1. QFT: Lecture notes on special topics

1.1 Conventions in Peskin and Schroeder (PS)

See pages xix - xxi in PS.
The flat spacetime metric: $g_{\mu\nu} = diag(1, -1, -1, -1)$ and thus $p \cdot x = p^0 t - \mathbf{p} \cdot \mathbf{r}$.
The notation $\eta_{\mu\nu}$ for the Minkowski metric is not used in PS.

1.2 QFT: Introduction and overview of the subject and the course (not in PS)

Why is non-relativistic QM not enough? Or even relativistic QM?

Two key reasons are (to be studied later):
1. Particles can be created and annihilated in scattering processes,
2. Physical processes obey causality.

QFT = QM (framework) + field theory (phenomenology)

- The QM framework provides
  - 1) unitarity (conservation of probability)
  - 2) real energies (eigenvalues of the Hamiltonian)
- Nature demands also (no violations ever detected)
  - 3) Poincaré invariance (Lorentz + translations, Lorentz invariance $\Rightarrow$ causality)
  - 4) stability (there must exist a state, the vacuum, with a lowest possible energy)
  - 5) CPT invariance (see later in the course)
  - 6) spin - statistics theorem: integer spin particles are bosons, half integer spin particles are fermions (the latter satisfy the Pauli exclusion principle)
  - 7) conservation of charge (electric and other kinds)
- We want also
  - 8) predictablity, that is renormalisability or finiteness (see later in the course)
  - 9) locality (here this means interacting field theories, see below)
- Note
  - 1), 2), 3) and 4)$\Rightarrow$ 5) and 6) plus gauge invariance for massless fields of spin 1 or higher (see below)
  - the above points 1) - 9) $\Rightarrow$ structure of the standard model of particle physics

– gravity is not included here: Einstein's theory of gravity is an "effective low energy field theory" which is not renormalisable (i.e., not compatible with QFT, more later in the course)
$\Rightarrow$ a consistent quantum gravity theory is needed, e.g., string/M theory. Any low energy field theory that is consistent when coupled to quantum gravity is called **UV complete**. In fact, general relativity appears (as a 2-dim. quantum effect) in string theory as a UV complete generalisation of Einstein's theory. UV complete theories belong to the **landscape** while non-complete ones belong to the **swampland**. This is hot research subject at the moment (2020).

Quantum methods

- 1. QFT: Field theory (elementary excitations = point particles)
  - i) 2nd quantized field theory (QFT) $\Rightarrow$ unitarity, stability ($E \geq E_0$):
    This course!
  - ii) path integrals $\Rightarrow$ Lorentz invariance (Weinberg, QFT, Vol 1, p. 376 - 377)
    - no general proof exists of the equivalence of these two methods
    - for certain restricted Lagrangians there is a proof.
- 2. String theory (elementary excitations = strings, perturbative, see below)
- 3. M-theory (fundamental objects = surfaces, non-perturbative, see below)

Field theories for various spins and their Lagrangians

- Aspects of Lagrangian field theories:
  The Lagrangian L will be very important in this course. In ordinary (Newtonian) mechanics for one particle the Lagrangian is

$$L(x, \dot{x}) := E_{kin} - E_{pot} = \frac{1}{2}m\dot{x}^2 - V(x), \tag{1.1}$$

which can be generalised to many particles, or even infinitely many, by just summing over them $L = \Sigma_{i=1}^{N} L_i$. This is certainly valid for the kinetic terms in E_{kin} while the potential E_{pot} might involve terms with many coordinates and therefore cannot be written as a simple sum like this. This can be further generalised to an integral by viewing the index i as a continuous variable. In field theory the role of this index is then played by the points in space $\mathbf{R}^3$ (or 3-momenta $\mathbf{p}$ as we will see later) which for a free massless scalar field becomes (with $x^\mu = (t, \mathbf{r})$ and $\dot{\phi} := \frac{\partial}{\partial t}\phi$)

$$\begin{aligned} L :&= E_{kin} - E_{pot} = \int d^3x \mathcal{L}(\phi(x), \dot{\phi}(x)) = \int d^3x (\frac{1}{2}\partial_\mu \phi \partial^\mu \phi) \\ &= \int d^3x (\frac{1}{2}\dot{\phi}(t, \mathbf{r})\dot{\phi}(t, \mathbf{r}) - \frac{1}{2}\nabla\phi(t, \mathbf{r}) \cdot \nabla\phi(t, \mathbf{r})), \end{aligned} \tag{1.2}$$

where the first term on the last line is the kinetic term E_{kin} and the second one is part of the potential energy E_{pot}. Using instead a Lorentz covariant language the whole term $\frac{1}{2}\partial_\mu\phi\partial^\mu\phi$ is referred to as the kinetic term. We see also how the overall sign of this term is related to the "mostly negative" signature used in PS.

The potential energy E_{pot} in L may also contain mass terms and higher powers of the fields in question. For a real scalar field these are

$$E_{pot} = \int d^3x V(\phi) = \int d^3x (\frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4). \tag{1.3}$$

The field theory Lagrangian, which is a density, is therefore in this case

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi) = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4. \tag{1.4}$$

That this theory cannot contain an odd power of ϕ as the highest one or higher powers than four are crucial facts that will be explained later.

Another key aspect of a theory defined in terms of a Lagrangian is the role played be the parameters in it, in the above example m and λ, and how they must be determined by experiments before the theory has any **predictive power**. Predictive power must be imposed on any field theory for it to be useful when explaining the outcome of experiments, a fact that puts heavy constrains on the theory. Trying to follow the same logic for general relativity fails as will be clear later in the course.

- Theories familiar from previous courses (at least to some extent).
  - spin 0: Klein-Gordon (ϕ real, Φ complex)
    $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2$ or $\mathcal{L} = \partial_\mu\Phi^\star\partial^\mu\Phi - m^2\Phi^\star\Phi$
  - spin 1/2: Dirac (Weyl, Majorana), derived later!
    $\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$
  - spin 1: Maxwell
    $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$
  - spin 2: Einstein's general relativity (GR)
    $\mathcal{L} = -\frac{1}{16\pi G_N}\sqrt{g}\,R$ (the minus sign is due to the signature used in PS)
    where $R = g^{\mu\nu}R_{\mu\nu}$, the Ricci tensor $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$ and the Riemann tensor
    $R^\mu{}_{\nu\rho\sigma} = \partial_\rho\Gamma^\mu_{\sigma\nu} - \partial_\sigma\Gamma^\mu_{\rho\nu} + \Gamma^\mu_{\rho\tau}\Gamma^\tau_{\sigma\nu} - \Gamma^\mu_{\sigma\tau}\Gamma^\tau_{\rho\nu}$
    $\Gamma^\mu_{\nu\rho} = \frac{1}{2}g^{\mu\tau}(\partial_\nu g_{\rho\tau} + \partial_\rho g_{\nu\tau} - \partial_\tau g_{\nu\rho})$

- Theories introduced in this course (mainly the first three cases)
  - self-interacting neutral spin 0:
    $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$ where ϕ is real

- self-interacting charged spin 0: the *covariant derivative* is $D_\mu = \partial_\mu + ieA_\mu$
  $\mathcal{L} = D_\mu \Phi^* D^\mu \Phi - m^2 \Phi^* \Phi - \frac{\lambda}{2}(\Phi^*\Phi)^2$ where $\Phi \in \mathbf{C}$ and A_μ is Maxwell
- self-interacting spin 1: Yang-Mills theory with gauge group G ($i = 1, 2, .., \dim G$)
  $\mathcal{L} = -\frac{1}{4} F^i_{\mu\nu} F^{i\mu\nu}$ where $F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g f^i{}_{jk} A^j_\mu A^k_\nu$
  * for $G = SU(2)$ the structure constants $f^i{}_{jk} = \epsilon^{ijk}$ (the 3d epsilon tensor)
- topological spin 1 in 4d: "2nd Chern class" (more later if time permits)
  $\mathcal{L} = \frac{1}{8\pi^2} \epsilon^{\mu\nu\rho\sigma} Tr F_{\mu\nu} F_{\rho\sigma}$, $\epsilon^{\mu\nu\rho\sigma}$ is the 4d epsilon tensor in Minkowski space
- topological spin 1 in 3d: "Chern-Simons theory", important in some condensed matter systems and in string/M theory (more later if time permits)
  $\mathcal{L} = \frac{k}{4\pi} \epsilon^{\mu\nu\rho} Tr(A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho)$, $(k \in \mathbf{Z})$,
  $\epsilon^{\mu\nu\rho}$ is the 3d epsilon tensor in Minkowski space

- Physics applications (studied in more advanced courses):
  - Standard model of particle physics: Yang-Mills plus spin 1/2 and 0 fields in 4d
  - Graphene: massless Dirac in 3d plus QED in 4d
  - Topological insulators and FQHE: Chern-Simons plus other fields, all in 3d
  - Superconductors at quantum critical points: conformal in 3d (CFT_3)
  - Phase transitions: CFT_2 and CFT_3
  - String theory: conformal in 2d (CFT_2)
  - M-theory: conformal in 3d (CFT_3) and in 6d (CFT_6)

Spacetime symmetries
In any dimension the symmetries of a field theory in flat spacetime can be

- Non-relativistic (time is absolute): Galilei
- Relativistic: Poincaré (Lorentz plus translations): **This course**
- Conformal: includes scale invariance (CFT_d=conformal field theory in d dimensions)

Coupling constant dependence of cross-sections (and energy levels etc)
A general expansion in coupling constant g of a cross-section $\sigma(g)$ is, for $0 \le g < 1$,

- $\sigma(g) = \sigma_0 + \sigma_1\, g + \sigma_2\, g^2 + ... + e^{-1/g^2}(\sigma_0^{(1)} + \sigma_1^{(1)}\, g + ...) + e^{-2/g^2}(\sigma_0^{(2)} + \sigma_1^{(2)}\, g + ...)...$
  - the $\sigma_n g^n$ terms are called *perturbative* (from perturbation theory: **this course**),
  - the terms $e^{-n/g^2}, n \ge 1$ are called *non-perturbative* (related to solitons and instantons). These terms do not have a power expansion around $g = 0$,
  - a very active research area is *resurgence*[1] which aims at deriving the non-perturbative terms from the perturbative ones.

[1] See, e.g., G. Dunne, ArXiv hep-th/1510.03435.

Duality[2]

The modern view on QFT is that the same physics can be described by different field theories where both the field content and the coupling constants may be different:

- Hamiltonian $H = H_0(\phi) + gH_{int}(\phi) = H'_0(\phi') + g'H'_{int}(\phi')$
  (if evaluated on a physical state, energies are the same since they are measurable!)
- Often the coupling constants are related as $g' = 1/g$ (strong-weak duality, AdS/CFT)
- The relation between the fields is often very complicated (even non-local)
- The role of elementary excitations and solitons are often interchanged (string theory)

This course: QFT from second quantised field theory

- Field theories for spin 0, 1/2, 1, plus comments on Einstein's theory of gravity (GR)
- Spontaneous symmetry breaking and the Higgs effect
- Perturbation theory
- Feynman graph expansion → scattering amplitudes at "tree" and "1-loop" level
- The physical interpretation of the Lagrangian in field theory: Renormalisation
- Running coupling constants and vacuum polarisation (β-functions if time permits): used to argue that the groups $U(1) \times SU(2) \times SU(3)$ in the standard model become unified at $E = 10^{17} GeV$ since the three coupling constants seem to converge to the same value at this scale.

[2] If you are interested, see Polchinski's review on **Dualities** hep-th/1412.5704.