Financial Time Series – Testing for nonlinearities

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- $H_0 = \{$the data is generated by the assumed (linear model)$\}$, i.e., H_0 is almost certainly false
- Find a test with good power against a *specific* H_1 like $H_1 = \{$the driving noise follows a GARCH process$\}$
- *Power* of a test is $P($reject $H_0 | H_1$ is true$)$

Nonlinearity tests

- What are the statistical assumptions for the test?
- Does a rejection mean that my model is not useful for the purpose I am working with?
- What aspect of my data lead to the test rejecting the null hypothesis?
- Against what H_1 does my test have good power? What does that suggest that I do next?
- *Nonparametric* and *parametric* tests

Nonparametric tests

The test below has good power for testing ARMA models driven by $Z = \mathrm{IID}\,\mathcal{N}(0,1)$ versus ARMA models driven by a GARCH process as the noise term Z. Based on the fact that the ACF of powers of IID noise is zero.

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$$Q(m) := n(n+2)\sum_{i=1}^{m}\frac{\hat{\rho}_i^2(Z_t^2)}{n-i},$$

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where n is the number of observations, m is "properly chosen".
Under H_0, that the linear model is adequate, $Q(m)$ is asymptotically χ^2_{m-p-q}-distributed.

Nonparametric tests

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- Additional assumptions: causal, $\mathbb{E}(|X_t^3|) < \infty$

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- Centered third-order moment:

$$\begin{aligned} c(u, v) :&= \mathbb{E}\left((X_t - \mu)(X_{t+u} - \mu)(X_{t+v} - \mu)\right) \\ &= \mathbb{E}(Z_t^3) \sum_{k=-\infty}^{\infty} \psi_k \psi_{k+u} \psi_{k+v} \end{aligned}$$

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- Fourier transform of c given by

$$b_3(w_1, w_2) := \frac{\mathbb{E}(Z_t^3)}{4\pi^2} \Gamma(-(w_1 + w_2))\Gamma(w_1)\Gamma(w_2),$$

  with $\Gamma(w) := \sum_{u=0}^{\infty} \psi_u \exp(-iwu)$

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- If Z is Gaussian, $\mathbb{E}(Z_t^3) = 0$.

Parametric tests

- *RESET test*: *Regression Equation Specification Error Test*
- Consider (causal) $\mathrm{AR}(p)$ model

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- Obtain the least-squares estimate $(\hat{\phi}_0, \hat{\phi}_1, \ldots, \hat{\phi}_p)$ (i.e., Hannan–Rissanen) and compute the fit

$$\hat{X}_t := \hat{\phi}_0 + \sum_{j=1}^{p} \hat{\phi}_j X_{t-j},$$

the residuals $\hat{Z}_t := X_t - \hat{X}_t$, and the sum of squared residuals

$$\mathsf{SSR}_0 := \sum_{t=p+1}^{n} \hat{Z}_t^2,$$

Parametric tests

- Consider the linear regression

$$\hat{Z}_t = \alpha_{10} + \sum_{j=1}^{p} \alpha_{1j} X_{t-j} + \sum_{i=1}^{s} \alpha_{2i} \hat{X}_t^{1+i} + V_t$$

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- Compute:

$$\hat{V}_t = \hat{Z}_t - \left(\hat{\alpha}_{10} + \sum_{j=1}^{p} \hat{\alpha}_{1j} X_{t-j} + \sum_{i=1}^{s} \hat{\alpha}_{2i} \hat{X}_t^{1+i} \right),$$

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- If $\mathrm{AR}(p)$ model is adequate, all α_{1i} and α_{2j} should be zero
- Under H_0,

$$F := \frac{(\mathsf{SSR}_0 - \mathsf{SSR}_1)(n-p-g)}{\mathsf{SSR}_1 g} \sim F(g, n-p-g),$$

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