2. Problems

Problem 1: The Higgs mechanism
Consider scalar QED, i.e., the theory consisting of a Maxwell field coupled to a complex scalar field $\Phi(x)$ with charge e, mass m and quartic self-interactions. The theory is described by the Lagrangian

$$\mathcal{L} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + D_\mu\Phi^\star D^\mu\Phi - m^2\Phi^\star\Phi - \tfrac{\lambda}{2}(\Phi^\star\Phi)^2. \tag{2.1}$$

a) Determine the exact form of the covariant derivative D_μ so that the Lagrangian is invariant under the gauge transformation $\Phi \to \Phi' = e^{ie\alpha(x)}\Phi$. Note the charge in the exponent!
b) Express $\Phi(x)$ as a real scalar $R(x)$ times a phase factor $e^{i\phi(x)}$ and show that if $\Phi(x)$ gets a VEV, i.e., $\langle\Phi(x)\rangle = v$, where the constant v is real, the theory will contain one massive scalar boson (Higgs) and one massless one (Goldstone).
c) Assume that the scalar potential $V(\Phi, \bar{\Phi})$ is of the *mexican hat* type which requires m^2 to be negative (set $m^2 = -\mu^2$). Show that there is a stable vacuum away from $\Phi = 0$ and determine v in terms of the parameters of the Lagrangian.
d) Explain now the masses of the Higgs and Goldstone scalar bosons found in b).
e) Show that the vector field can absorb ("eat") the Goldstone boson in which process the Goldstone boson disappears from the Lagrangian and the vector field becomes massive, i.e., it has now three degrees of freedom. Determine this mass and explain why its sign is physically correct.

2. Vector and spinor representations
Show that

$$S^i = \frac{1}{2}\sigma^i, \tag{2.2}$$

and

$$(T^i)_{jk} = -i\epsilon_{ijk}, \tag{2.3}$$

satisfy the same Lie algebra and hence that

$$su(2) \approx so(3). \tag{2.4}$$

The Lie algebras of the two groups $SU(2)$ and $SO(3)$ are therefore said to be **isomorphic**. These two sets of matrices are regarded as different **representations** (spin 1/2 and spin 1, respectively) of the same Lie algebra.

3. How to obtain the field strength in gauge theories
Derive the Maxwell field strength by computing the commutator

$$[D_\mu, D_\nu]\Phi(x), \tag{2.5}$$

where $\Phi(x)$ is a charged (complex) scalar field.