.43, -2.86, and -2.57, respectively. Test the null hypothesis $H_0 : \phi_1^* = 0$ and reject according to the chosen level. If a root is detected, repeat the procedure with the differenced process until rejection to determine d.