Financial Time Series – Nonlinear models

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TMS088/MSA410 – May 2020

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Nonlinear models

- Linear model: $X_t = \sum_{j \in \mathbb{Z}} \psi_j Z_{t-j}$, where $Z \sim \mathrm{WN}(0, \sigma^2)$ and $(\psi_j, j \in \mathbb{Z})$ is a sequence of real numbers

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- Conditional mean/variance models:

$$X_t = g(\mathcal{F}_{t-1}) + \sqrt{h(\mathcal{F}_{t-1})} Z_t,$$

  where $(\mathcal{F}_t, t \in \mathbb{Z})$ is a *filtration*

$$\mu_t = \mathbb{E}(X_t|\mathcal{F}_{t-1}) =: g(\mathcal{F}_{t-1})$$

  and

$$\sigma_t^2 = \mathsf{Var}(X_t|\mathcal{F}_{t-1}) = \mathbb{E}\left(\left(X_t - \mathbb{E}(X_t|\mathcal{F}_{t-1})\right)^2 |\mathcal{F}_{t-1}\right) =: h(\mathcal{F}_{t-1}).$$

- $\mathcal{F}_{t-1}$ represents the information available at time $t-1$, e.g, all linear combinations of $(Z_s, s \le t)$

Example: Bilinear models

- Idea: Second-order Taylor expansions of $f(Z_s, X_s, s \leq t)$
- Introduced by Grander and Andersen in 1978
- Let $X = (X_t, t \in \mathbb{Z})$ be given by

$$X_t = c + \sum_{i=1}^{p} \phi_i X_{t-i} - \sum_{j=1}^{q} \theta_j Z_{t-j} + \sum_{i=1}^{m} \sum_{j=1}^{s} \beta_{ij} X_{t-i} Z_{t-j} + Z_t,$$

  where p, q, m, and s are nonnegative integers, the other parameters are real-valued, and Z is a white noise

Example: SETAR model

A time series $X = (X_t, t \in \mathbb{Z})$ follows a *k-regime self-exciting threshold autoregressive (SETAR) model* if it satisfies

$$X_t = c_j + \sum_{i=1}^{p} \phi_{ji} X_{t-i} + Z_{jt} \text{ if } X_{t-d} \in [\gamma_{j-1}, \gamma_j),$$

where $d \in \mathbb{N}$ and the parameters $c_j, \phi_{ji}, \gamma_{j-1}, \gamma_j$ are real valued with $i = 1, \ldots, p$, $j = 1, \ldots, k$ and the series $Z_1, \ldots, Z_k$ with $Z_j = (Z_{jt}, t \in \mathbb{Z})$ are mutually independent $\mathrm{IID}(0, \sigma_j^2)$ noises.

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Estimate by minimizing

$$S(\theta) := \sum_{t=p+1}^{n} \left(X_t - \mathbb{E}_\theta(X_t | \mathcal{F}_{t-1})\right)^2$$

where the conditional expectation is computed with respect to $\theta := c_j, \phi_{ji}, \gamma_{j-1}, \gamma_j$

Definition

A stochastic process $X = (X_t, t \in \mathbb{T})$ on some index set $\mathbb{T}$ is a *Markov process* if its conditional distribution function satisfies

$$P(X_h | X_s, s \leq t) = P(X_h | X_t)$$

for arbitrary $h > t$. If X is a discrete-time stochastic process, i.e., $\mathbb{T} = \mathbb{N}$ or $\mathbb{Z}$, then the property becomes

$$P(X_h | X_t, X_{t-1}, \ldots) = P(X_h | X_t)$$

for arbitrary $h > t$ and the process is also known as (first-order) *Markov chain*.

Example: Markov switching AR(p) model

Let S be a Markov chain taking values in $\{1, 2\}$ with

$$P(S_t = 2 | S_{t-1} = 1) = w_1,$$
$$P(S_t = 1 | S_{t-1} = 2) = w_2$$

with $w_1, w_2 \in [0, 1]$. $X = (X_t, t \in \mathbb{Z})$ follows a *Markov switching autoregressive model* (MSA) with two states if

$$X_t = \begin{cases} c_1 + \sum_{i=1}^{p} \phi_{1i} X_{t-i} + Z_{1t} & \text{if } S_t = 1, \\ c_2 + \sum_{i=1}^{p} \phi_{2i} X_{t-i} + Z_{2t} & \text{if } S_t = 2. \end{cases}$$

Here $Z_1 = (Z_{1t}, t \in \mathbb{Z})$ and $Z_2 = (Z_{2t}, t \in \mathbb{Z})$ are $\mathrm{IID}(0, \sigma^2)$ noises for finite σ^2 and independent of each other.