Exercises for ARCH and GARCH models

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We recall the formulas for a finite and infinite geometric sum. Let $r \neq 1$ and $n \in \mathbb{N}$. Then

$$\sum_{k=0}^{n-1} r^k = \frac{1-r^n}{1-r}$$

hence, if $|r| < 1$,

$$\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}.$$

These formulas are useful in some of the following exercises on ARCH and GARCH models. Note that all (G)ARCH processes below are implicitly assumed to be stationary.

Ex. 1 — Let the stochastic process $X = (X_t, t \in \mathbb{Z})$ be a *causal* ARCH(1) *process* with

$$X_t = \sigma_t Z_t,$$

where $Z \sim \mathrm{IID}\,\mathcal{N}(0,1)$,

$$\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2,$$

with $\alpha_0 > 0$, $0 < \alpha_1 < 1$.

a) Is it true that $\mathsf{Cov}(X_t, f(X_{t-h})) = 0$ for any $t \in \mathbb{Z}$, $h \in \mathbb{N}$ and any (measurable) function f?

b) Find the *conditional variance* of the ARCH(1)-model $\mathsf{Var}(X_t|X_{t-1}) = \mathbb{E}[(X_t - \mathbb{E}[X_t])^2|X_{t-1}]$.

Ex. 2 — Let X and Z be as in Exercise 1.

a) Show that

$$X_t^2 = \alpha_0 \sum_{j=0}^{n} \alpha_1^j Z_t^2 Z_{t-1}^2 \cdots Z_{t-j}^2 + \alpha_1^{n+1} X_{t-n-1}^2 Z_t^2 Z_{t-1}^2 \cdots Z_{t-n}^2$$

for all $n \in \mathbb{N}$.

b) One can show that the previous task implies that

$$X_t^2 = \lim_{n\to\infty} \alpha_0 \sum_{j=0}^{n} \alpha_1^j Z_t^2 Z_{t-1}^2 \cdots Z_{t-j}^2$$

almost surely. The so called *monotone convergence theorem* says that if $(Y_n, n \in \mathbb{N})$ are non-negative increasing random variables such that $\lim_{n\to\infty} Y_n = Y$ almost surely then $\mathbb{E}[Y] = \mathbb{E}[\lim_{n\to\infty} Y_n] = \lim_{n\to\infty} \mathbb{E}[Y_n]$. Use this to find $\mathbb{E}[X_t^2]$ in another way than in the lecture notes.

c) (Harder exercise) Use the fact that the fourth moment of the white noise $E[Z_t^4] = 3$ to evaluate $\mathbb{E}[X_t^4]$ using the monotone convergence theorem of the previous exercise. Deduce that $\mathbb{E}[X_t^4] < \infty \iff 3\alpha_1^2 < 1$.

Ex. 3 — Let the stochastic process $X = (X_t, t \in \mathbb{Z})$ be a *causal* ARCH(1) *process* given by

$$X_t = \sigma_t Z_t$$

and

$$\sigma_t^2 = \frac{1}{2} + \frac{1}{4} X_{t-1}^2$$

for $t \in \mathbb{Z}$, where $Z \sim \mathrm{IID}\,\mathcal{N}(0,1)$ (with 4th moment equal to 3).

a) Show that $X^2 := (X_t^2, t \in \mathbb{Z})$ is weakly stationary by doing the following:
  (i) Show that $\mathbb{E}(X_t^2) = \mathbb{E}(\sigma_t^2)$.
  (ii) Use the previous result and the fact that $\mathbb{E}(X_t^2)$ does not depend on t to compute $\mathbb{E}(X_t^2)$ explicitly.
  (iii) Assume that $\mathbb{E}(X_t^4)$ is constant for all $t \in \mathbb{Z}$ (this follows from Exercise 2 and also the fact that X is strictly stationary). Compute $\mathbb{E}(X_t^4)$.
  (iv) Use the previous result to show that $\mathsf{Cov}(X_t^2, X_{t+h}^2)$ does not depend on t for $h > 0$.
  (v) Conclude that you have shown stationarity and write down the autocovariance function.

b) Show that $\tilde{Z} := (\tilde{Z}_t, t \in \mathbb{N})$ with $\tilde{Z}_t := X_t^2 - \sigma_t^2$ is white noise with mean zero and variance $40/39$.

c) Show that $(X_t^2, t \in \mathbb{Z})$ is a causal AR(1) process with mean $2/3$. (*Hint:* Use the result of the previous task.)

Ex. 4 — Let the stochastic process $X = (X_t, t \in \mathbb{Z})$ be a causal ARCH(p) process with $\mathbb{E}[Z_t^4] < \infty$ and $\mathbb{E}[X_t^4] < \infty$ constant for all $t \in \mathbb{Z}$.

a) Show that $Y_t = X_t^2/\alpha_0$ satisifies the equations

$$Y_t = Z_t^2\big(1 + \sum_{i=1}^{p} \alpha_i Y_{t-i}\big).$$

b) Show that Y_t satisifies the AR(p) equations

$$\phi(B)Y_t = 1 + \tilde{Z}_t$$

for some white noise $\tilde{Z} := (\tilde{Z}_t, t \in \mathbb{N})$, where $\phi_i = \alpha_i$ for $i = 1, \ldots, p$. *Hint:* Consider the solution to Exercise 3. Can you do something similar here?

Ex. 5 — Let the stochastic process $X = (X_t, t \in \mathbb{Z})$ be a GARCH(p, q) *process* with $\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j < 1$, $\mathbb{E}[Z_t^4] < \infty$ and $\mathbb{E}[X_t^4] < \infty$ constant for all $t \in \mathbb{Z}$.

a) Let $\tilde{Z}_t = X_t^2 - \sigma_t^2$. Show that $(\tilde{Z}_t, t \in \mathbb{Z})$ is mean zero white noise.

b) Show that $(X_t^2, t \in \mathbb{Z})$ satisfies the ARMA(m, q)-equation

$$\phi(B)X_t^2 = \alpha_0 + \theta(B)\tilde{Z}_t$$

where $m = \max(p, q)$ and the ARMA coefficients satisfies $\phi_i = \alpha_i + \beta_i$ for $i = 1, \ldots, m$ and $\theta_i = -\beta_i$ for $i = 1 \ldots q$ where $\beta_i = 0$ for $i > q$ and $\alpha_i = 0$ for $i > p$.

Ex. 6 — Let the stochastic process $X = (X_t, t \in \mathbb{Z})$ be a causal GARCH$(1, 1)$ *process* with $\mathbb{E}[Z_t^4] < \infty$ and $\mathbb{E}[X_t^4] < \infty$ constant for all $t \in \mathbb{Z}$ and $\alpha_1 + \beta_1 < 1$.

a) Find $\mathbb{E}[X_t^4]$.

b) Confirm that the kurtosis of X_t, $t \in \mathbb{Z}$, is given by

$$\frac{\mathbb{E}(X_t^4)}{\mathbb{E}(X_t^2)^2} = \frac{\mu_4(1 - (\alpha_1 + \beta_1)^2)}{1 - \beta_1^2 - 2\alpha_1\beta_1 - \mu_4\alpha_1^2},$$

where $\mu_4 = \mathbb{E}(Z_t^4)/\,\mathbb{E}(Z_t^2)^2$ is the kurtosis of Z_t, $t \in \mathbb{Z}$, provided that the denominator is positive.

c) If $Z_t \sim \mathcal{N}(0,1)$, $\mu_4 = 3$. Show that if this Gaussian assumption holds, X_t has *excess kurtosis*, i.e.,

$$\frac{\mathbb{E}(X_t^4)}{\mathbb{E}(X_t^2)^2} \geq 3.$$

Ex. 7 — Let $Z \sim \mathrm{IID}(0,1)$ and assume that the distribution of Z_t is symmetric, i.e., $Z_t \stackrel{d}{=} -Z_t$.

a) Show that $g(Z) \sim \mathrm{IID}(0, 1 + \lambda^2 \operatorname{\mathsf{Var}}(|Z_t|))$, where $g(Z)$ is the process defined by $g(Z_t) = Z_t + \lambda(|Z_t| - \mathbb{E}(|Z_t|))$, $\lambda \in \mathbb{R}$, for all $t \in \mathbb{Z}$.

b) Using the previous result, show that the process $\ell = (\ell_t, t \in \mathbb{Z})$ defined by

$$\ell_t = \alpha_0 + \alpha(B)g(Z_t) + \beta(B)\ell_t$$

where

$$\begin{aligned}\alpha(z) &:= 1 + \alpha_1 z + \cdots + \alpha_p z^p,\\ \beta(z) &:= \beta_1 z + \cdots + \beta_q z^q,\end{aligned}$$

is an ARMA(q,p) process with mean with mean $\mu = \alpha_0/(1 - \beta(1))$ if $1 - \beta(z) \neq 0$ for all $z \in \mathbb{C}$ with $|z| = 1$ and $1 - \beta$ and α have no common zeros.