Financial Time Series – GARCH processes

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Modeling stock prices/returns

$$dP_t = \mu P_t \, dt + \sigma P_t \, dB_t, P(0) = P_0$$

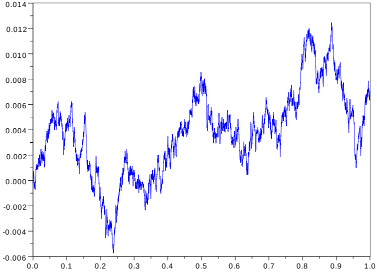

Figure: Path of a Brownian motion.

Modeling stock prices/returns

$$dP_t = \mu P_t\, dt + \sigma P_t\, dB_t, P(0) = P_0$$

$$P_t = P_0 \exp((\mu - 2^{-1}\sigma^2)t + \sigma B_t).$$

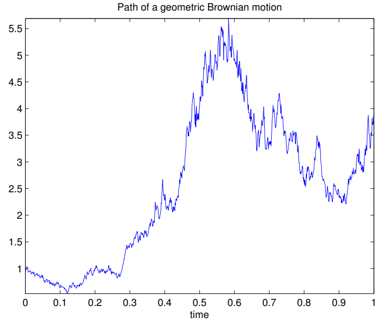

Figure: Path of a geometric Brownian motion.

Modeling stock prices/returns

- Log-returns $x = (X_t, t \in Z)$ fulfills

$$X_t = \log(P_{t+1}) - \log(P_t) = \mu - \frac{\sigma^2}{2} + \sigma(B_{t+1} - B_t).$$

- Assume mean has been removed (or that $\mu = \frac{\sigma^2}{2}$)
- $X_t = \sigma Z_t$, where $Z = (Z_t, t \in \mathbb{Z}) \sim \mathrm{IID}\,\mathcal{N}(0,1)$

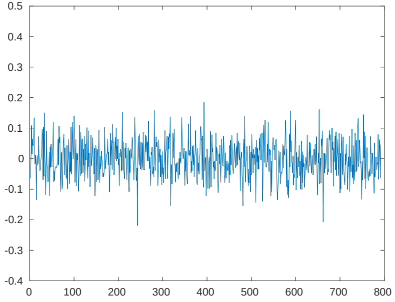

Figure: Observations from the model $X_t = \sigma Z_t$.

Modeling stock prices/returns

Problems when X is meant to be real-world data!

1. X_t and X_s seem only uncorrelated, not independent.
2. σ does not appear to be constant in time.
3. X_t has fait tails, i.e., the *kurtosis* $\frac{\mathbb{E}(X_t^4)}{\mathbb{E}(X_t^2)^2} > 3$.

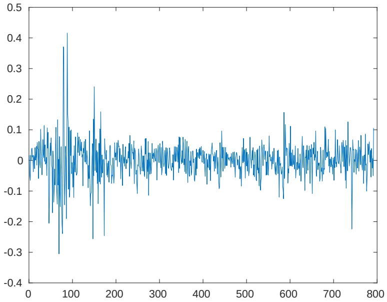

Figure: Returns of the S&P500 index.

Random variance models

Definition

A stochastic process $X = (X_t, t \in \mathbb{Z})$ is said to follow a *random variance model* if

$$X_t = \sigma_t Z_t \tag{1}$$

for all $t \in \mathbb{Z}$, where $Z = (Z_t, t \in \mathbb{Z})$ is $\mathrm{IID}(0,1)$ and $\sigma = (\sigma_t, t \in \mathbb{Z})$ is an unspecified stochastic process called the *volatility*. If X_t can be written as a deterministic function of $(Z_s, s \leq t)$ for all $t \in \mathbb{Z}$, then X is said to be *causal*.

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The *realized volatility*:

$$\hat{\sigma}_t^2 := (\tau - 1)^{-1} \sum_{j=t-\tau}^{t} (x_j - \bar{x}_t)^2$$

for observed data $(x_1, \ldots, x_n)$, fixed $\tau < n$, and $\tau < t \leq n$, where

$$\bar{x}_t := \tau^{-1} \sum_{j=t-\tau}^{t} x_j.$$

Conditional heteroscedasticity and ARCH models

Conditional heteroscedasticity:

$$\mathrm{Var}(X_t|X_{t-1}, X_{t-2}, \ldots) = \mathbb{E}((X_t - \mathbb{E}(X_t))^2|X_{t-1}, X_{t-2}, \ldots) \neq \text{ constant}$$

Conditional heteroscedasticity and ARCH models

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Autoregressive conditional heteroscedasticity:

Definition

A stochastic process $X = (X_t, t \in \mathbb{Z})$ is called an $\mathrm{ARCH}(p)$ *process* if it is stationary and if it satisfies the ARCH equations

$$X_t = \sigma_t Z_t,$$

where $Z \sim \mathrm{IID}(0, 1)$,

$$\sigma_t^2 = \alpha_0 + \sum_{j=1}^{p} \alpha_j X_{t-j}^2,$$

$\alpha_0 > 0$, $\alpha_j \geq 0$ for $j = 1, \ldots, p$.

$$\alpha(z) := \alpha_1 z + \cdots + \alpha_p z^p.$$

Conditional heteroscedasticity and ARCH models

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For causal X:

$$\mathbb{E}(X_t^2) = \mathbb{E}(\sigma_t^2) = \alpha_0 + \sum_{j=1}^{p} \alpha_j \, \mathbb{E}(X_t^2) = \alpha_0 + \alpha(1) \, \mathbb{E}(X_t^2).$$

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$$\mathbb{E}(X_t^2) = \frac{\alpha_0}{1 - \alpha(1)}.$$

GARCH models

Definition

A stochastic process $X = (X_t, t \in \mathbb{Z})$ is called a $\mathrm{GARCH}(p,q)$ *process* if it is a stationary solution to the GARCH equations

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$$\sigma_t^2 = \alpha_0 + \sum_{j=1}^{p} \alpha_j X_{t-j}^2 + \sum_{i=1}^{q} \beta_i \sigma_{t-i}^2,$$

with $\alpha_0 > 0$, $\alpha_j \geq 0$ for $j = 1, \ldots, p$, $\beta_i \geq 0$ for $i = 1, \ldots, q$.

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with $\alpha_0 > 0$, $\alpha_j \geq 0$ for $j = 1, \dots, p$, $\beta_i \geq 0$ for $i = 1, \dots, q$.

Usually $Z_t \sim \mathcal{N}(0, 1)$ or $\sqrt{\nu/(\nu - 2)} Z_t \sim t_\nu$ for all $t \in \mathbb{Z}$. The factor $\sqrt{\nu/(\nu - 2)}$ yields $\mathrm{Var}(Z_t) = 1$, Z_t follows a *generalized* or *non-standardized* t-distribution.

GARCH(1,1)

Proposition

If $\alpha_1 + \beta_1 < 1$*, there exists a stationary solution* $X = (X_t, t \in \mathbb{Z})$ *to the* $\mathrm{GARCH}(1,1)$ *equations that is given by the equation*

$$X_t = \sigma_t Z_t,$$

where $Z \sim \mathrm{IID}(0,1)$ *and*

$$\sigma_t^2 = \alpha_0 \left(1 + \sum_{i=1}^{\infty} (\alpha_1 Z_{t-1}^2 + \beta_1)(\alpha_1 Z_{t-2}^2 + \beta_1) \cdots (\alpha_1 Z_{t-i}^2 + \beta_1) \right). \tag{2}$$

It is unique (P*-a.s.), strictly stationary and causal. Conversely, if* $\alpha_1 + \beta_1 \geq 1$*, then there no non-zero stationary solution to the* $\mathrm{GARCH}(1,1)$ *equations for which* σ_t *can be written as a deterministic function of* $(Z_s, s < t)$ *for all* $t \in \mathbb{Z}$*.*

GARCH(1,1)

Proof.

$$\mathbb{E}(|\sigma_t^2|) = \mathbb{E}(\sigma_t^2) = \alpha_0\Big(1 + \sum_{j=1}^{\infty} \mathbb{E}\left((\alpha_1 Z_{t-1}^2 + \beta_1)(\alpha_1 Z_{t-2}^2 + \beta_1) \cdots (\alpha_1 Z_{t-j}^2 + \beta_1)\right)\Big)$$

$\square$

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$$= \alpha_0 \left(1 + \sum_{j=1}^{\infty} (\alpha_1 + \beta_1)^j\right) = \alpha_0 \frac{1}{1 - \alpha_1 - \beta_1} < \infty$$

$\square$

GARCH(1,1)

Proof.

$$\alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2 = \alpha_0 + \alpha_1 \sigma_{t-1}^2 Z_{t-1}^2 + \beta_1 \sigma_{t-1}^2 = \alpha_0 + (\alpha_1 Z_{t-1}^2 + \beta_1)\sigma_{t-1}^2$$

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$$= \alpha_0 + \alpha_0 \Big(\alpha_1 Z_{t-1}^2 + \beta_1 + \sum_{i=1}^{\infty} (\alpha_1 Z_{t-1}^2 + \beta_1)(\alpha_1 Z_{t-2}^2 + \beta_1) \cdots (\alpha_1 Z_{t-1-i}^2 + \beta_1) \Big)$$

$\square$

GARCH(1,1)

Proof.

$$
\begin{aligned}
&\alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2 = \alpha_0 + \alpha_1 \sigma_{t-1}^2 Z_{t-1}^2 + \beta_1 \sigma_{t-1}^2 = \alpha_0 + (\alpha_1 Z_{t-1}^2 + \beta_1)\sigma_{t-1}^2 \\
&= \alpha_0 + \alpha_0 \Big(\alpha_1 Z_{t-1}^2 + \beta_1 + \sum_{i=1}^{\infty} (\alpha_1 Z_{t-1}^2 + \beta_1)(\alpha_1 Z_{t-2}^2 + \beta_1) \cdots (\alpha_1 Z_{t-1-i}^2 + \beta_1) \Big) \\
&= \alpha_0 \Big(1 + \alpha_1 Z_{t-1}^2 + \beta_1 + \sum_{i=2}^{\infty} (\alpha_1 Z_{t-1}^2 + \beta_1)(\alpha_1 Z_{t-2}^2 + \beta_1) \cdots (\alpha_1 Z_{t-i}^2 + \beta_1) \Big)
\end{aligned}
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□

Proof.

$$\begin{aligned}\mathbb{E}(X_t^2) = \mathbb{E}(\sigma_t^2) &= \alpha_0 + \alpha_1\,\mathbb{E}(X_{t-1}^2) + \beta_1\,\mathbb{E}(\sigma_{t-1}^2)\\ &= \alpha_0 + \mathbb{E}(X_t^2)(\alpha_1 + \beta_1)\end{aligned}$$

□

GARCH(p,q)

$$\sigma_t^2 = \alpha_0 + \alpha(B)X_t^2 + \beta(B)\sigma_t^2$$

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Proposition (Existence of a $\mathrm{GARCH}(p,q)$ process)

If

$$\alpha(1) + \beta(1) = \sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j < 1,$$

there exists a unique weakly and strictly stationary causal solution $X = (X_t, t \in \mathbb{Z})$ to the $\mathrm{GARCH}(p,q)$ equations and a real-valued sequence $(\psi_j)_{j=0}^{\infty}$ such that $\sum_{j=0}^{\infty} |\psi_j| < \infty$ and σ^2 can be expressed by

$$\sigma_t^2 = \psi_0 + \sum_{j=1}^{\infty} \psi_j X_{t-j}^2.$$

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Conversely, if

$$\alpha(1) + \beta(1) = \sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j \geq 1,$$

then no stationary and causal solution to the $\mathrm{GARCH}(p,q)$ equations for which σ_t can be written as a deterministic function of $(Z_s, s < t)$ for all $t \in \mathbb{Z}$ exists.

GARCH squares as an ARMA process

As before, $\alpha(1) + \beta(1) < 1 \implies$

$$\mathbb{E}(X_t^2) = \alpha_0 + (\alpha(1) + \beta(1))\,\mathbb{E}(X_t^2)$$

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Exercise: If $\mathbb{E}(\sigma_t^4) < +\infty$, $X^2 = (X_t^2, t \in \mathbb{Z}) \sim \mathrm{ARMA}(\max\{p, q\}, q)$ with

$$\phi(z) = 1 - \alpha(z) - \beta(z),$$

$$\theta(z) = 1 - \beta(z)$$

and mean $\alpha_0(1 - \alpha(1) - \beta(1))^{-1}$.

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$$\theta(z) = 1 - \beta(z)$$

and mean $\alpha_0(1 - \alpha(1) - \beta(1))^{-1}$. Alternatively:

$$X_t^2 - \sum_{i=1}^{\max\{p,q\}} (\alpha_i + \beta_i) X_{t-i}^2 = \alpha_0 + \eta_t - \sum_{j=1}^{q} \beta_j \eta_{t-j},$$

where $\alpha_i := 0$ for $i > p$, $\beta_i := 0$ for $i > q$ and $\eta_t := X_t^2 - \sigma_t^2$ is white noise.