CHALMERS | UNIVERSITY OF GOTHENBURG

Financial Time Series TMS088/MSA410 – LP4 2019/20

PROJECT 1 : Some Takeaways

M. PEREIRA

1 Random vectors and joint distribution

A **random vector** $\boldsymbol{X} = (X_1, \ldots, X_n)$ is a vector whose entries $X_1, \ldots, X_n$ are random variables.

Just like we characterize the "randomness" of a random variable using its distribution function, for a random vector we use the **joint distribution function** of its entries, which is the function $F_{\boldsymbol{X}} : \mathbb{R}^n \to [0,1]$ defined by

$$F_{\boldsymbol{X}}(x_1, \ldots, x_n) = \mathbb{P}(X_1 \leq x_1, \ldots, X_n \leq x_n), \quad x_1, \ldots, x_n \in \mathbb{R}.$$

In other words, $F_{\boldsymbol{X}}(x_1, \ldots, x_n)$ is the probability that "simultaneously" $X_1 \leq x_1$, $X_2 \leq x_2$, ..., and $X_n \leq x_n$.

In some cases, there exists a function $f_{\boldsymbol{X}} : \mathbb{R}^n \mapsto \mathbb{R}$ such that $\forall x_1, \ldots, x_n \in \mathbb{R}$,

$$F_{\boldsymbol{X}}(x_1, \ldots, x_n) = \int_{-\infty}^{x_n} \cdots \int_{-\infty}^{x_1} f_{\boldsymbol{X}}(y_1, \ldots, y_n) dy_1 \ldots dy_n, \quad x_1, \ldots, x_n \in \mathbb{R},$$

Such a function is called **joint probability density function** (or simply joint density) $(X_1, \ldots, X_n)$. In particular, $F_{\boldsymbol{X}}$ and $f_{\boldsymbol{X}}$ are then linked by the relation:

$$f_{\boldsymbol{X}}(x_1, \ldots, x_n) = \frac{\partial^n F}{\partial x_1 \ldots \partial x_n}(x_1, \ldots, x_n), \quad x_1, \ldots, x_n \in \mathbb{R}.$$

Note: In the case $n = 1$, the joint distribution function and the joint density introduced above correspond to the distribution function and the density of the sole entry of the vector.

2 Independence and uncorrelatedness

Two random variables X_1 and X_2 are **independent** if for any values $x_1, x_2 \in \mathbb{R}$, the probability that "simultaneously" $X_1 \leq x_1$ and $X_2 \leq x_2$ is equal to the product of the probability that $X_1 \leq x_1$ with the probability that $X_2 \leq x_2$:

$$\mathbb{P}(X_1 \leq x_1, X_2 \leq x_2) = \mathbb{P}(X_1 \leq x_1) \cdot \mathbb{P}(X_2 \leq x_2)$$

In this case, the joint distribution function of (X_1, X_2) is equal to the product of the distribution functions of X_1 and X_2. If X_1 and X_2 are independent, then the outcome of X_1 has no effect on the outcome of X_2 (and vice-versa).

Remark: The notion of independence can be generalized to more than two variables. We say that $n \geq 2$ random variables $X_1, \ldots, X_n$ are mutually independent if the joint distribution function of $(X_1, \ldots, X_n)$ is equal to the product of the distribution functions of the variables X_i.

On the other hand, two random variables X_1 and X_2 are called **uncorrelated** if

$$\mathrm{Cov}(X_1, X_2) = \mathbb{E}[X_1 \cdot X_2] - \mathbb{E}[X_1] \cdot \mathbb{E}[X_2] = 0$$

Note that the covariance is just a *measure* of *linear* dependence between two random variables. When faced with two uncorrelated random variables X_1, X_2, the only thing that we can safely say is that there is no linear dependence between X_1 and X_2, i.e. that there is no constant $a \in \mathbb{R}$ such that $X_2 = aX_2$. It it is for instance perfectly possible that there exists some other non-linear relationship between X_1 and X_2.

Example. *Let X be a random variable following a uniform distribution on $[-1,1]$, meaning that its probability density function is*

$$f_X(x) = \begin{cases} 1/2 & \text{if } x \in [-1,1] \\ 0 & \text{otherwise} \end{cases}$$

Let Y be the random variable defined by $Y = X^2$.

$$Cov(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = \mathbb{E}[X^3] - \mathbb{E}[X]\mathbb{E}[X^2]$$

where

$$\mathbb{E}[X] := \int_{\mathbb{R}} x f_X(x) dx = \int_{-1}^{1} x \cdot \frac{1}{2} dx = \frac{1}{2}\left[\frac{x^2}{2}\right]_{-1}^{1} = 0$$

$$\mathbb{E}[X^3] := \int_{\mathbb{R}} x^3 f_X(x) dx = \int_{-1}^{1} \frac{x^3}{2} dx = \frac{1}{2}\left[\frac{x^4}{4}\right]_{-1}^{1} = 0$$

Hence $Cov(X,Y) = 0$, and therefore X and Y are uncorrelated. However, X and Y are clearly not independent since Y is a function of X. The covariance was not able to detect the nonlinear relation between X and Y...

Advice: You should keep in mind that independence is a much more stronger assumption than uncorrelatedness. Also:

$$X_1 \text{ and } X_2 \text{ are independent} \Rightarrow X_1 \text{ and } X_2 \text{ are uncorrelated}$$

but in general,

$$X_1 \text{ and } X_2 \text{ are uncorrelated} \nRightarrow X_1 \text{ and } X_2 \text{ are independent}$$

3 Gaussian variables, vectors and processes

3.1 Gaussian variables and vectors

A **Gaussian variable** (with mean μ and variance σ^2) is a random variable X whose probability density function f_X is the function defined by

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad x \in \mathbb{R}$$

In this case, we write $X \sim \mathcal{N}(\mu, \sigma^2)$.

If $X \sim \mathcal{N}(\mu, \sigma^2)$, then for any constants $a, b \in \mathbb{R}$, $aX + b \sim \mathcal{N}(a\mu + b, a^2\sigma^2)$.

Let $n \geq 2$. If $X_1 \sim \mathbb{N}(\mu_1, \sigma_1^2), \dots, X_n \sim \mathbb{N}(\mu_n, \sigma_n^2)$ ***and*** $X_1, \dots, X_n$ ***are (mutually) independent***, then for any constants $a_1, \dots, a_n \in \mathbb{R}$

$$\sum_{i=1}^{n} a_i X_i \sim \mathcal{N}\left(\sum_{i=1}^{n} a_i \mu_i, \sum_{i=1}^{n} a_i^2 \sigma_i^2\right)$$

A **Gaussian vector** is a random vector $\boldsymbol{X} = (X_1, \dots, X_n)$ whose joint probability density function $f_{\boldsymbol{X}}$ is the function defined by

$$f_{\boldsymbol{X}}(y_1, \dots, y_n) = \frac{1}{\sqrt{(2\pi)^n \det \boldsymbol{\Sigma}}} \exp\left(-\frac{1}{2}\begin{pmatrix} y_1 - \mu_1 \\ \vdots \\ y_n - \mu_n \end{pmatrix}^T \boldsymbol{\Sigma}^{-1} \begin{pmatrix} y_1 - \mu_1 \\ \vdots \\ y_n - \mu_n \end{pmatrix}\right)$$

where $\mu_i = \mathbb{E}[X_i]$ $(1 \leq i \leq n)$ and $\boldsymbol{\Sigma}$ is the covariance matrix of $\boldsymbol{X}$, i.e. the $n \times n$ matrix defined by whose entry (i, j) is $\mathrm{Cov}(X_i, X_j)$ $(1 \leq i, j \leq n)$.

$\boldsymbol{X} = (X_1, \dots, X_n)$ is a Gaussian vector $\iff$ For any coefficients $a_1, \dots, a_n \in \mathbb{R}$, the random variable defined by

$$\sum_{i=1}^{n} a_i X_i \tag{1}$$

is a Gaussian variable.

Remark: The entries of a Gaussian vector are Gaussian variables (just take $a_i = 1$ and setting all the other coefficients to 0 in (1)). However, the fact that a vector $\boldsymbol{X}$ is composed of entries which are Gaussian variable is in general NOT enough to conclude that $\boldsymbol{X}$ is Gaussian vector! You must also have that (1) is satisfied for any coefficients!

TIP

To show that a random vector $\boldsymbol{X}$ is a Gaussian vector, you can show that for any choice of coefficients $a_1, \dots, a_n$ the random variable (1) is a Gaussian variable.

If $\boldsymbol{X} = (X_1, \dots, X_n)$ is a Gaussian vector, then for any entries X_i and X_j,

$$\mathrm{Cov}(X_i, X_j) = 0 \Rightarrow X_i \text{ and } X_j \text{ are independent.}$$

TIP

To show that two random variables X_1 and X_2 are independent, you can show that the vector (X_1, X_2) is a Gaussian vector and that $\mathrm{Cov}(X_1, X_2) = 0$.

Note: To use this tip, it is not enough to show that X_1 and X_2 are Gaussian variables! Indeed, two uncorrelated Gaussian random variables are not necessarily independent[1]. But if, besides, they form a Gaussian vector, then they will be independent.

3.2 Gaussian processes

A **Gaussian process** is a process $(X_t, t \in \mathbb{Z})$ such that for any $n \geq 1$ times $t_1, \ldots, t_n \in \mathbb{Z}$ the vector $(X_{t_1}, \ldots, X_{t_n})$ is a Gaussian vector.

Note: For a process $(X_t, t \in \mathbb{Z})$ to be a Gaussian process it is NOT enough to only check that for any $t \in \mathbb{Z}$, X_t is a Gaussian variable.

> $(X_t, t \in \mathbb{Z})$ is a Gaussian process $\iff$ For any number $n \geq 1$ of arbitrary times $t_1, \ldots, t_n \in \mathbb{Z}$ and any choice of coefficients $a_1, \ldots, a_n \in \mathbb{R}$, the random variable
>
> $$\sum_{i=1}^{n} a_i X_{t_n} \tag{2}$$
>
> is a Gaussian variable.

TIP
To show that $(X_t, t \in \mathbb{Z})$ is a Gaussian process, you have can show that for any $n \geq 1$, any times $t_1, \ldots, t_n \in \mathbb{Z}$, and any coefficients $a_1, \ldots, a_n \in \mathbb{R}$, the random variable (2) is a Gaussian variable.

> If $(X_t, t \in \mathbb{Z})$ is a Gaussian process, then for any times $t_1, t_2 \in \mathbb{Z}$,
>
> $$\mathrm{Cov}(X_{t_1}, X_{t_2}) = 0 \Rightarrow X_{t_1} \text{ and } X_{t_2} \text{ are independent.}$$

[1] cf. `https://en.wikipedia.org/wiki/Normally_distributed_and_uncorrelated_does_not_imply_independent`