1.8 Some basic aspects of field theory

Here we discuss two key aspects of Lagrangian field theory: 1) Noether's theorem, and 2) gauging of symmetries. This will provide enough background to understand the Higgs mechanism which will the subject of the first home problem (see Canvas).

Lagrangian field theory has many nice features as for instance providing an easy way to derive the field equations and a fundamental connection between global symmetries and conservations laws. After having discussed these aspects of field theory we will turn the global symmetries into local ones, also called gauge symmetries, as a means of constructing interactions between various fields. This will turn out to be a key property that can be used to construct models for elementary particle physics.

1.8.1 Noether's theorem

We start by introducing a free complex scalar field Φ with the Lagrangian

$$\mathcal{L} = \partial_\mu \Phi^\star \partial^\mu \Phi - m^2 \Phi^\star \Phi. \tag{1.148}$$

This Lagrangian has a **global** $U(1)$ (or phase) symmetry

$$\Phi(x) \to \Phi'(x) = e^{i\alpha} \Phi(x), \tag{1.149}$$

where α is a real constant parameter (angle). The fact that this transformation does not affect the coordinates x^μ means that symmetry is an *internal* one. For small α the variation of the field $\delta\Phi$ becomes

$$\delta\Phi(x) := (\Phi'(x) - \Phi(x))|_{\mathcal{O}(\alpha)} = i\alpha\, \Phi(x). \tag{1.150}$$

The variation of the *action functional* $S[\Phi, \Phi^\star] = \int d^4x \mathcal{L}(\Phi(x), \Phi^\star(x))$ is then

$$\delta S[\Phi, \Phi^\star] := (S[\Phi', \Phi'^\star] - S[\Phi, \Phi^\star])|_{\mathcal{O}(\delta\Phi, \delta\Phi^\star)} = \int d^4x \left(\mathcal{L}(\Phi'(x), \Phi'^\star(x)) - \mathcal{L}(\Phi(x), \Phi^\star(x)) \right) |_{\mathcal{O}(\delta\Phi, \delta\Phi^\star)}. \tag{1.151}$$

Inserting the relation $\Phi' = \Phi + \delta\Phi$ and its c.c. into the above variation we get

$$\begin{aligned}
&\delta S[\Phi, \Phi^\star] \\
&= \int d^4x \left(\partial_\mu (\Phi^\star + \delta\Phi^\star) \partial^\mu (\Phi + \delta\Phi) - m^2 (\Phi^\star + \delta\Phi^\star)(\Phi + \delta\Phi) - (\partial_\mu \Phi^\star \partial^\mu \Phi - m^2 \Phi^\star \Phi) \right)|_{\mathcal{O}(\delta\Phi, \delta\Phi^\star)} \\
&= \int d^4x \left(\partial_\mu \delta\Phi^\star \partial^\mu \Phi - m^2 \delta\Phi^\star \Phi + c.c. \right).
\end{aligned} \tag{1.152}$$

The last step is to perform an integration by parts so that the derivatives do not act on the field variations. This gives (recall that $\Box := \partial_\mu \partial^\mu = \partial_t^2 - \nabla^2$ in the metric used here)

$$\begin{aligned}
\delta S[\Phi, \Phi^\star] = &\int d^4x \left(-\delta\Phi^\star (\Box \Phi + m^2 \Phi) - (\Box \Phi^\star + m^2 \Phi^\star) \delta\Phi \right) \\
&+ \int d^4x\, \partial_\mu (\delta\Phi^\star \partial^\mu \Phi + \delta\Phi \partial^\mu \Phi^\star),
\end{aligned} \tag{1.153}$$

where the first line is called the *bulk* term and the second line the *boundary* term.

One can also perform a constant coordinate transformation $x^\mu \to x'^\mu = x^\mu + \xi^\mu$ (where ξ^μ are constant real parameters) which means that a scalar field transforms[19] as $\delta\phi = -\xi^\mu \partial_\mu \phi$ (and similarly for complex fields Φ and $\Phi^\star$). Such a transformation is called *external*. For a real scalar field the Lagrangian then behaves as follows:

$$\delta_\xi \mathcal{L} = \delta_\xi(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi) = \partial_\mu\phi\partial^\mu\delta_\xi\phi = -\partial_\mu\phi\partial^\mu(\xi^\nu\partial_\nu\phi) = -\frac{1}{2}\xi^\nu\partial_\nu(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi) = -\xi^\nu\partial_\nu\mathcal{L}, \tag{1.154}$$

from which we can conclude that the Lagrangian is a scalar quantity transforming the same way as the scalar field ϕ. The point here is, however, that the Lagrangian does transform showing that the transformation is an external one. The action is, however, invariant. We will return to this below.

The variation of the action above can be expressed in general terms as follows,

$$\delta S[\phi, \delta\phi] = \int d^4x \left(\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\delta\phi + \int d^4x \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right), \tag{1.155}$$

where ϕ can now represent any kind of field and the mass term above can be part of a more general potential term $V(\phi)$, or even more general interaction terms containing different kinds of fields like $ieA_\mu(\partial^\mu\Phi^\star)\Phi + c.c.$.

We emphasise at this point that ϕ and $\delta\phi$, which appear in $\delta S[\phi, \delta\phi]$, are both arbitrary functions and completely independent of each other. The result above can then be used in two fundamentally different ways, 1) to derive field equations and 2) to establish the connection between global symmetries and conserved charges. We will later show that these conserved charges are actually **generators** of the symmetries that gave rise to them in the first place. This is a important fact used many times in this course.

1) **Hamilton's principle**

This principle states that the equations of motion, in the form of the Euler-Lagrange (EL) equations, follow by demanding that the action be stationary under variations of the fields that vanish at the initial and final time of $S = \int_{t_i}^{t_f} L$. Thus imposing $\delta S = 0$ on the variation above has two implications:

The vanishing of the bulk term implies EL:

$$\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = 0, \tag{1.156}$$

while the vanishing of the boundary term implies one of two boundary conditions (b.c.) in the space directions

$$\text{Dirichlet b.c.} : \delta\phi|_{space} = 0, \text{ or Neumann b.c. } \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}|_{space} = 0. \tag{1.157}$$

[19] A scalar field transforms as $\phi'(x') = \phi(x)$ which implies that $\delta\phi(x) := \phi'(x) - \phi(x) = \phi'(x' - \xi) - \phi(x) = \phi'(x') - \xi^\mu\partial_\mu\phi'(x') - \phi(x) = -\xi^\mu\partial_\mu\phi'(x') = -\xi^\mu\partial_\mu\phi(x)$ to first order in small quantities.

2) **Noether's theorem**[20]

This theorem follows by combining a) and b):

a) Let $\delta_s\phi$ be a symmetry transformation. Then, for any field configuration $\phi(x)$, $\delta_s\phi$ is a symmetry iff for some J we have (recall the coordinate transformations above)

$$\delta S[\phi, \delta_s\phi] = \int d^4x \partial_\mu J^\mu(\phi, \delta_s\phi). \tag{1.158}$$

b) Now let instead $\phi = \bar{\phi}$ be an on-shell field configuration (i.e., satisfying the EL equations). Then for any variation $\delta\phi$ we have

$$\delta S[\bar{\phi}, \delta\phi] = \int d^4x \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}|_{\phi=\bar{\phi}} \delta\phi \right). \tag{1.159}$$

We can now set $\phi = \bar{\phi}$ and $\delta\phi = \delta_s\phi$ in both results above, a) and b), and then subtract them. This leads to Noether's theorem

$$\partial_\mu j^\mu = 0, \text{ where } j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta_s\phi - J^\mu. \tag{1.160}$$

Thus on-shell any global (continuous) symmetry implies a conserved current.

Comment 1 Conserved currents (i.e., divergence-free currents $\partial_\mu j^\mu = 0$) lead to the conservation of the corresponding charges $Q = \int d^3x\, j^0$ (recall that j^0 is the charge density). In fact, $\partial_\mu j^\mu = 0$ if written out in components gives the continuity equation. Thus it implies the conservation of charge as follows:

$$\dot{Q} = \partial_t Q = \partial_t \int_V d^3x j^0 = \int_V d^3x \dot{j}^0 = -\int_V d^3x \nabla \cdot \mathbf{j} = -\int_{\partial V} \mathbf{j} \cdot d\mathbf{S}, \tag{1.161}$$

where we have used Gauss' law in the last equality. Here V is the volume containing the charges and ∂V its boundary. Since the current $\mathbf{j}$ is assumed to be zero far away from the system (or at infinity) the charge Q is a fixed, time independent, number.

Comment 2 As we will see more in detail later the charge in the previous comment is also the generator of the symmetry that gave rise to the charge. A rather direct and clear example of this is the momentum operator $p^\mu = (p^0, \mathbf{p}) = i\partial^\mu$. Note that in the signature in PS this gives the usual QM relation $\mathbf{p} = -i\nabla := -i\partial_i$. Then we see that $e^{i\xi^i p^i} = e^{\xi^i \partial_i}$ which when acting on a function gives $e^{\xi^i\partial_i} f(x) = f(x + \xi)$ showing that $\mathbf{p}$ is the generator of translations in space. But $\mathbf{p}$ is also the conserved charge coming from the global translation symmetry in space directions via the stress tensor and Noether's theorem. The same argument in the time direction leads directly to the Schoedinger equation in QM which is the infinitesimal version of the above statement for the time translation operator H, the Hamiltonian, (using Poisson or commutator brackets). In general charges like H and $\mathbf{p}$ are called *generators* while their exponential relatives $e^{i\delta t H}$ and $e^{i\xi^i p^i}$ are *group elements* (for momenta p^μ this is an abelian group since partial derivatives commute).

[20] The following approach to Noether's theorem can be found in a review by Bañados, hep-th/1601.03616.

1.8.2 The gauge principle

Turning global symmetries into local ones, or gauge symmetries, is of fundamental importance in all field theory descriptions of elementary particles, gravity and often even of condensed matter systems. An argument for taking such a step, i.e., of "gauging a global symmetry", can be made by invoking Lorentz invariance and unitarity as we will see later in the course. At this point, let us accept the importance of this step as a means of constructing interacting field theories and discuss it in the example of a complex scalar field.

As already mentioned above the Lagrangian

$$\mathcal{L} = \partial_\mu \Phi^\star \partial^\mu \Phi - m^2 \Phi^\star \Phi. \tag{1.162}$$

has a global $U(1)$ symmetry

$$\Phi(x) \to \Phi'(x) = e^{i\alpha}\Phi(x), \tag{1.163}$$

which is obvious since the parameter α is a constant (i.e., spacetime independent). Turning α into a function of spacetime $\alpha(x)$, i.e., "gauging the symmetry", implies immediately that the Lagrangian is no longer invariant under the new, gauged, transformation: It is not gauge invariant. The problem resides in the kinetic term since the derivatives makes it impossible to cancel the phase factors $e^{i\alpha(x)}$ against each other. To fix this problem one has to introduce a gauge field A_μ and a covariant derivative

$$D_\mu = \partial_\mu + ieA_\mu, \tag{1.164}$$

where e is the charge of the complex field D_μ is acting on. By defining the gauge field to transform under gauge transformations as $A_\mu \to A'_\mu = A_\mu - \partial_\mu \alpha(x)$ we find that the kinetic term above becomes gauge invariant due to the defining property of the covariant derivative (namely that it transforms the same as the field does): $\Phi \to \Phi' = e^{ie\alpha(x)}\Phi$ implies (note the difference to PS eq 4.6, the transformation of A_μ cannot depend on ϕ)

$$\begin{aligned} D_\mu \Phi \to (D_\mu \Phi)' : &= \partial_\mu \Phi' + ieA'_\mu \Phi' = \partial_\mu (e^{ie\alpha(x)}\Phi) + ie(A_\mu - \partial_\mu \alpha(x))e^{ie\alpha(x)}\Phi \\ &= e^{ie\alpha(x)}(\partial_\mu \Phi + ieA_\mu \Phi) = e^{ie\alpha(x)} D_\mu \Phi. \end{aligned} \tag{1.165}$$

Consider now the gauge invariant Lagrangian of **scalar QED**: (recall $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$)

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + D_\mu \Phi^\star D^\mu \Phi - m^2 \Phi^\star \Phi. \tag{1.166}$$

By writing out explicitly the scalar field kinetic term we discover its interactions with the gauge field A_μ:

$$D_\mu \Phi^\star D^\mu \Phi = \partial_\mu \Phi^\star \partial^\mu \Phi + ieA_\mu \left((\partial^\mu \Phi^\star)\Phi - \Phi^\star(\partial^\mu \Phi)\right) + e^2 A_\mu A^\mu \Phi^\star \Phi. \tag{1.167}$$

There are two important implications of this result:

1. The charge current j^μ is defined by identifying the second term with $A_\mu j^\mu$. Thus, j^μ is a real and conserved ($\partial_\mu j^\mu = 0$ on-shell) quantity

$$j^\mu := ie\left((\partial^\mu \Phi^\star)\Phi - \Phi^\star(\partial^\mu \Phi)\right). \tag{1.168}$$

Exercise: Show that j^μ is real and conserved on-shell (use the field equations obtained from $\mathcal{L}$ above).
2. From the Lagrangian we see that Φ is a massive field and A_μ is massless. However, let us assume that Φ has a constant real background vacuum expectation value, a VEV, denoted v. This means that $\Phi(x) = v + \varphi(x)$ where $\varphi(x)$ is the deviation, or fluctuation, away from the constant value v. A standard way to write this VEV is $v = \langle\Phi\rangle$. Then in the kinetic term for the scalar field we find the term

$$\mathcal{L}(A_\mu, \varphi, \varphi^\star; v) = ... + e^2 v^2 A_\mu A^\mu, \tag{1.169}$$

which shows that also the vector field is massive if $v \neq 0$. This simple fact is the basis of the Higgs effect that gives masses to some of the vector gauge fields (Z and $W^\pm$) and the spin 1/2 matter fields in the standard model of particle physics. The Higgs particle itself is another result of this process. For the theory to accomplish this by itself one has to add to the above Lagrangian a fourth order scalar potential term which forces the theory into a stable vacuum away from $\Phi = 0$, i.e., to go through a *phase transition*.
Exercise: Does the sign of the mass term above for the vector field make sense? (Mass terms must add positive contributions to the energy!)