Financial Time Series – An exam example

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Problem 1

Let the process $X = (X_t, t \in \mathbb{Z})$ be given by

$$X_t = Z_t - \frac{4\theta}{3} Z_{t-2} + \frac{\theta^2}{3} Z_{t-4} \tag{1}$$

for all $t \in \mathbb{Z}$ where $\theta \neq 0$ and $Z = (Z_t, t \in \mathbb{Z}) \sim \mathrm{IID}\, \mathcal{N}(0,1)$ is a sequence of independent identically distributed Gaussian random variables with zero mean and unit variance.

Problem 1

(a) Show that X is weakly stationary and find $\gamma_X(h)$, the autocovariance function of X, for all lags $h \in \mathbb{Z}$.

Problem 1

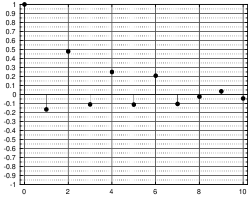
(b) Is X strictly stationary?

Problem 1

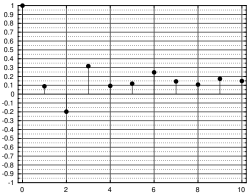
(c) State the definition of a causal and of an invertible process. For which θ is X causal and/or invertible?

Problem 1

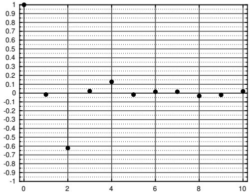
(d) Below you can find sample autocorrelation functions for 4 different samples of time series, each of size $n = 400$. Based on the plots, which, if any, of the time series would you choose to model with X?



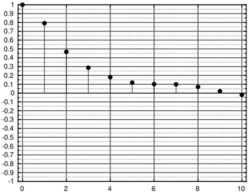
(a) Series 1



(b) Series 2



(c) Series 3



(d) Series 4

Problem 1

(d) Below you can find sample autocorrelation functions for 4 different samples of time series, each of size $n = 400$. Based on the plots, which, if any, of the time series would you choose to model with X?

Problem 1

(e) Compute $b^{\ell}_{n+1}(X^n)$, the best linear predictor of X_{n+1} given $X^n := (X_{n-3}, X_{n-1})$, $n \in \mathbb{N}$, when $\theta = 3$.

Problem 1

(f) Compute $\mathbb{E}[(X_{n+1} - b^{\ell}_{n+1}(X^n))^2]$, the mean squared error of the predictor, when $\theta = 3$.

Problem 1

(g) Compute 95% prediction bounds for $b^{\ell}_{n+1}(X^n)$, when $\theta = 3$.

(h) Suppose that the time series Z in (1) is replaced by an $\mathrm{ARCH}(1)$ process $\tilde{Z}$ given by

$$\tilde{Z}_t = \sigma_t Z_t,$$

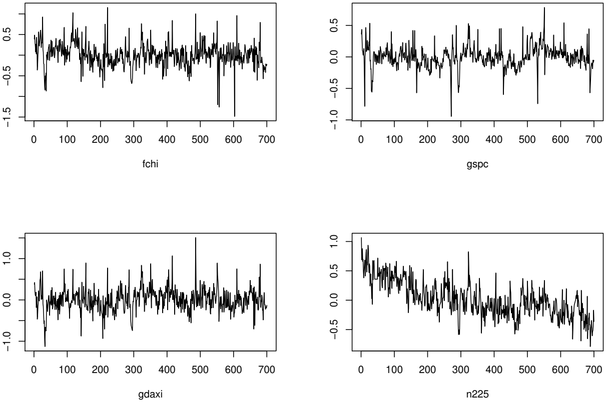
where $Z = (Z_t, t \in \mathbb{Z}) \sim \mathrm{IID}\,\mathcal{N}(0,1)$, and

$$\sigma_t^2 = \alpha_0 + \alpha_1 \tilde{Z}_{t-1}^2$$

with $0 < \alpha_0 < 1$ and $\alpha_1 = 1 - \alpha_0$, $t \in \mathbb{Z}$. Does this change (and if so how) your answers to Problems 1 (a) and 1 (b)? *Hint: You might want to solve Problem 2 (c) below before answering this one.*

Problem 2

The figure below contains 4 plots of time series consisting of log-transformed daily trading volumes from stock indices of French, American, German and Japanese markets, respectively. The data was sampled on 700 working days between November 2016 and July 2019.



Problem 2

(a) How many, if any, of these time series do you think can be modeled as stationary processes? Motivate your answer.

Problem 2

(b) During the modeling of one of the data sets, denoted by $X = (X_t, t = 1, \ldots, 700)$, it was suggested that X can be modeled by the equation $X_t = Y_t + s_t + \mu$, where $Y = (Y_t, t \in \mathbb{Z})$ is a weakly stationary time series with zero mean, $s = (s_t, t \in \mathbb{Z})$ is a seasonal component and $\mu \in \mathbb{R}$. The researchers working with this data set believed that s has a period of 5 days. Find a linear filter $\xi(B) = \sum_{j=0}^{N} a_j B^j$, where $N \in \mathbb{N}$, that you can show eliminates s and μ. Show that $\tilde{X} := \xi(B)X$ is a stationary time series and express $\gamma_{\tilde{X}}$, the autocovariance function of $\tilde{X}$, in terms of γ_Y, the autocovariance function of Y.

Problem 2

Problem 2

(c) After further study, it was determined that $\tilde{X}$ could be modeled as an $\mathrm{ARCH}(1)$ process given by

$$\tilde{X}_t = \sigma_t Z_t,$$

where $Z = (Z_t, t \in \mathbb{Z}) \sim \mathrm{IID}\,\mathcal{N}(0,1)$, and

$$\sigma_t^2 = \alpha_0 + \alpha_1 \tilde{X}_{t-1}^2$$

with $\alpha_0 > 0$ and $0 < \alpha_1 < 1$, $t \in \mathbb{Z}$. Recall that this implies that $\tilde{X}$ is causal as well as strictly and weakly stationary. Find the conditional variance Var$(\tilde{X}_t|\tilde{X}_{t-1}) = \mathbb{E}[(\tilde{X}_t - \mathbb{E}[\tilde{X}_t])^2|\tilde{X}_{t-1}]$ and the unconditional variance Var$(\tilde{X}_t)$ for all $t \in \mathbb{Z}$.

Problem 2